Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 164, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

QUASILINEAR ASYMPTOTICALLY LINEAR SCHRÖDINGER PROBLEM IN $\mathbb{R}^N$ WITHOUT MONOTONICITY

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Communicated by Pavel Drabek

Abstract. We establish existence and non-existence results for a quasilinear asymptotically linear Schrödinger problem. In the first result, we prove that a minimization problem constrained to the Pohozaev manifold is not achieved. In the second, the main argument consists in a splitting lemma for a functional constrained to the Pohozaev manifold. Because of the lack of the monotonicity we are not able to project to the usual Nehari manifold any longer, and this approach is crucial in order to compare the critical level to reach a contradiction. This argument was used in [21, 24, 32] for semilinear equations and in [11] for quasilinear equations.

1. Introduction

Consider the quasilinear Schrödinger problem

$$iz_t = -\Delta z + W(x)z - a(x)h(|z|^2)z - k\Delta l(|z|^2)l'(|z|^2)z, x \in \mathbb{R}^N, \tag{1.1}$$

where $z:\mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $a, W:\mathbb{R}^N \to \mathbb{R}$ is a given potential, k is real constant, and l and h are real functions. The above quasilinear equations have been accepted as models of several physical phenomena corresponding to various types of l; we refer to [14] and references given there for for physical applications of this problems. Specifically, we would like to mention, the superfluid film equation in plasma physics has this structure for $l(s) = s$ (see e.g. [22]), while in the case $l(s) = (1 + s)^{1/2}$, (1.1) models the self-channeling of a high-power ultrashort laser in matter (see e.g. [23]).

The standing waves solutions of (1.1); that is, solutions of the type $z(t, x) = \exp(-iEt)u(x)$ where $E \in \mathbb{R}$ and $u > 0$ is a real function. Inserting z into (1.1), with $l(s) = s$ and $l(s) = (1 + s^2)^{1/2}$, turns, respectively, the following equations:

$$-\Delta u + V_\infty u - k\Delta(u^2))u = a(x)h(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

$$-\Delta u + V_\infty u - k\Delta((1 + u^2)^{1/2})\frac{u}{(1 + u^2)^{1/2}} = a(x)h(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $V_\infty = W - E$.

2010 *Mathematics Subject Classification.* 35J10, 35J20, 35J60, 35Q55.
Key words and phrases. Quasilinear Schrödinger equations; variational methods; asymptotically linear.

Submitted August 27, 2017. Published September 11, 2018.

For (1.2), semilinear case corresponding to $k = 0$ has been studied extensively in recent years, see e.g. [35, 46] for a review. Suppose $k > 0$ and $h(s)$ behaves like a polynomial function $|s|^{r-1}s$, with $r + 1 < 2 \cdot 2^*$ (subcritical) and $r + 1 = 2 \cdot 2^*$ (critical), where $2 \cdot 2^* = 4N/(N-2)$, with $N \geq 3$. The number $2 \cdot 2^*$ is called the *critical exponent* for the equation (1.2), see [30, Remark 3.13]. For the subcritical case, which was started, up to our knowledge, in [34] and extended in [29], variational methods and constrained minimization arguments were used to provide existence of positive solutions results with an unknown Lagrange multiplier λ in front of the nonlinear term. The second method, due to the authors in [30], used a change of variables which allowed to rewrite the functional in semilinear form. Then, they were able to work with a functional well defined in a usual Sobolev space. In this framework, the new problem becomes a nonhomogeneous problem bringing a new difficulty to handle this equation. See also [15]. The critical case was studied, among others, by [20, 28, 33, 40, 44, 45, 31] and in the references therein.

For (1.3), still with $k > 0$, few results are known. In [18] the authors treated existence and uniqueness, but did not study the existence of standing waves. For this class of the solutions, for superlinear perturbation with subcrtical or critical growth, we refer to [14, 13, 38, 19, 43] and references therein. In this situation, we recall that the change of variables as in (1.2), in general, cannot be generalized to this case. In [38] a change of known variable is made and the existence of nontrivial solution is proved.

The purpose of this article is to investigate the existence of positive solutions for an asymptotically linear quasilinear elliptic problem; that is, when h behalves at infinity like s^2. When $k = 0$, the autonomous and quasilinear asymptotically linear problem (1.2) was treated in a pioneering work [8]. In [6, 16, 26, 42] the nonautonomous case was considered. In [1, 17] the problems were studied imposing periodicity conditions on the nonlinearities. In the situation where the involved nonlinearities are homogeneous or preserve some monotonicity conditions, many authors, in order to search for critical points, studied the Euler-Lagrange functional restricted to the Nehari manifold (see [46]). But, it was remarked in [16] that not all functions can be projected on the Nehari manifold, if the nonlinearity is nonhomogeneous and quasilinear asymptotically linear. This projection is very important to compare the critical level of the associated functional. In [5, 21, 32] were studied minimization problems restricted to the Pohozaev manifold. This argument was used, firstly, in [37]. Using this idea in [24] was obtained existence results without monotonicity condition. The main argument is to apply a splitting lemma due to [41], with Cerami condition, see [12], in a functional restricted to the Pohozaev manifold. In this paper, we extend former result in [24], with $k = 0$, for more general quasilinear equations. For a similar argument with Nehari manifold, see [3]. Recently, in [2] and [11] are treated a class of quasilinear asymptotically linear problem slightly different from that studied by us, which are associated to the first problem (1.2).

We will consider the following assumptions on the functions a and h:

- (A1) $a \in C^2(\mathbb{R}^N, \mathbb{R}^+)$, with $\inf_{x \in \mathbb{R}^N} a(x) > 0$;
- (A2) $\lim_{|x| \to \infty} a(x) = a_\infty > V_\infty > 0$, important condition to ensure the existence and non-existence of solution;

(A3) $\nabla a(x)\cdot x \geq 0$, for all $x \in \mathbb{R}^N$, with the strict inequality holding on a subset of positive Lebesgue measure of $\mathbb{R}^N$;
(A4) $a(x) + \frac{\nabla a(x)\cdot x}{N} < a_\infty$, for all $x \in \mathbb{R}^N$;
(A5) $\nabla a(x)\cdot x + \frac{x\cdot\mathcal{H}\cdot x}{N} \geq 0$, for all $x \in \mathbb{R}^N$, where $\mathcal{H}$ represents the hessian matrix of the function a.
(A6) $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\lim_{s\to 0} \frac{h(s)}{s} = 0$;
(A7) $\lim_{s\to\infty} \frac{h(s)}{s^2} = 1$;
(A8) If $H(s) = \int_0^s h(t)\,\mathrm{d}t$ and $Q(s) = \frac{1}{2\sqrt{2}}h(s)s - H(s)$, then there exists a constant $D \geq 1$, such that $0 < Q(s) \leq DQ(t)$, for all $0 < s \leq t$, and $\lim_{s\to\infty} Q(s) = +\infty$.

Next we establish a nonexistence result on the Pohozaev manifold which is defined in (2.7).

Theorem 1.1. *Under assumptions* (A1)–(A8), $p = \inf_{u\in\mathcal{P}} I(u)$ *is not a critical level for the functional* I*. In particular, the infimum* p *is not achieved.*

For stating our existence result, we assume that $|a - a_\infty|$ is not too large.

(A9) Assume that a satisfies
$$\|a_\infty - a\|_{L^\infty} < \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\overline{\theta}^N \|\omega\|_2^2 C},$$
where $\overline{\theta} = \sup_{y\in\mathbb{R}^N} \theta_y$.

Theorem 1.2. *Assume* (A1)-(A9) *and that*

(1) $h \in C^1(\mathbb{R}) \cap Lip(\mathbb{R}, \mathbb{R}^+)$ *and there exists* $\tau > 0$ *such that* $\lim_{s\to 0^+} \frac{h'(s)}{s^\tau} = 0$;
(2) *the least energy level* c_∞ *of* (3.1) *is an isolated radial critical level for* I_∞ *or, equivalently, equation* (3.1) *admits a unique positive solution which is radially symmetric about some point.*

Then the non-autonomous problem (2.4) *admits a positive solution* $v \in H^1(\mathbb{R}^N)$.

Remark 1.3. As an example of a function h such that $h(s)/s^2$ is not increasing and satisfies (A6)–(A8) we have
$$h(s) = \begin{cases} \frac{s^8 - 1{,}5s^6 + 2s^4}{1+s^6} & \text{for } s \geq 0, \\ 0 & \text{for } s \leq 0. \end{cases}$$
As an example for $a(x)$, we have $a(x) = a_\infty - 1/(|x|+k)$, $k > (1/a_\infty)$ with $a_\infty > V_\infty$ positive constants.

2. Preliminaries

We note that the solutions of (1.3) with $k = 1$ are the critical points of the functional
$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \Big(1 + \frac{u^2}{1+u^2}\Big)|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty u^2\,\mathrm{d}x - \int_{\mathbb{R}^N} a(x)H(u)\,\mathrm{d}x.$$
Since the functional $I(u)$ may not be well defined in the usual Sobolev spaces $H^1(\mathbb{R}^N)$, we make the change of variables
$$v = G(u) = \int_0^u g(t)\,\mathrm{d}t, \quad \text{where } g(t) = \sqrt{1 + t^2/(1+t^2)}. \tag{2.1}$$

Since $g(t)$ is bounded and increasing with respect to $|t|$, the inverse function $G^{-1}(t)$ exists. Next we give some properties of the change of variables.

Lemma 2.1 (See [14])**.**
(1) $\sqrt{1/2}\, t \le |G^{-1}(t)| \le t$ *for all* $t \ge 0$*;*
(2) $|(G^{-1}(t))'| \le 1$ *for all* $t \in \mathbb{R}$*;*
(3) $\lim_{t\to 0} |G^{-1}(t)|/t = 1$*;*
(4) $\lim_{t\to\infty} |G^{-1}(t)|/t = \sqrt{1/2}$*;*
(5) $tg'(t)/g(t) \le 3 - \sqrt{8}$ *for all* $t \in \mathbb{R}$*;*
(6) $\sqrt{\frac{1}{2}} G^{-1}(t) \le t(G^{-1}(t))' \le G^{-1}(t)$ *for all* $t \ge 0$*.*

Then after the change of variables, $I(u)$ can be written as

$$\overline{I}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty |G^{-1}(v)|^2 \, dx - \int_{\mathbb{R}^N} a(x) H(G^{-1}(v)) \, dx,$$

for $v \in H^1(\mathbb{R}^N)$. By Lemma 2.1, $\overline{I}(v)$ is well defined in $H^1(\mathbb{R}^N)$ and $\overline{I}(v) \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. If u is a nontrivial solution of (1.3), then for all $\varphi \in C_0^\infty(\mathbb{R}^N)$ it should satisfy

$$\int_{\mathbb{R}^N} \big(g^2(u)\nabla u \nabla\varphi + g(u)g'(u)|\nabla u|^2\varphi + V_\infty u\varphi - a(x)h(u)\varphi\big) \, dx = 0. \tag{2.2}$$

We show that (2.2) is equivalent to

$$\overline{I}'(v)\psi = \int_{\mathbb{R}^N} \Big(\nabla u \nabla\psi + \frac{V_\infty G^{-1}(v)}{g(G^{-1}(v))}\psi - \frac{a(x)h(G^{-1}(v))}{g(G^{-1}(v))}\psi\Big) \, dx = 0, \tag{2.3}$$

for all $\psi \in C_0^\infty(\mathbb{R}^N)$. Indeed, if we choose $\varphi = \big(\frac{1}{g(u)}\big)\psi$ in (2.2), then we obtain (2.3), since $g(t)$ is bounded, increasing function and positive. On the other hand, since $u = G^{-1}(v)$, if we let $\psi = g(u)\varphi$ in (2.3), we obtain (2.2). Therefore, to find the nontrivivial solutions of (1.3), it suffices to study the existence of the nontrivial solutions to the equation

$$-\Delta v = -\frac{V_\infty G^{-1}(v)}{g(G^{-1}(v))} + \frac{a(x)h(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N,\ v \in H^1(\mathbb{R}^N). \tag{2.4}$$

This means, the critical points of functional $\overline{I}(v)$, associated to equation (2.4), are weak solutions of (2.4).

Conditions (A6) and (A7) imply that given $\varepsilon > 0$ and $3 \le p \le 2^*$, there exists a positive constant $C = C(\varepsilon, p)$ such that for all s in $\mathbb{R}$,

$$|H(s)| \le \frac{\varepsilon}{2}|s|^2 + C|s|^p, \quad \forall s \in \mathbb{R}\,. \tag{2.5}$$

We also obtain the estimate

$$|h(s)| \le \varepsilon|s| + C|s|^{p-1}, \quad \forall s \in \mathbb{R}.$$

In what follows we denote by $\|v\|^2 = \int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty v^2) \, dx$ the norm in $H^1(\mathbb{R}^N)$.

Now we consider the Pohozaev identity

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, dx &= N\int_{\mathbb{R}^N} \Big(a(x)H(G^{-1}(v)) - \frac{V_\infty}{2}(G^{-1}(v))^2\Big) dx \\ &\quad + \int_{\mathbb{R}^N} \nabla a(x)\cdot x \cdot H(G^{-1}(v)) \, dx, \end{aligned} \tag{2.6}$$

where $H(s) = \int_0^s h(t) \, dt$ and define $\overline{G}(x, G^{-1}(v)) = a(x)H(G^{-1}(v)) - \frac{V_\infty}{2}(G^{-1}(v))^2$.

We also define the Pohozaev manifold associated to (2.4) by

$$\mathcal{P} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : u \text{ satisfies (2.6)}\}. \tag{2.7}$$

Lemma 2.2. *Let the functional $J{:}H^1(\mathbb{R}^N) \to \mathbb{R}$ be*

$$J(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - N\int_{\mathbb{R}^N} \overline{G}(x, G^{-1}(v)) \,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla a(x).x.H(G^{-1}(v)) \,\mathrm{d}x.$$

Then:

(a) *$\{v \equiv 0\}$ is an isolated point of $J^{-1}(\{0\})$;*
(b) *$\mathcal{P} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : J(v) = 0\}$ is a closed set;*
(c) *$\mathcal{P}$ is a C^1 manifold;*
(d) *there exists $\sigma > 0$ such that $\|v\| > \sigma$, for all $v \in \mathcal{P}$.*

Proof. (a) Using condition (A4), we have

$$J(v) > \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - Na_\infty \int_{\mathbb{R}^N} H(G^{-1}(v)) \,\mathrm{d}x + N\int_{\mathbb{R}^N} \frac{V_\infty (G^{-1}(v))^2}{2} \,\mathrm{d}x.$$

By the Sobolev embedding, condition (2.5) and Lemma 2.1(1), we obtain

$$\begin{aligned}
J(v) &\geq \Big(\frac{N-2}{4}\Big)\int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty v^2) \,\mathrm{d}x \\
&\quad - Na_\infty \int_{\mathbb{R}^N} \big(\frac{\varepsilon}{2}|G^{-1}(v)|^2 + C|G^{-1}(v)|^p\big) \,\mathrm{d}x \\
&\geq \frac{N-2}{4}\|v\|^2 - \frac{Na_\infty}{V_\infty}\int_{\mathbb{R}^N} \big(\frac{V_\infty \varepsilon |v|^2}{2} + C|v|^p\big) \,\mathrm{d}x \\
&\geq \frac{1}{2}\Big(\frac{N-2}{2} - \frac{Na_\infty \varepsilon}{V_\infty}\Big)\|v\|^2 - Na_\infty C\|v\|^p.
\end{aligned}$$

If $\varepsilon > 0$ is sufficiently small, there exists $\rho \in (0,1)$ such that $J(v) > 0$ if $0 < \|u\| < \rho$.

(b) Since $J(v)$ is a C^1 functional, $\mathcal{P} \cup \{0\} = J^{-1}(\{0\})$ is a closed subset. Moreover, $\{u \equiv 0\}$ is an isolated point in $J^{-1}(\{0\})$ and claim in (b) follows.

(c) Consider the derivative of J at v and applied to v, thus obtain

$$\begin{aligned}
&J'(v)v \\
&= (N-2)\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - N\int_{\mathbb{R}^N} \big(a(x) + \frac{\nabla a(x).x}{N}\big) h(G^{-1}(v))(G^{-1}(v))'v \,\mathrm{d}x \\
&\quad + N\int_{\mathbb{R}^N} V_\infty G^{-1}(v)(G^{-1}(v))'v \,\mathrm{d}x.
\end{aligned} \tag{2.8}$$

Since $v \in \mathcal{P}$, inserting (2.6) into (2.8) with Lemma 2.1(1-6) and (A8) we obtain

$$\begin{aligned}
J'(v)v &\leq 2N\int_{\mathbb{R}^N} \Big(a(x) + \frac{\nabla a(x).x}{N}\Big)\Big(H(G^{-1}(v)) - \frac{h(G^{-1}(v))(G^{-1})'v}{2}\Big) \,\mathrm{d}x \\
&\leq 2N\int_{\mathbb{R}^N} \Big(a(x) + \frac{\nabla a(x).x}{N}\Big)\Big(H(G^{-1}(v)) - \frac{h(G^{-1}(v))(G^{-1}(v))}{2\sqrt{2}}\Big) \,\mathrm{d}x < 0.
\end{aligned}$$

This shows that $\mathcal{P}$ is a C^1 manifold.

(d) Since $v \in \mathcal{P}$, by condition (A4), (2.5) and Lemma 2.1(1), we obtain

$$(N-2)\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + N\int_{\mathbb{R}^N} V_\infty (G^{-1}(v))^2 \,\mathrm{d}x$$

$$
\begin{aligned}
&= 2N \int_{\mathbb{R}^N} \Big(a(x) + \frac{\nabla a(x).x}{N}\Big) H(G^{-1}(v))\,\mathrm{d}x \\
&< 2N \int_{\mathbb{R}^N} a_\infty H(G^{-1}(v))\,\mathrm{d}x \\
&\le 2N a_\infty \int_{\mathbb{R}^N} \Big(\frac{\varepsilon}{2}|G^{-1}(v)|^2 + C|G^{-1}(v)|^p\Big)\,\mathrm{d}x \\
&\le \frac{2Na_\infty}{V_\infty} \int_{\mathbb{R}^N} \frac{V_\infty \varepsilon}{2}|v|^2\,\mathrm{d}x + 2Na_\infty C \int_{\mathbb{R}^N} |v|^p\,\mathrm{d}x.
\end{aligned}
$$

Thus, using again Lemma 2.1(1), we obtain

$$
(N-2)\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + NV_\infty\Big(\frac{1}{2} - \frac{a_\infty \varepsilon}{V_\infty}\Big)\int_{\mathbb{R}^N} |v|^2\,\mathrm{d}x \le 2Na_\infty C \int_{\mathbb{R}^N} |v|^p\,\mathrm{d}x.
$$

Therefore, by the Sobolev embedding, we obtain $0 < \sigma^{p-2} = C\|v\|^{p-2}$ for all v. $\square$

3. Nonexistence result

We begin by presenting the main relations between the Pohozaev manifold $\mathcal{P}$ associated with the non-autonomous problem (2.4), and the Pohozaev manifold $\mathcal{P}_\infty$ associated with the autonomous problem at infinity

$$
-\Delta v = (G^{-1}(v))'[a_\infty h(G^{-1}(v)) - V_\infty G^{-1}(v)], x \in \mathbb{R}^N, \quad v \in H^1(\mathbb{R}^N). \tag{3.1}
$$

We have the Pohozaev manifold

$$
\mathcal{P}_\infty := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : J_\infty(v) = 0\},
$$

with

$$
J_\infty(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x - N\int_{\mathbb{R}^N} G_\infty(v)\,\mathrm{d}x,
$$

where

$$
G_\infty(G^{-1}(v)) = a_\infty H(G^{-1}(v)) - \frac{V_\infty (G^{-1}(v))^2}{2}.
$$

For our purposes, we need to consider the functional I_∞ associated with $(\mathcal{P}_\infty)$ and given by

$$
I_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^2 + V_\infty (G^{-1}(v))^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} a_\infty H(G^{-1}(v))\,\mathrm{d}x, \quad v \in H^1(\mathbb{R}^N).
$$

We set of paths

$$
\Gamma_\infty = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0,\ I_\infty(\gamma(1)) < 0\ \},
$$

and define the mini-max mountain pass level

$$
c_\infty := \min_{\gamma \in \Gamma_\infty} \max_{0 \le t \le 1} I_\infty(\gamma(t)).
$$

Note that (A3) and (A4) imply that $I_\infty(u) < \overline{I}(u)$ for all u in $H^1(\mathbb{R}^N) \setminus \{0\}$. We will show at the end of this section that

$$
p := \inf_{u \in \mathcal{P}} \overline{I}(u) = c_\infty,
$$

and that this is not achieved, which means that this is not a critical level for the functional $\overline{I}$.

Lemma 3.1. *Suppose that $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x > 0$. Then there exist unique $\theta_1 > 0$ and $\theta_2 > 0$ such that $v(\cdot/\theta_1) \in \mathcal{P}$ and $v(\cdot/\theta_2) \in \mathcal{P}_\infty$.*

Proof. The case of projecting on $\mathcal{P}_\infty$ is already known [32]. We will verify the case of $\mathcal{P}$. First we define the function

$$\begin{aligned}\psi(\theta) &:= \overline{I}\big(v(\frac{x}{\theta})\big) = \frac{\theta^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2\ \mathrm{d}x - \int_{\mathbb{R}^N} a(x)H\left(G^{-1}\big(v(\frac{x}{\theta})\big)\right)\ \mathrm{d}x \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty\left(G^{-1}\big(v(\frac{x}{\theta})\big)\right)^2\ \mathrm{d}x \\ &= \frac{\theta^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla v|^2\ \mathrm{d}x - \theta^N\int_{\mathbb{R}^N} a(\theta x)H(G^{-1}(v(x)))\ \mathrm{d}x \\ &\quad + \frac{\theta^N}{2}\int_{\mathbb{R}^N} V_\infty(G^{-1}(v(x)))^2\ \mathrm{d}x.\end{aligned}$$

Taking the derivative of $\psi(\theta)$ and recalling that we are considering $N \geq 3$, we obtain

$$\begin{aligned}\psi'(\theta) &= \theta^{N-3}\Big[\big(\frac{N-2}{2}\big)\int_{\mathbb{R}^N}|\nabla v|^2\ \mathrm{d}x - N\theta^2\int_{\mathbb{R}^N} a(\theta x)H(G^{-1}(v(x))\ \mathrm{d}x \\ &\quad - \theta^2\int_{\mathbb{R}^N}\nabla a(\theta x)\cdot\theta x\cdot H(G^{-1}(v(x)))\ \mathrm{d}x + \frac{N\theta^2}{2}\int_{\mathbb{R}^N} V_\infty(G^{-1}(v(x)))^2\ \mathrm{d}x\Big].\end{aligned}$$

Hence, $v(\cdot/\theta) \in \mathcal{P}$ if and only if $\psi'(\theta) = 0$, for some $\theta > 0$. Note that, by condition (A2) and the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}&\lim_{\theta\to\infty}\int_{\mathbb{R}^N}\Big(a(\theta x)H(G^{-1}(v(x))) - V_\infty\frac{(G^{-1}(v(x)))^2}{2}\Big)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^N}\Big(a_\infty H(G^{-1}(v(x))) - \frac{V_\infty(G^{-1}(v(x)))^2}{2}\Big)\ \mathrm{d}x \\ &= \int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\ \mathrm{d}x.\end{aligned}$$

By (A2)–(A4), we obtain $\nabla a(x)x \to 0$ if $|x| \to +\infty$. Thus

$$\lim_{\theta\to\infty}\int_{\mathbb{R}^N}\nabla a(\theta x)\cdot\theta x\cdot H(G^{-1}(v))\ \mathrm{d}x = 0.$$

Therefore, if $\theta > 0$ is sufficiently large, then

$$\psi'(\theta) = \theta^{N-3}\Big\{\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\ \mathrm{d}x - N\theta^2\Big(\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\ \mathrm{d}x + o_\theta(1)\Big)\Big\}.$$

Since $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\ \mathrm{d}x > 0$, it follows that $\psi'(\theta) < 0$, for $\theta > 0$ sufficiently large.

On the other hand, by condition (A4), conditions (A2) and (A3) yield

$$\begin{aligned}-\frac{V_\infty}{2}\int_{\mathbb{R}^N}(G^{-1}(v))^2\ \mathrm{d}x &\leq \int_{\mathbb{R}^N}\Big(a(\theta x) + \frac{\nabla a(\theta x)(\theta x)}{N}\Big)H(G^{-1}(v))\ \mathrm{d}x \\ &\quad - \frac{V_\infty}{2}\int_{\mathbb{R}^N}(G^{-1}(v))^2\ \mathrm{d}x.\end{aligned}$$

By (2.5) Lemma 2.1(1), and taking $\theta > 0$ sufficiently small we have

$$-\frac{V_\infty}{2}\int_{\mathbb{R}^N}(G^{-1}(v))^2\ \mathrm{d}x \leq \int_{\mathbb{R}^N}\Big(a_\infty H(G^{-1}v) - \frac{V_\infty(G^{-1}(v))^2}{2}\Big)\ \mathrm{d}x \leq C\|v\|_2^2.$$

The previous inequalities imply that there exist positive constants A and B, independent of θ, such that

$$-A \leq \int_{\mathbb{R}^N} \Big[\Big(a(\theta x) + \frac{\nabla a(\theta x)\cdot \theta x}{N}\Big) H(G^{-1}(v)) - \frac{V_\infty (G^{-1}(v))^2}{2}\Big]\,\mathrm{d}x \leq B.$$

Thus, taking $\theta > 0$ sufficiently small in the expression of $\psi'(\theta)$, we obtain $\psi'(\theta) > 0$. Since ψ' is continuous, there exists at least one $\theta_1 = \theta_1(v)$, $\theta_1 > 0$, such that $\psi'(\theta_1) = 0$, which means that $v(\cdot/\theta_1) \in \mathcal{P}$.

To show the uniqueness of θ_1, note that $\psi'(\theta) = 0$ implies

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x &= N\theta^2 \int_{\mathbb{R}^N} \Big(a(\theta x) + \frac{\nabla a(\theta x)(\theta x)}{N}\Big) H(G^{-1}(v))\,\mathrm{d}x \\ &\quad - \frac{N\theta^2 V_\infty}{2}\int_{\mathbb{R}^N} (G^{-1}(v))^2\,\mathrm{d}x,\end{aligned}$$

with $\theta > 0$, or equivalently

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x = N\theta^2 \varphi(\theta),$$

giving

$$\varphi(\theta) = \int_{\mathbb{R}^N} \Big[a(\theta x) + \frac{\nabla a(\theta x)\cdot(\theta x)}{N}\Big] H(G^{-1}(v))\,\mathrm{d}x - V_\infty \int_{\mathbb{R}^N} \frac{(G^{-1}(v))^2}{2}\,\mathrm{d}x.$$

Taking the derivative of φ and using the properties of the functions involved,

$$\varphi'(\theta) = \frac{1}{\theta}\int_{\mathbb{R}^N} \Big(\nabla a(\theta x)\cdot(\theta x) + \frac{(\theta x)\cdot \mathcal{H}\cdot(\theta x)}{N} + \frac{\nabla a(\theta x).(\theta x)}{N}\Big) H(G^{-1}(v))\,\mathrm{d}x.$$

Hypotheses (A3) and (A5), with the conditions on function H, imply that $\varphi'(\theta) > 0$. Therefore, $\varphi(\theta)$ is an increasing of θ and hence there exists a unique $\theta > 0$ such that

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x = N\theta^2 \varphi(\theta).$$

□

Lemma 3.2. *Let* $\mathcal{O} = \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x > 0\}$ *be an open subset of* $H^1(\mathbb{R}^N)$. *The function* $\theta_1 : \mathcal{O} \to \mathbb{R}^+$ *defined by* $v \mapsto \theta_1(v)$, *such that* $v(\cdot/\theta_1(v)) \in \mathcal{P}$, *is continuous.*

Proof. Consider $(v_n) \subset \mathcal{O}$ such that $v_n \to v$, as $n \to \infty$. We will show that $\theta_1(v_n) \to \theta_1(v)$, as $n \to \infty$. First note that $\theta_1(v_n)$ is bounded. Indeed, consider the expression $\psi'(\theta) = 0$ in the proof of the previous lemma applied to v_n and $\theta_1(v_n)$

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,\mathrm{d}x \\ &= N\theta_1^2(v_n)\int_{\mathbb{R}^N} a(\theta_1(v_n)x) H(G^{-1}(v_n))\,\mathrm{d}x - \frac{N\theta_1^2(v_n)}{2}\int_{\mathbb{R}^N} V_\infty (G^{-1}(v_n))^2\,\mathrm{d}x \\ &\quad + N\theta_1^2(v_n)\int_{\mathbb{R}^N} \frac{\nabla a(\theta_1(v_n)x)\cdot(\theta_1(v_n)x).H(G^{-1}(v_n))}{N}\,\mathrm{d}x.\end{aligned}$$

Since $\theta_1(v_n) > 0$, we suppose by contradiction that $\theta_1(v_n) \to +\infty$, as $n \to \infty$. Thus the right-hand side of the equation above goes to infinity while the left hand side tends to $\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x < \infty$. Therefore, we conclude that $\theta_1(v_n)$ is bounded sequence and thus has a convergent subsequence, let us say $\theta_1(v_n) \to \overline{\theta}_1$, as $n \to \infty$. Again using Lebesgue Dominated Convergence Theorem, as $n \to \infty$, we have

$$\int_{\mathbb{R}^N} a(\theta_1(v_n)x)H(G^{-1}(v_n))\,\mathrm{d}x \to \int_{\mathbb{R}^N} a(\overline{\theta}_1 x)H(G^{-1}(v))\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \frac{\nabla a(\theta_1(v_n)x)\cdot(\theta_1(v_n)x)}{N}H(G^{-1}(v_n))\,\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{\nabla a(\overline{\theta}_1 x)(\overline{\theta}_1 x)}{N}H(G^{-1}(v))\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} |\nabla v_n|^2\,\mathrm{d}x \to \int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x,$$

$$\int_{\mathbb{R}^N} v_n^2\,\mathrm{d}x \to \int_{\mathbb{R}^N} v^2\,\mathrm{d}x,$$

since $v_n \to v$ in $H^1(\mathbb{R}^N)$, as $n \to \infty$. Thus, we obtain

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x\\ &= N\overline{\theta}_1^2\int_{\mathbb{R}^N}\Big[\Big(a(\overline{\theta}_1 x)+\frac{\nabla a(\overline{\theta}_1 x)(\overline{\theta}_1 x)}{N}\Big)H(G^{-1}(v)) - \frac{V_\infty}{2}(G^{-1}(v))^2\Big]\,\mathrm{d}x\,.\end{aligned}$$

It follows that $\overline{\theta}_1$ is such that $v(\cdot/\overline{\theta}_1) \in \mathcal{P}$. The uniqueness of the projection in $\mathcal{P}$ implies that $\overline{\theta}_1 = \theta_1(v)$. Hence, $\theta_1(v_n) \to \theta_1(v)$ in $\mathbb{R}$, as $n \to \infty$. □

Lemma 3.3. *If $v \in \mathcal{P}_\infty$, then there exists $\theta > 0$ such that $v(\cdot/\theta) \in \mathcal{P}$ and $\theta > 1$.*

Proof. Let $v \in \mathcal{P}_\infty$, then $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x > 0$ and Lemma 3.1 asserts the existence of a unique θ such that $v(\cdot/\theta) \in \mathcal{P}$. Now we obtain

$$\begin{aligned}0 = \theta^{N-2}\Big[&\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x - N\theta^2\Big(\int_{\mathbb{R}^N} a(\theta x)H(G^{-1}(v) - \frac{V_\infty(G^{-1}(v))^2}{2}\,\mathrm{d}x\\ &+\int_{\mathbb{R}^N}\frac{\nabla a(\theta x)\cdot\theta x}{N}H(G^{-1}(v))\,\mathrm{d}x\Big)\Big]\end{aligned}$$

and since $\theta > 0$, by (A4), it follows that

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x < N\theta^2\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x,$$

or equivalently

$$(2^*)^{-1}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x < \theta^2\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x.$$

But since $v \in \mathcal{P}_\infty$, we have

$$(2^*)^{-1}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x.$$

Therefore, the inequality above is true if and only if $\theta > 1$. □

Lemma 3.4. *If $v \in \mathcal{P}$, then there exists $\theta > 0$ such that $v(\cdot/\theta) \in \mathcal{P}_\infty$ and $\theta < 1$.*

Proof. First, we must verify that if $v \in \mathcal{P}$, then $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x > 0$. In fact, using condition (A4), if $v \in \mathcal{P}$, then v satisfies

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x &< N\int_{\mathbb{R}^N}\Big(a_\infty H(G^{-1}(v) - \frac{V_\infty(G^{-1}(v))^2}{2}\Big)\,\mathrm{d}x\\ &= N\int_{\mathbb{R}^N} G_\infty(G^{-1}(v))\,\mathrm{d}x.\end{aligned}$$

Since $v \not\equiv 0$ and $v \in H^1(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x > 0$; therefore we also have $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x > 0$. The existence of an unique $\theta > 0$ such that $v(\cdot/\theta) \in \mathcal{P}_\infty$ is guaranteed by Lemma 3.1. In order to show that $\theta < 1$, we note that

$$\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x < \int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x.$$

However, if $v(\cdot/\theta) \in \mathcal{P}_\infty$, then θ satisfies

$$\theta^2 = \frac{(2^*)^{-1} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x} < \frac{\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x}{\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x} = 1.$$

Therefore $\theta < 1$. $\square$

Remark 3.5. An immediate consequence of the previous lemmas is that $v \in H^1(\mathbb{R}^N)\backslash\{0\}$ can be projected on $\mathcal{P}$ and on $\mathcal{P}_\infty$ if and only if $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x > 0$.

Lemma 3.6. *If $v \in \mathcal{P}_\infty$, then $v(\cdot - y) \in \mathcal{P}_\infty$, for all $y \in \mathbb{R}^N$. Moreover, there exists $\theta_y > 1$ such that $v(\frac{\cdot - y}{\theta_y}) \in \mathcal{P}$ and $\lim_{|y|\to+\infty} \theta_y = 1$.*

Proof. If $v \in \mathcal{P}_\infty$, then it follows from the translation invariance of I_∞ that $v(\cdot - y) \in \mathcal{P}_\infty$, for all $y \in \mathbb{R}^N$. Furthermore, from Lemma 3.1 there exists $\theta_y > 1$ such that $v(\frac{\cdot - y}{\theta_y}) \in \mathcal{P}$. Suppose now, by contradiction, that there exists a sequence $y_n \in \mathbb{R}^N$ such that $|y_n| \to +\infty$ and $\theta_{y_n} \to A > 1$ or $+\infty$, as $n \to \infty$. Let us define

$$K(\theta_{y_n} x + y_n) := a(\theta_{y_n} x + y_n) + \frac{\nabla a(\theta_{y_n} x + y_n) \cdot (\theta_{y_n} x + y_n)}{N}.$$

From (2.5) and (A4) we have that

$$\begin{aligned} K(\theta_{y_n} x + y_n) H(G^{-1}(v)) - \frac{V_\infty (G^{-1}(v))^2}{2} &< a_\infty H(G^{-1}(v)) - \frac{V_\infty (G^{-1}(v))^2}{2} \\ &\le a_\infty (1 + C) |G^{-1}(v)|^2 \in L^1(\mathbb{R}^N). \end{aligned}$$

Hence, using the Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned} &\lim_{y_n \to \infty} \int_{\mathbb{R}^N} K(\theta_{y_n} x + y_n) H(G^{-1}(v)) \,\mathrm{d}x - V_\infty \int_{\mathbb{R}^N} \frac{(G^{-1}(v))^2}{2} \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x. \end{aligned}$$

But for each y_n it follows that $v(\frac{\cdot - y_n}{\theta_{y_n}}) \in \mathcal{P}$ with $\theta_{y_n} > 1$, which implies

$$\begin{aligned} &\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x \\ &= N\theta_{y_n}^2 \int_{\mathbb{R}^N} \Big[K(\theta_{y_n} x + y_n) H(G^{-1}(v)) - \frac{V_\infty (G^{-1}(v))^2}{2} \Big] \,\mathrm{d}x. \end{aligned} \tag{3.2}$$

The right-hand side of (3.2) approaches infinity or $NA^2 \int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x$, however the left-hand side is fixed on $\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x$. Since $v \in \mathcal{P}_\infty$ and $A > 1$ or $+\infty$, we arrive at a contradiction. $\square$

Lemma 3.7. $\sup_{y \in \mathbb{R}^N} \theta_y = \overline{\theta} < \infty$ *and* $\overline{\theta} > 1$.

Proof. From Lemma 3.6, given $\varepsilon = 1$ there exists $R > 0$ such that $|\theta_y| \le 2$ if $|y| > R$. We will show that there exists $M > 0$ such that $\sup_{0\le|y|\le R} \theta_y \le M$. Suppose that this supremum is not finite, or equivalently, that there exists a sequence $y_n \in \mathbb{R}^N$ with $|y_n| \in [0, R]$ such that $\theta_{y_n} \to \infty$, as $n \to \infty$. As in the previous lemma, but now with $\theta_{y_n} \to \infty$,as $n \to \infty$, we can prove that

$$\lim_{\theta_{y_n}\to\infty} \int_{\mathbb{R}^N} \Big[K(\theta_{y_n} x + y_n)H(G^{-1}(v)) - \frac{V_\infty G^{-1}(v)}{2}\Big] \,\mathrm{d}x = \int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x.$$

Therefore, it follows from (3.2) that

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x = N\theta_{y_n}^2 \Big(\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) + o_{y_n}(1)\Big) \,\mathrm{d}x.$$

But, since $\theta_{y_n} \to \infty$, as $n \to \infty$, and the left-hand side is a fixed number, this is an absurd, unless the supremum exists. □

Lemma 3.8. *There exists a real number $\widehat{\sigma} > 0$ such that $\inf_{u\in\mathcal{P}} \|\nabla v\|_2 \ge \widehat{\sigma}$.*

Proof. Let $v \in \mathcal{P}$, then v satisfies Pohozaev identity and by condition (A4) we have

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x < N \int_{\mathbb{R}^N} \Big[a_\infty H(G^{-1}(v)) - \frac{V_\infty (G^{-1}(v))^2}{2}\Big] \,\mathrm{d}x.$$

On the other hand, from condition (2.5), with $p = 2^*$, by Lemma 2.1(1), and a given $\frac{V_\infty}{a_\infty} > \varepsilon > 0$, we obtain

$$\frac{1}{2^*} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x < a_\infty C \int_{\mathbb{R}^N} |G^{-1}(v)|^{2^*} \,\mathrm{d}x < a_\infty C \int_{\mathbb{R}^N} |v|^{2^*} \,\mathrm{d}x$$

and using Sobolev-Gagliardo-Nirenberg Theorem (see [10]) in the above inequality, we obtain

$$0 < \frac{1}{2^* a_\infty C} < \|\nabla v\|_2^{2^*-2}.$$

Hence, $\inf_{v\in\mathcal{P}} \|\nabla v\|_2 \ge \widehat{\sigma}$, with $\widehat{\sigma} > \big(\frac{1}{2^* a_\infty C}\big)^{\frac{1}{2^*-2}} > 0$. □

Lemma 3.9. $p := \inf_{v\in\mathcal{P}} \overline{I}(v) > 0$.

Proof. Let $v \in \mathcal{P}$, with (A3) and Lemma 3.8, then $\overline{I}(v)$ satisfies

$$\begin{aligned}
\overline{I}(v) &= \frac{1}{N}\Big(\int_{\mathbb{R}^N} \nabla a(x) x H(G^{-1}(v)) \,\mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x\Big) \\
&\ge \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x \\
&\ge \frac{1}{N} \widehat{\sigma}^2 > 0.
\end{aligned}$$

Thus we obtain that $p > 0$. □

Using $g(t)$ is bounded and increasing function in relation to $|t|$, the inverse function $G^{-1}(t)$ and some algebraic manipulations, we obtain the following result.

Remark 3.10. In [21] is proved that $\inf_{v\in\mathcal{P}_\infty} I_\infty(v) = c_\infty$.

Remark 3.11. If $v \in H^1(\mathbb{R}^N)$, with $\int_{\mathbb{R}^N} G_\infty(G^{-1}(v)) \,\mathrm{d}x > 0$ and $\theta > 0$ is such that $v(\cdot/\theta) \in \mathcal{P}_\infty$, then we can write

$$I_\infty(v(x/\theta)) = \frac{\theta^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x. \tag{3.3}$$

Lemma 3.12. *With the above notation $p = c_\infty$.*

Proof. Let $\omega \in H^1(\mathbb{R}^N)$ be the ground state solution of the problem at infinity, $\omega \in \mathcal{P}_\infty$ and $I_\infty(\omega) = c_\infty$. Given any $y \in \mathbb{R}^N$, we define $\omega_y := \omega(x-y)$. From the translation invariance of the integrals, we obtain $\omega_y \in \mathcal{P}_\infty$ and $I_\infty(\omega_y) = c_\infty$. From Lemma 3.3, for any $y \in \mathbb{R}^N$, there exists a $\theta_y > 1$ such that $\overline{\omega}_y = \omega_y(\cdot/\theta_y) \in \mathcal{P}$. Therefore,

$$\begin{aligned} |\overline{I}(\overline{\omega}_y) - c_\infty| &= |\overline{I}(\overline{\omega}_y) - I_\infty(\omega_y)| \\ &\le \frac{|\theta_y^{N-2} - 1|}{2} \int_{\mathbb{R}^N} |\nabla \omega|^2 \, \mathrm{d}x + |\theta_y^N - 1| \int_{\mathbb{R}^N} \frac{V_\infty (G^{-1}(\omega))^2}{2} \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |H(G^{-1}(\omega))||a_\infty - \theta_y^N a(x\theta_y + y)| \, \mathrm{d}x. \end{aligned}$$

Since $\theta_y \to 1$, if $|y| \to \infty$, it follows that

$$|\overline{I}(\overline{\omega}_y) - c_\infty| \le o_y(1) + o_y(1) + \int_{\mathbb{R}^N} |H(G^{-1}(\omega))||a_\infty - a(x+y)| \, \mathrm{d}x,$$

and since $a(x+y) \to a_\infty$, as $|y| \to \infty$, it follows that

$$\lim_{|y|\to\infty} \overline{I}(\overline{\omega}_y) = c_\infty.$$

Therefore, $p = \inf_{v \in \mathcal{P}} \overline{I}(v) \le c_\infty$.

On the other hand, consider $v \in \mathcal{P}$ and $0 < \theta < 1$ such that $v(\cdot/\theta) \in \mathcal{P}_\infty$. Since $v \in \mathcal{P}$, then using (3.3) and (A3) we obtain

$$\bar{I}(v) > \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x \ge \frac{\theta^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x = I_\infty(v(x/\theta)) \ge c_\infty.$$

□

Thus, for any $v \in \mathcal{P}$, $\overline{I}(v) > c_\infty$ and hence $\inf_{v \in \mathcal{P}} \overline{I}(v) \ge c_\infty$. We conclude that $p = c_\infty$.

Now we are ready to prove Theorem 1.1, which is the main result in this section.

Proof of Theorem 1.1. Suppose, by contradiction, that there exists $\varsigma \in H^1(\mathbb{R}^N)$, a critical point of the functional $\overline{I}$ at level p. In particular, that $\varsigma \in \mathcal{P}$ and $\overline{I}(\varsigma) = p$. Let $\theta \in (0,1)$ be such that $\varsigma(x/\theta) \in \mathcal{P}_\infty$. Then using (A3) and (3.3), we obtain

$$\begin{aligned} p = \overline{I}(\varsigma) &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \varsigma|^2 \, \mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} \nabla a(x).x.H(G^{-1}(\varsigma)) \, \mathrm{d}x \\ &> \frac{1}{N} \int_{\mathbb{R}^N} |\nabla \varsigma|^2 \, \mathrm{d}x > \frac{\theta^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla \varsigma|^2 \, \mathrm{d}x \\ &= I_\infty(\varsigma(x/\theta)) \ge c_\infty. \end{aligned}$$

Therefore $p > c_\infty$, which contradicts the previous lemma. □

4. Existence of a positive solution

This section we will show the existence of a positive solution for the equation (1.3) by showing the existence of a positive solution for the dual equation (2.4). By the previous theorem, we should look for solutions which have energy levels above of c_∞. We start by showing that the min-max levels of Mountain Pass Theorem, see [4], for the functional $\overline{I}$ and I_∞ are equal. It is easy to verify that the functional

$\overline{I}$ satisfies the geometrical hypotheses of the Mountain Pass Theorem. For instance, the condition (2.5) give us a local minimun at the origin, while if we take ω the ground state solution of the problem at infinity, then $z_1 = \omega(\frac{\cdot - y}{t})$, for $|y|$ and t sufficiently large, by (A2) we obtain $\overline{I}(z_1) < 0$.

Let c be the min-max mountain pass level for the functional $\overline{I}$ given by

$$c := \min_{\gamma\in\Gamma} \max_{0\le t\le 1} \overline{I}(\gamma(t)), \tag{4.1}$$

where

$$\Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \overline{I}(\gamma(1)) < 0\}.$$

Lemma 4.1. *For the above quantities, $p = c = c_\infty$*

The proof of the above lemma is analogous to [24, Lemmas 4.1 and 4.2]. Now we observe the following property of the Pohozaev manifold $\mathcal{P}$ with respect to the paths in the Mountain Pass Theorem.

Lemma 4.2. *If $\gamma \in \Gamma$, then there exists $s \in (0,1)$ such that $\gamma(s)$ intersects $\mathcal{P}$.*

Proof. From Lemma 2.2(a) there exists $\rho > 0$ such that, if $0 < \|v\| < \rho$, then $J(v) > 0$. Furthermore, observe that

$$J(v) = N\overline{I}(v) - \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \nabla a(x).x.H(G^{-1}(v)) \,\mathrm{d}x.$$

From condition (A3) it follows that $J(v) < N\overline{I}(v)$. Therefore, if $\gamma \in \Gamma$, on the one hand we have $J(\gamma(0)) = 0$, and on the other hand $J(\gamma(1)) < N\overline{I}(\gamma(1)) < 0$, since $\overline{I}(\gamma(1)) < 0$, and we conclude that there exists $s \in (0,1)$, for which $\|\gamma(s)\| > \rho$ and such that $J(\gamma(s)) = 0$. The function $\gamma(s)$ satisfies $\gamma(s) \in \mathcal{P}$, which shows that every path $\gamma \in \Gamma$ intersects $\mathcal{P}$. ☐

We recall that a sequence (v_n) is said to be a Cerami sequence for the functional $\overline{I}$ at level d in $\mathbb{R}$, denote by $(Ce)_d$, if

$$\overline{I}(v_n) \to d \quad \text{and} \quad \|\overline{I'}(v_n)\|(1 + \|v_n\|) \to 0, \quad \text{as } n \to \infty. \tag{4.2}$$

Now we show that, if $d > 0$, then any $(Ce)_d$ sequence for the functional $\overline{I}$ is bounded, up to a sequence.

Lemma 4.3. *If (v_n) is a $(Ce)_d$ sequence with $d > 0$, then it has a bounded subsequence.*

Proof. For any $v \in H^1(\mathbb{R}^N)$,

$$\overline{I}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty |G^{-1}(v)|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} a(x)H(G^{-1}(v)) \,\mathrm{d}x \tag{4.3}$$

and

$$\begin{aligned} &\overline{I'}(v)G^{-1}(v)g(G^{-1}(v)) \\ &= \int_{\mathbb{R}^N} \big[1 + \frac{G^{-1}(v)}{g(G^{-1}(v))} g'(G^{-1}(v))\big] |\nabla v|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V_\infty |G^{-1}(v)|^2 \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} a(x)h(G^{-1}(v))G^{-1}(v) \,\mathrm{d}x. \end{aligned} \tag{4.4}$$

Now, by the previous arguments, we know that there is $C > 0$ such that

$$\|G^{-1}(v_n)g(G^{-1}(v_n))\| \le C\|v_n\|, \quad \forall\, n \in \mathbb{N}.$$

Thus, the above inequality combined with (4.2), (4.3) and (4.4) implies that

$$
\begin{aligned}
&2\sqrt{2}\, d + o_n(1) \\
&= 2\sqrt{2}\, \overline{I}(v_n) - \overline{I'}(v_n)G^{-1}(v_n)g(G^{-1}(v_n)) \\
&= \int_{\mathbb{R}^N} \Big[\frac{2\sqrt{2}-2}{2} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))\Big] |\nabla v_n|^2 \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \Big(a(x)h(G^{-1}(v_n))G^{-1}(v_n) - 2\sqrt{2}\, a(x)H(G^{-1}(v_n))\Big)\,\mathrm{d}x \\
&\quad + \frac{2\sqrt{2}-2}{2} \int_{\mathbb{R}^N} V_\infty (G^{-1}(v_n))^2 \,\mathrm{d}x.
\end{aligned}
\tag{4.5}
$$

Now, using the hypothesis (A8) and Lemma 2.1(5) in (4.5), we obtain

$$
\begin{aligned}
2\sqrt{2}\, d + o_n(1) &\geq \int_{\mathbb{R}^N} \Big[\frac{2\sqrt{2}-2}{2} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} g'(G^{-1}(v_n))\Big] |\nabla v_n|^2 \,\mathrm{d}x \\
&= \big(\frac{6\sqrt{2}-8}{2}\big) \int_{\mathbb{R}^N} |\nabla v_n|^2 \,\mathrm{d}x.
\end{aligned}
$$

It follows that

$$
\lim_{n\to\infty} \sup \|\nabla v_n\|_2^2 \leq \frac{2\sqrt{2}\, d}{3\sqrt{2}-4}. \tag{4.6}
$$

By the Sobolev embedding, we derive that

$$
\lim_{n\to\infty} \sup \int_{\mathbb{R}^N} |v_n|^{2^*} \,\mathrm{d}x \leq \Big(\frac{2\sqrt{2}\, d}{3\sqrt{2}-4}\Big)^{2^*}, \quad \forall n \in \mathbb{N}. \tag{4.7}
$$

From 2.5, with $p = 2^* - 1$, as $n \to \infty$, $\overline{I'}(v_n)v_n = o_n(1)$ gives

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big[|\nabla v_n|^2 + V_\infty \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} v_n\Big] \,\mathrm{d}x \\
&\leq a_\infty \int_{\mathbb{R}^N} \Big(\frac{\frac{\varepsilon}{2}|G^{-1}(v_n)| + C|G^{-1}(v_n)|^{2^*-1}}{g(G^{-1}(v_n))}\Big) v_n \,\mathrm{d}x.
\end{aligned}
$$

Using Lemma 2.1(1) and that $1 \leq g(t) \leq 2$ for all $t \in \mathbb{R}$, we obtain

$$
C_1 \int_{\mathbb{R}^N} |v_n|^2 \,\mathrm{d}x \leq C_2 \int_{\mathbb{R}^N} |v_n|^{2^*} \,\mathrm{d}x.
$$

Then, by (4.7),

$$
\lim_{n\to\infty} \sup \int_{\mathbb{R}^N} |v_n|^2 \,\mathrm{d}x \leq \frac{C_2}{C_1} \Big(\frac{2\sqrt{2}\, d}{3\sqrt{2}-4}\Big)^{2^*}, \quad \forall n \in \mathbb{N}.
$$

From (4.6) and (4.7), it follows that (v_n) is bounded in $H^1(\mathbb{R})$. □

The next step is to show the existence of a Cerami sequence for the functional $\overline{I}$ at level c.

Lemma 4.4. *Let c be as in* (4.1)*, then there exists a $(Ce)_c$ sequence $(v_n) \subset H^1(\mathbb{R}^N)$.*

The proof of the above lemma can be found in [24, Lemma 4.5]. Next we present a splitting lemma on $\overline{I}$.

Lemma 4.5. *Let $(v_n) \in H^1(\mathbb{R}^N)$ be a bounded sequence such that*

$$\overline{I}(v_n) \to d > 0 \quad \text{and} \quad \|\overline{I'}(v_n)\|(1 + \|v_n\|) \to 0.$$

Replacing (v_n) by a subsequence, if necessary, there exists a solution $\overline{v}$ of (2.4), a number $k \in \mathbb{N} \cup \{0\}$, k functions $v^1, v^2, \dots, v^k$ and k sequences of points $(y_n^j) \in \mathbb{R}^N$, $1 \le j \le k$, such that as $n \to \infty$, they satisfy:

(a) $v_n \to \overline{v}$ *in* $H^1(\mathbb{R}^N)$ *or*
(b) v^j *are nontrivial solutions of* (3.1)*;*
(c) $|y_n^j| \to \infty$ *and* $|y_n^j - y_n^i| \to \infty$, $i \ne j$*;*
(d) $v_n - \sum_{i=1}^k v^i(x - y_n^i) \to \overline{v}$*;*
(e) $\overline{I}(v_n) \to \overline{I}(\overline{v}) + \sum_{i=1}^k I_\infty(v^i)$.

Remark 4.6. One can also mimic the proof of [46, Theorem 8.4]. Nowadays the proof of this lemma is standard and is a version of the concentration compactness of Lions [27] and found in [41]. The main ingredients are Lions lemma and Brezis-Lieb Lemma [9].

Corollary 4.7. *If $\overline{I}(v_n) \to c_\infty$ and $\|\overline{I'}(v_n)\|(1 + \|v_n\|) \to 0$, then either (v_n) is relatively compact or the splitting lemma holds with $k = 1$ and $\overline{v} = 0$.*

Let us set

$$c_\sharp := \inf\{c > c_\infty : c \text{ is a radial critical value of } I_\infty\}.$$

Lemma 4.8. *Assume that c_∞ is an isolated radial critical level for I_∞. Then $c_\sharp > c_\infty$ and $\overline{I}$ satisfies condition $(Ce)_d$ at level $d \in (c_\infty, \min\{c_\sharp, 2m_\infty\})$. Assume now that the limiting problem (3.1) admits a unique positive radial solution. Then $\overline{I}$ satisfies condition $(Ce)_d$ at level $d \in (c_\infty, 2c_\infty)$.*

The proof of the above lemma can be found in [25, Lemma 5.9].

Lemma 4.9. *If $\overline{I}(v_n) \to d > 0$, as $n \to \infty$, and $(v_n) \subset \mathcal{P}$, then the sequence (v_n) is bounded.*

Proof. The convergence $\overline{I}(v_n) \to d > 0$, as $n \to \infty$, implies $\overline{I}(v_n)$ is bounded in $\mathbb{R}$. If $v_n \in \mathcal{P}$ then

$$\begin{aligned} d + 1 \ge \overline{I}(v_n) &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + \frac{1}{N}\int_{\mathbb{R}^N} \nabla a(x).x.H(G^{-1}(v_n)) \, dx \\ &> \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx, \end{aligned}$$

hence $\|\nabla v_n\|_2$ is bounded. By Sobolev-Gagliardo-Nirenberg inequality, it follows that $\|v_n\|_{2^*}$ is also bounded. Now using (2.5), Lemma 2.1(1) with $\|a\|_\infty \varepsilon < \frac{V_\infty}{2}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} a(x)H(G^{-1}(v_n)) \, dx &\le \int_{\mathbb{R}^N} a(x)\Big(\frac{\varepsilon}{2}|G^{-1}(v_n)|^2 + C|G^{-1}(v_n)|^{2^*}\Big) \, dx \\ &\le \frac{\|a\|_\infty \varepsilon}{2}\|v_n\|_2^2 + C\|v_n\|_{2^*}^{2^*}. \end{aligned}$$

Replacing this in the expression of $\overline{I}$ and using Lemma 2.1(1), we have

$$d + 1 \ge \overline{I}(v_n) \ge \frac{1}{2}\|\nabla v_n\|_2^2 + \Big(\frac{1}{4}V_\infty - \frac{\|a\|_\infty \varepsilon}{2}\Big)\|v_n\|_2^2 - C\|v_n\|_{2^*}^{2^*}$$

since $\left(\frac{1}{4}V_\infty - \frac{\|a\|_\infty \varepsilon}{2}\right) > 0$, if $\|v_n\|_2 \to \infty$, as $n \to \infty$, this would gives us a contradiction. $\square$

Next we introduce the barycenter function, see [3, 41] which is going to be critical for proving the existence of a solution of problem (2.4).

Definition 4.10. For a given function $u \neq 0 \in H^1(\mathbb{R}^N)$, let

$$\mu(u)(x) = \frac{1}{|B_1|} \int_{B_1(x)} |u(y)| \,\mathrm{d}y,$$

with $\mu(u) \in L^\infty(\mathbb{R}^N)$ a continuous function. Subsequently, take

$$\widehat{u}(x) = \left[\mu(u)(x) - \frac{1}{2} \max \mu(u)\right]^+.$$

It follows that $\widehat{u} \in C_0(\mathbb{R}^N)$. Now define the *barycenter* of u by

$$\beta(u) = \frac{1}{|\widehat{u}|_{L^1}} \int x\widehat{u}(x) \,\mathrm{d}x \ \in \mathbb{R}^N.$$

Since $\widehat{u}$ has compact support, by definition, $\beta(u)$ is well defined. The function β satisfies the following properties:

(a) β is a continuous function in $H^1(\mathbb{R}^N) \setminus \{0\}$;
(b) If u is radial, then $\beta(u) = 0$;
(c) Given $y \in \mathbb{R}^N$ and defining $u_y := u(x-y)$, then $\beta(u_y) = \beta(u) + y$.

We also need the following lemma.

Lemma 4.11. *Assume that (u_n), $(v_n) \subset H^1(\mathbb{R}^N)$ are sequences such that $\|u_n - v_n\| \to 0$ and $\overline{I'}(v_n) \to 0$, as $n \to \infty$, with (v_n) bounded. Then $\overline{I'}(u_n) \to 0$ as $n \to \infty$.*

Proof. For any test function $\varphi \in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned}
&[\overline{I'}(u_n) - \overline{I'}(v_n)]\varphi \\
&= \int_{\mathbb{R}^N} \nabla(u_n - v_n)\nabla\varphi \,\mathrm{d}x + V_\infty \int_{\mathbb{R}^N} \Big(\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\Big)\varphi \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} a(x)\Big(\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))} - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\Big)\varphi \,\mathrm{d}x.
\end{aligned}$$

We first observe that the term $\int_{\mathbb{R}^N} \nabla(u_n - v_n)\nabla\varphi \,\mathrm{d}x = o_n(1)$, as $n \to \infty$, since $\|u_n - v_n\| \to 0$, as $n \to \infty$.

Now we verify that

(i) $\int_{\mathbb{R}^N} \left(\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\right)\varphi \,\mathrm{d}x = o(1)$, as $n \to \infty$;
(ii) $\int_{\mathbb{R}^N} a(x) \left(\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))} - \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\right) \varphi \,\mathrm{d}x = o(1)$, as $n \to \infty$.

(i) Note that $s = G(t)$ if, and only if $t = G^{-1}(s)$. By Lemma 2.1(2), we have

$$(G^{-1})'(s) = \frac{1}{G'(t)} = \frac{1}{g(t)} \leq 1, \quad \forall t \in \mathbb{R}. \tag{4.8}$$

On the other hand, as (v_n) is bounded and $\|u_n - v_n\| = o(1)$, as $n \to \infty$, then (u_n) is bounded. Up to a subsequence, we can assume, as $n \to \infty$, that

$$u_n, v_n \to u \text{ a.e. weakly} \quad \text{and} \quad u_n, v_n \to u \text{ strongly in } \operatorname{supp}\varphi.$$

Also, there exists $\psi \in L^r$, $r < 2^*$, such that $|u_n|, |v_n| \leq \psi(x)$ for all $x \in \mathbb{R}^N$. Notice that, as $n \to \infty$,

$$\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))}\varphi \to \frac{G^{-1}(u)}{g(G^{-1}(u))}\varphi \quad \text{and} \quad \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi \to \frac{G^{-1}(u)}{g(G^{-1}(u))}\varphi.$$

By (4.8) and Lemma 2.1(2), we have

$$\Big|\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))}\Big| \leq 1, \quad \forall n \in \mathbb{N}.$$

So that $\big|\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))}\varphi\big| \leq \varphi \in L^r$, $r < 2^*$. Therefore, by Lebesgue Dominated Convergence Theorem, we have (i).

(ii) First, note that $a(x)$ is bounded. We can assume there exists $q \in [3, 2^*)$ such that $|h(s)| \leq \varepsilon|s| + C|s|^{q-1}$, for all $s \in \mathbb{R}$. Now using the Lemma 2.1(1), we obtain

$$|h(G^{-1}(u_n))| \leq \varepsilon|u_n| + C|u_n|^{q-1} \leq \varepsilon|\psi| + C|\psi|^{q-1}, \tag{4.9}$$

where $\psi \in L^{q-1}$ is obtained in the above convergence of u_n, v_n such that $|u_n|, |v_n| \leq \psi$.

So that, as $n \to \infty$,

$$\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))}\varphi \to \frac{h(G^{-1}(u))}{g(G^{-1}(u))}\varphi \quad \text{and} \quad \frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))}\varphi \to \frac{h(G^{-1}(u))}{g(G^{-1}(u))}\varphi,$$

using (4.8) and (4.9), we obtain

$$\Big|\frac{h(G^{-1}(u_n))}{g(G^{-1}(u_n))}\varphi\Big| \leq |h(G^{-1}(u_n))\varphi| \leq C(\varepsilon|\psi| + C|\psi|^{q-1}) \in L^1.$$

Therefore, by Lebesgue Dominated Convergence Theorem, we have (ii). $\square$

Now we define

$$b := \inf\{\overline{I}(v) : v \in \mathcal{P} \text{ and } \beta(v) = 0\}. \tag{4.10}$$

It is clear that $b \geq c_\infty$, and the following result holds.

Lemma 4.12. $b > c_\infty$.

The proof of the above lemma is analogous to [25, Lemma 5.13]. Let us consider a positive, radially symmetric, ground state solution $\omega \in H^1(\mathbb{R}^N)$ of the autonomous problem at infinity. We define the operator $\Pi : \mathbb{R}^N \to \mathcal{P}$ by

$$\Pi[y](x) = \omega\Big(\frac{x-y}{\theta_y}\Big),$$

where θ_y is exactly the real number θ which projects $\omega(\cdot - y)$ onto the Pohozaev manifold $\mathcal{P}$. Π is a continuous function of y because θ_y is unique and $\theta_y(\omega(\cdot - y))$ is a continuous function of $\omega(\cdot - y)$. We will verify some properties of this operator Π:

Lemma 4.13. *$\beta(\Pi[y](x)) = y$ and $\overline{I}(\Pi[y]) \to c_\infty$, if $|y| \to \infty$.*

The proof of the above lemma can be found in [24, Lemmas 4.13 and 4.14].

Lemma 4.14. *Let C be a positive constant such that $|H(G^{-1}(s))| \leq C|s|^2$, and assume* (A9). *Then $\overline{I}(\Pi[y]) < \min\{c_\sharp, 2c_\infty\}$.*

Proof. Noting that I_∞ is translation invariant, the maximum of $t \mapsto I_\infty(\omega(\cdot/t))$ is attained at $t = 1$ and that $\theta_y > 1$ and using (A9), we obtain

$$\begin{aligned}\overline{I}(\Pi[y]) &= I_\infty(\Pi[y]) + \overline{I}(\Pi[y]) - I_\infty(\Pi[y]) \\ &\le I_\infty(\omega) + \int_{\mathbb{R}^N} (a_\infty - a(x)) H(G^{-1}(\Pi[y]))\, \mathrm{d}x \\ &< c_\infty + \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\overline{\theta}^N \|\omega\|_2^2 C} \theta_y^N C \|\omega\|_2^2 \\ &< \min\{c_\sharp, 2c_\infty\}.\end{aligned}$$

□

We will need a version of the Linking Theorem with Cerami condition [7, Theorem 2.3], which we state here for the sake of completeness.

Definition 4.15. Let S be a closed subset of a Banach space X, and Q a sub manifold of X with relative boundary ∂Q. We say that S and ∂Q link if:

(1) $S \cap \partial Q = \emptyset$;
(2) for any $h \in C^0(X, X)$ such that $h|_{\partial Q} = id$, we have $h(Q) \cap S \neq \emptyset$.

If S and Q are as above and B is a subset of $C^0(X, X)$, then S and ∂Q with respect to B if (1) and (2) hold for any $h \in B$.

Theorem 4.16 (Linking)**.** *Suppose that $\overline{I} \in C^1(X, \mathbb{R})$ is a functional satisfying (Ce) condition. Consider a closed subset $S \subset X$ and a submanifold $Q \subset X$ with relative boundary ∂Q; suppose also that:*

(a) *S and ∂Q "link";*
(b) *$\alpha = \inf_{u \in S} \overline{I}(v) > \sup_{v \in \partial Q} \overline{I}(v) = \alpha_0$;*
(c) *$\sup_{v \in Q} \overline{I}(v) < +\infty$.*

If $B = \{h \in C^0(X, X); h|_{\partial Q} = id\}$, then the real number $\tau = \inf_{h \in B} \sup_{v \in Q} \overline{I}(h(v))$ defines a critical value of $\overline{I}$, with $\tau \ge \alpha$.

We refer the reader to [39, 36] for similar versions of the Linking Theorem with Cerami condition.

Proof of Theorem1.2. Condition (A4) implies $I_\infty(v) < \overline{I}(v)$ for all $v \in H^1(\mathbb{R}^N) \setminus \{0\}$. In particular, $I_\infty(\Pi[y]) < \overline{I}(\Pi[y])$, for any $y \in \mathbb{R}^N$. Since $b > c_\infty$, from Lemma 4.12, and $\overline{I}(\Pi[y]) \to \infty$ if $|y| \to \infty$, from Lemma 4.13, then there exists $\overline{\rho} > 0$ such that for every $\rho \ge \overline{\rho}$,

$$c_\infty < \max_{|y| = \rho} \overline{I}(\Pi[y]) < b. \tag{4.11}$$

To apply the Linking Theorem 4.16, we set

$$Q := \Pi(\overline{B_{\overline{\rho}}(0)}) \quad \text{and} \quad S := \{v \in H^1(\mathbb{R}^N) : v \in \mathcal{P}, \beta(v) = 0\},$$

and will show that ∂Q and S "link". Since $\beta(\Pi[y]) = y$, from Lemma 4.13, we have that $\partial Q \cap S = \emptyset$, because if $v \in S$, then $\beta(v) = 0$, and if $v \partial Q$, then $\beta(v) = y \neq 0$, due to the equality $|y| = \overline{\rho}$. Now we need to show that $h(Q) \cap S \neq \emptyset$, for any $h \in \mathbb{H}$, where

$$\mathbb{H} = \{h \in C(Q, \mathcal{P}) : h|_{\partial Q} = \mathrm{id}\}.$$

Given $h \in \mathbb{H}$, let us define $T : \overline{B_{\overline{\rho}}(0)} \to \mathbb{R}^N$ for $T(y) = \beta \circ h \circ \Pi[y]$. The function T is continuous, because it is the composition of continuous functions. Moreover, for any $|y| = \overline{\rho}$, we have that $\Pi[y] \in \partial Q$, thus $h \circ \Pi[y] = \Pi[y]$, because $h|_{\partial Q} = id$, and hence from Lemma 4.13 we obtain $T(y) = y$. By the Fixed Point Theorem of Brouwer, we conclude that there exists $\tilde{y} \in B_{\overline{\rho}}(0)$ such that $T(\tilde{y}) = 0$, which implies $h(\Pi[\tilde{y}]) \in S$. Therefore $h(Q) \cap S \neq \emptyset$ and S and ∂Q "link".

Furthermore, from the definitions of b and Q and the inequalities (4.11), we may write

$$b = \inf_S \overline{I} > \max_{\partial Q} \overline{I}.$$

Let us define

$$d = \inf_{h \in \mathbb{H}} \max_{v \in Q} \overline{I}(h(v)).$$

Then we have $d \geq b$. Indeed, we have already proved that $h(Q) \cap S \neq \emptyset$, for all $h \in \mathbb{H}$. If h is fixed, then there exists $\omega \in S$ such that ω also belongs to $h(Q)$, which means that $\omega = h(u)$ for some $u \in \Pi(\overline{B_{\overline{\rho}}(0)})$. Therefore,

$$\overline{I}(\omega) \geq \inf_{v \in S} \overline{I}(v) \quad \text{and} \quad \max_{v \in Q} \overline{I}(h(v)) \geq \overline{I}(h(u)).$$

This gives

$$\max_{v \in Q} \overline{I}(h(v)) \geq \overline{I}(h(u)) = \overline{I}(\omega) \geq \inf_{v \in S} \overline{I}(v) = b,$$

and hence

$$\inf_{h \in \mathbb{H}} \max_{v \in Q} \overline{I}(h(v)) \geq b.$$

In particular, it follows that $d > c_\infty$, because from Lemma 4.12 we know that $b > c_\infty$. Furthermore, if we take $h = id$, then

$$\inf_{h \in \mathbb{H}} \max_{v \in Q} \overline{I}(h(v)) < \max_{v \in Q} \overline{I}(v) < \min\{c_\sharp, 2c_\infty\},$$

by Lemma 4.14. This implies $d < \min\{c_\sharp, 2c_\infty\}$. The two inequalities give $d \in (c_\infty, \min\{c_\sharp, 2c_\infty\})$, thus from Lemma 4.5 (Ce) condition is satisfied at level d. Therefore, we can apply the Linking Theorem and conclude that d is a critical level for the functional $\overline{I}$. This guarantees the existence of a nontrivial solution $v \in H^1(\mathbb{R}^N)$ of the equation (2.4). Reasoning as usual, because of the hypotheses on h and G^{-1}, and using the maximum principle we may conclude that v is positive, which implies the proof of the theorem. □

Acknowledgments. O. H. Miyagaki was partially supported by INCTmat/Brazil and CNPq/Brazil.

References

[1] S. Alama, Y. Y. Li; On "multi bump" bound states for certain semilinear elliptic equations, *Indiana Univ. Math. J.*, **41** (1992), 983-1026.
[2] C. O. Alves, Y. Wang, Y. Shen; Soliton solutions for a class of quasilinear Schrödinger equations with a parameter. arXiv: 1309.4606.
[3] A. Ambrosetti, G. Cerami, D. Ruiz; Solitons of linearly coupled systems of semilinear non-autonomous equations on $\mathbb{R}^N$, *J. Func. Anal.*, **254** (2008), 2816-2845.
[4] A. Ambrosetti, P. H. Rabinowitz; Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381.
[5] A. Azzollini, A. Pomponio; On the Schrödinger equation in $\mathbb{R}^N$ under the effect of a general nonlinear term, *Indiana Univ. Math. J.*, **58** (3) (2009), 1361-1378.
[6] A. Bahri, Y. Y. Li; On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$, *Rev. Mat. Iberoamericana*, **6** (1-2) (1990), 1-15.
[7] P. Bartolo, V. Benci, D. Fortunato; Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, *Nonlinear Anal.*, **7** (1983), 981-1012.

[8] H. Berestycki, P. L. Lions; Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (4) (1983), 313-345.

[9] H. Brezis, E. H. Lieb; A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, **88** (3) (1983), 486-490.

[10] H. Brezis; Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Universitext, Springer*, Berlin, 2010.

[11] P. C. Carrião, R. Lehrer, O. H. Miyagaki; Existence of solutions to a class of asymptotically linear Schrödinger equations in $\mathbb{R}^N$ via Pohozaev manifold, *Journal of Mathematical Analysis and Applications,* **428** (2015), 165-183.

[12] G. Cerami, An existence criterion for the critical points on unbounded manifolds. (Italian) *Istit. Lombardo Accad. Sci. Lett. Rend.*, A **112** (1978)(2) 332-336 (1979).

[13] Y. Cheng, J. Yang; Positive solution to a class of relativistic nonlinear Schrödinger equation *J. Math. Analysis and Applications*, **411** (2014), 665-674.

[14] Y. Cheng, J. Yang; The existence and uniqueness result for a relativistic nonlinear Schrödinger equation. *Abstr. Appl. Anal.*, 2014, Art. ID 362985, 10 pp.

[15] M. Colin, L. Jeanjean; Solutions for a quasilinear Schrödinger equation: a dual approach, *Nonlinear Anal.*, **56** (2004), 213-226.

[16] D. G. Costa, H. Tehrani; On a Class of Asymptotically Linear Elliptic Problems in $\mathbb{R}^N$, *J. Differential Equations*, **173** (2001), 470-494.

[17] V. Coti-Zelati, P. H. Rabinowitz; Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$, *Comm. Pure Appl. Math.*, **45** (1992), 1217-1269.

[18] A. De Bouard, N. Hayashi, J. Saut, Global existence of small solutions to a relativistic nonlinear Schrödinger equation, *Communications in Mathematical Physics,* **189** (1997) 73-105.

[19] Y. Deng, S. Peng, S. Yan; Positive soliton solutions for generalized quasilinear Schrödinger equations with critical growth *J. Differential Equations*, **258** (2015) 115-147.

[20] J. M. do Ó, O. H. Miyagaki, S.H.M. Soares; Soliton solutions for quasilinear Schrödinger equations with critical growth, *J. Differential Equation*, **248** (2010), 722-744.

[21] L. Jeanjean, K. Tanaka; A remark on least energy solutions in $\mathbb{R}^N$, *Proc. Amer. Math. Soc.*, **131** (2002), 2399-2408.

[22] S. Kurihara; Large-amplitude quasi-solitons in superfluid films, *Journal of the Physical Society of Japan,* **50** (10), 3262-3267.

[23] E. W. Laedke, K. H. Spatschek, L. Stenflo; Evolution theorem for a class of perturbed envelope soliton solutions, *Journal of Mathematical Physics,* **24** (12) (1982), 2764-2769.

[24] R. Lehrer, L. A. Maia, Positive solutions of asymptotically linear equations via Pohozaev Manifold. *J. Funct. Anal.*, **266** (2014), 213–246.

[25] R. Lehrer, L. A. Maia, M. Squassina; Asymptotically Linear Fractional Schrödinger Equations, *Complex Var. and Ellip. Eq.*, **60** (2015), 529–558.

[26] G. Li, H.-S. Zhou; The existence of a positive solution to asymptotically linear scalar field equations, *Proc. R. Soc. Edinb.* A, **130** (2000), 81-105.

[27] P. L. Lions, The concentration-compactness principle in the calculus of variations. the locally compact case, part 2. *Ann. Inst. H. Poncaré-AN* ,**1** (1984), 223–283.

[28] X. Liu, J.-Q.Liu, Z.-Q. Wang; Ground states for quasilinear Schrödinger equations with critical growth, *Calc. Var. Partial Dif. Equations*, **46** (2013), 641-669.

[29] J. Liu, Z.-Q. Wang; Soliton solutions for quasilinear Schrödinger equations I, *Proc. Amer. Math. Soc.*, **131** (2003), 441-448.

[30] J. Liu, Y. Wang, Z.-Q. Wang; Soliton solutions for quasilinear Schrödinger equations II, *J. Differential Equations*, **187** (2003), 473-493.

[31] X.-Q. Liu, J.-Q. Liu, Z.-Q. Wang; Quasilinear elliptic equations with critical growth via perturbation method, *J. Differential Equation*, **248** (2013), 102-124.

[32] L.A. Maia, R. Ruviaro; Sing-changing solutions for a Schrödinger equation with saturable nonlinearity. *Milan J. Math*, **79** (2011), 259–271.

[33] A. Moameni; Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in R^N, *J. Differential Equations*, **229** (2006), 570-587.

[34] M. Poppenberg, K. Schmitt, Z.-Q Wang; On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Dif. Equations*, **14** (2002), 329-344.

[35] P. H. Rabinowitz; On a class of nonlinear Schrödinger equations. *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* **43** (1992), 270–291.

[36] M. Schechter; Linking methods in critical point theory, *Birkhäuser,* Boston, 1999.
[37] J. Shatah; Unstable Ground State of Nonlinear Klein-Gordon Equations, *Trans. Amer. Math. Soc.* **290** (2) (1985), 701-710.
[38] Y. Shen, Y. Wang; Soliton solutions for generalized quasilinear Schrödinger equations, *Nonlinear Analysis*, **80** (2013), 194-201.
[39] E. A. B. Silva; Subharmonic solutions for subquadratic Hamiltonian systems, *J. Differential Equations*, **115** (1995), 120-145.
[40] E. A. B. Silva, G. F. Vieira; Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Partial Dif. Equations*, **39** (2010), 1-33.
[41] M. Struwe; A global compactness result for elliptic boundary value problems involving limiting nonliarities, *Math. Z.*, **187**(4) (1984), 511-517.
[42] C. A. Stuart, H. S. Zhou; Applying the Mountain Pass Theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$, *Commum. Partial Diff. Eq.*, **9-10** (1999), 1731-1758.
[43] J. Yang, Y. Wang, A. A. Abdelgadir; Soliton solutions for quasilinear Schrödinger equations *Journal Mat. Phy*, **54** (2013), 071502, 19pp.
[44] Y. Wang; Multiple solutions for quasilinear elliptic equations with critical growth, *J. Korean Math. Soc.*, **48** (2011), 1269-1283.
[45] Y. J. Wang, Y. M. Zhang, Y.T. Shen; Multiple solutions for quasilinear Schrödinger equations involving critical exponent, *Appl. Math. Comp.*, **216** (2010), 849-856.
[46] M. Willem; Minimax Theorems, Volume 24, *Birkhauser*, Boston, 1996.

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