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INVERSE NODAL PROBLEM FOR A p-LAPLACIAN STURM-LIOUVILLE EQUATION WITH POLYNOMIALLY BOUNDARY CONDITION

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Abstract. In this article, we extend solution of inverse nodal problem for one-dimensional p-Laplacian equation to the case when the boundary condition is polynomially eigenparameter. To find the spectral data as eigenvalues and nodal parameters, a Prüfer substitution is used. Then, we give a reconstruction formula of the potential function by using nodal lengths. This method is similar to used in [24], and our results are more general.

1. Introduction

Consider p-Laplacian Sturm-Liouville eigenvalue problem

$$-\big(y'^{(p-1)}\big)' = (p-1)\big(\lambda - q_m(x)\big)y^{(p-1)}, \quad 0 \le x \le 1, \tag{1.1}$$

with the boundary conditions

$$\begin{gathered} y(0)=0, \quad y'(0)=1, \\ y'(1,\lambda)+f(\lambda)y(1,\lambda)=0, \end{gathered} \tag{1.2}$$

where $p>1$,

$$f(\lambda)=a_1\sqrt{\lambda}+a_2(\sqrt{\lambda})^2+\dots+a_m(\sqrt{\lambda})^m, \quad a_i\in\mathbb{R},\ a_m\neq 0,\ m\in\mathbb{Z}^+, \tag{1.3}$$

λ is a spectral parameter and $y^{(p-1)}=|y|^{(p-2)}y$. Throughout this study, we suppose that $q_m(x)$ is a real-valued $C[0,1]$-function defined on the interval $0\le x\le 1$ for each $m\in\mathbb{Z}^+$ and $y(x,\lambda)$ denotes the solution of the problem (1.1)-(1.2). When $p=2$, Equation (1.1) becomes the well-known Sturm-Liouville equation. The idea of inverse eigenvalue problems with an eigenparameter together with the boundary conditions is of great interest to many problems of mathematical physics and mechanics. These type problems have many physical applications. For instance, Sturm-Liouville equation including spectral parameter with the boundary conditions arises in heat and one-dimensional wave equation by seperation of variables. There are many literatures on these type of problems (see [2, 3, 6, 7, 8, 9, 18, 19, 22, 25]).

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Inverse spectral problem involves recovering differential equation from its spectral parameters like eigenvalues, norming constants and nodal points (zeros of eigenfunctions). These type of problems have been divided into two parts; inverse eigenvalue problem and inverse nodal problem. They play an important role and also have many applications in applied mathematics. Inverse nodal problem was firstly studied by McLaughlin in 1988. She showed that the knowledge of a dense subset of nodal points is sufficient to determine the potential function of Sturm-Liouville problem up to a constant [16]. Also, some numerical results about this problem were given in [10]. Nowadays, many authors have given some interesting results about inverse nodal problems for different type of operators (see [4, 12, 13, 15, 17, 21, 26]).

In this study, we devote our effort with the inverse nodal problem for p-Laplacian Sturm-Liouville equation with boundary condition polynomially dependent on spectral parameter. Essentially, we give asymptotics of eigenparameters and reconstruction formula for potential function. Note that inverse eigenvalue problems for different p-Laplacian operators have been studied by several authors (see [1, 5, 11, 14, 20, 21]).

The zero set $X_n = \{x_{j,m}^n\}_{j=1}^{n-1}$ of the eigenfunction $y_{n,m}(x)$ corresponding to $\lambda_{n,m}$ is called the set of nodal points. And, $l_{j,m}^n = x_{j+1,m}^n - x_{j,m}^n$ is referred as the nodal length of $y_{n,m}$. The eigenfunction $y_{n,m}(x)$ has exactly $n-1$ nodal points in $(0,1)$, say $0 = x_{0,m}^{(n)} < x_{1,m}^{(n)} < \dots < x_{n-1,m}^{(n)} < x_{n,m}^{(n)} = 1$.

Let us now recall some important results. Firstly, we need to introduce the generalized sine function S_p which is the solution of the initial value problem

$$\begin{gathered} -\big(S_p'^{(p-1)}\big)' = (p-1)S_p^{(p-1)}, \\ S_p(0) = 0,\ S_p'(0) = 1. \end{gathered} \tag{1.4}$$

S_p and S_p' are periodic functions which satisfy the identity

$$|S_p(x)|^p + |S_p'(x)|^p = 1,$$

for any $x \in \mathbb{R}$. These functions are p-analogues of classical sine and cosine functions. It is well known that

$$\hat{\pi} = \int_0^1 \frac{2}{(1-t^p)^{\frac{1}{p}}} dt = \frac{2\pi}{p\sin(\frac{\pi}{p})},$$

is the first zero of S_p in positive axis [5].

Lemma 1.1 ([5]). *(a) For $S_p' \neq 0$,*

$$(S_p')' = -|\frac{S_p}{S_p'}|^{p-2} S_p.$$

(b)

$$(S_p S_p'^{(p-1)})' = |S_p'|^p - (p-1)S_p^p = 1 - p|S_p|^p = (1-p) + p|S_p'|^p.$$

Using $S_p(x)$ and $S_p'(x)$, the generalized tangent function $T_p(x)$ can be defined as follows [5]

$$T_p(x) = \frac{S_p(x)}{S_p'(x)}, \quad \text{for } x \neq \big(k + \frac{1}{2}\big)\hat{\pi}.$$

The remaining part of this study is organized as follows; In section 2, we give some asymptotic formulas for eigenvalues and nodal parameters for p-Laplacian

Sturm-Liouville eigenvalue problem (1.1)-(1.2) with boundary condition polynomially dependent on spectral parameter by using modified Prüfer substitution. In section 3, we give a reconstruction for the potential function of the problem (1.1)-(1.2).

2. Asymptotic behavior of some eigenparameters

In this section, we present some results on (1.1)-(1.2). One of them is the Prüfer's transformation which is one of the most powerful method for solving inverse problem. Recall that the Prüfer's transformation for a nonzero solution y of (1.1) takes the form

$$\begin{aligned} y(x) &= R(x)S_p(\lambda^{1/p}\theta(x,\lambda)), \\ y'(x) &= \lambda^{1/p}R(x)S_p'(\lambda^{1/p}\theta(x,\lambda)), \end{aligned} \tag{2.1}$$

or

$$\frac{y'(x)}{y(x)} = \lambda^{1/p}\frac{S_p'(\lambda^{1/p}\theta(x,\lambda))}{S_p(\lambda^{1/p}\theta(x,\lambda))}, \tag{2.2}$$

where $R(x)$ is amplitude and $\theta(x)$ is the Prüfer variable [23]. Standard manipulations [21] yield

$$\theta'(x,\lambda) = 1 - \frac{q_m(x)}{\lambda}S_p^p\big(\lambda^{1/p}\theta(x,\lambda)\big). \tag{2.3}$$

Lemma 2.1 ([21]). *Define $\theta(x,\lambda_n)$ as in (2.1) and $\phi_n(x) = S_p^p(\lambda_n^{1/p}\theta(x,\lambda_n)) - \frac{1}{p}$. Then, for any $g \in L^1(0,1)$,*

$$\int_0^1 \phi_n(x)g(x)dx = 0.$$

Theorem 2.2. *The eigenvalues $\lambda_{n,m}$ of the p-Laplacian Sturm-Liouville eigenvalue problem given in problem (1.1)-(1.2) have the form*

$$\lambda_{n,1}^{1/p} = n\hat{\pi} - \frac{1}{a_1(n\hat{\pi})^{\frac{p-2}{2}}} + \frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q_1(x)dx + O\big(\frac{1}{n^{p-2}}\big), \quad \text{for } m = 1, \tag{2.4}$$

$$\begin{aligned} \lambda_{n,2}^{1/p} &= n\hat{\pi} - \frac{1}{a_1(n\hat{\pi})^{\frac{p-2}{2}} + a_2(n\hat{\pi})^{p-1}} + \frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q_2(x)dx \\ &\quad + O\big(\frac{1}{n^{2p-1}}\big), \quad \text{for } m = 2, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \lambda_{n,m}^{1/p} &= n\hat{\pi} - \frac{1}{a_1(n\hat{\pi})^{\frac{p-2}{2}} + \dots + a_m(n\hat{\pi})^{\frac{mp-2}{2}}} \\ &\quad + \frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q_m(x)dx + O\big(\frac{1}{n^{2p-1}}\big), \quad \text{for } m \geq 3, \end{aligned} \tag{2.6}$$

as $n \to \infty$.

Proof. Let $\theta(0,\lambda) = 0$ for (1.1)-(1.2). Integrating both sides of (2.3) with respect to x from 0 to 1, we obtain

$$\theta(1,\lambda) = 1 - \frac{1}{\lambda}\int_0^1 q_m(x)S_p^p(\lambda^{1/p}\theta(x,\lambda))dx.$$

By Lemma 2.1,

$$\int_0^1 q_m(x)\{S_p^p(\lambda^{1/p}\theta(x,\lambda)) - \frac{1}{p}\}dx = o(1), \quad \text{as } n \to \infty.$$

Hence, we obtain

$$\theta(1,\lambda)=1-\frac{1}{p\lambda}\int_0^1 q_m(x)dx+O\big(\frac{1}{\lambda^2}\big). \tag{2.7}$$

Let $\lambda_{n,m}$ be an eigenvalue of the problem (1.1)-(1.2). For $m=1$, by (1.2), we have

$$\lambda_{n,1}^{1/p}R(1)S_p'(\lambda_{n,1}^{1/p}\ \theta(1,\lambda_{n,1}))+a_1\sqrt{\lambda_{n,1}}R(1)S_p\big(\lambda_{n,1}^{1/p}\theta(1,\lambda_{n,1})\big)=0,$$

or

$$-\frac{\lambda_{n,1}^{\frac{1}{p}-\frac{1}{2}}}{a_1}=\frac{S_p(\lambda_{n,1}^{1/p}\theta(1,\lambda_{n,1}))}{S_p'(\lambda_{n,1}^{1/p}\theta(1,\lambda_{n,1}))}=T_p\big(\lambda_{n,1}^{1/p}\theta(1,\lambda_{n,1})\big).$$

As n is sufficiently large, it follows that

$$\lambda_{n,1}^{1/p}\theta(1,\lambda_{n,1})=T_p^{-1}\Big(-\frac{\lambda_{n,1}^{\frac{1}{p}-\frac{1}{2}}}{a_1}\Big)=n\hat{\pi}-\frac{\lambda_{n,1}^{\frac{1}{p}-\frac{1}{2}}}{a_1}+o(\lambda_{n,1}^{\frac{2}{p}-1}). \tag{2.8}$$

By considering (2.7) and (2.8) together, we obtain

$$\lambda_{n,1}^{1/p}=n\hat{\pi}-\frac{1}{a_1(n\hat{\pi})^{\frac{p-2}{2}}}+\frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q_1(x)dx+O\big(\frac{1}{n^{p-2}}\big).$$

For $m=2$, by (1.2), using the same process as in $m=1$, we can easily obtain

$$\lambda_{n,2}^{1/p}R(1)S_p'(\lambda_{n,2}^{1/p}\ \theta(1,\lambda_{n,2}))+(a_1\sqrt{\lambda_{n,2}}+a_2(\sqrt{\lambda_{n,2}})^2)R(1)S_p(\lambda_{n,2}^{1/p}\theta(1,\lambda_{n,2}))=0,$$

or

$$-\frac{\lambda_{n,2}^{\frac{1}{p}}}{a_1\sqrt{\lambda_{n,2}}+a_2(\sqrt{\lambda_{n,2}})^2}=\frac{S_p(\lambda_{n,2}^{1/p}\theta(1,\lambda_{n,2}))}{S_p'(\lambda_{n,2}^{1/p}\theta(1,\lambda_{n,2}))}=T_p(\lambda_{n,2}^{1/p}\theta(1,\lambda_{n,2})). \tag{2.9}$$

Therefore,

$$\lambda_{n,2}^{1/p}=n\hat{\pi}-\frac{1}{a_1(n\hat{\pi})^{\frac{p-2}{2}}+a_2(n\hat{\pi})^{p-1}}+\frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q_2(x)dx+O\big(\frac{1}{n^{2p-1}}\big).$$

Finally, by (1.2), we have

$$\begin{aligned}&\lambda_{n,m}^{1/p}R(1)S_p'(\lambda_{n,m}^{1/p}\theta(1,\lambda_{n,m}))+(a_1\sqrt{\lambda_{n,m}}+\dots\\&+a_m(\sqrt{\lambda_{n,m}})^m)R(1)S_p(\lambda_{n,m}^{1/p}\theta(1,\lambda_{n,m}))=0,\end{aligned}$$

or

$$\begin{aligned}-\frac{\lambda_{n,m}^{\frac{1}{p}}}{a_1\sqrt{\lambda_{n,m}}+\dots+a_m(\sqrt{\lambda_{n,m}})^m}&=\frac{S_p(\lambda_{n,m}^{1/p}\theta(1,\lambda_{n,m}))}{S_p'(\lambda_{n,m}^{1/p}\theta(1,\lambda_{n,m}))}\\&=T_p\big(\lambda_n^{1/p}\theta(1,\lambda_{n,m})\big),\end{aligned} \tag{2.10}$$

for $m\geq 3$, by considering (2.7) and (2.10) together, we deduce that

$$\begin{aligned}\lambda_{n,m}^{1/p}&=n\hat{\pi}-\frac{1}{a_1(n\hat{\pi})^{\frac{p-2}{2}}+\dots+a_m(n\hat{\pi})^{\frac{mp-2}{2}}}\\&+\frac{1}{p(n\hat{\pi})^{p-1}}\int_0^1 q_m(x)dx+O(\frac{1}{n^{2p-1}}).\end{aligned}$$

$\square$

Theorem 2.3. *The nodal points for problem* (1.1)-(1.2) *satisfy the following asymptotic estimates:*

$$x_{j,1}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^p} + \frac{j}{pn^{p+1}\hat{\pi}^p} \int_0^1 q_1(t)dt + \int_0^{x_{j,1}^n} \frac{q_1(t)}{(n\hat{\pi})^p} S_p^p dt + O\big(\frac{j}{n^p}\big), \quad \text{for } m = 1, \tag{2.11}$$

$$\begin{aligned} x_{j,2}^n &= \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + a_2 n^{p+1}\hat{\pi}^p} + \frac{j}{pn^{p+1}\hat{\pi}^p} \int_0^1 q_2(t)dt \\ &\quad + \int_0^{x_{j,2}^n} \frac{q_2(t)}{(n\hat{\pi})^p} S_p^p dt + O\big(\frac{j}{n^{2p+1}}\big), \quad \text{for } m = 2, \end{aligned} \tag{2.12}$$

$$\begin{aligned} x_{j,m}^n &= \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + \dots + a_m n^{\frac{mp+2}{2}} \hat{\pi}^{\frac{mp}{2}}} + \frac{j}{pn^{p+1}\hat{\pi}^p} \int_0^1 q_m(t)dt \\ &\quad + \int_0^{x_{j,m}^n} \frac{q_m(t)}{(n\hat{\pi})^p} S_p^p dt + O\big(\frac{j}{n^{2p+1}}\big), \quad \text{for } m \geq 3, \end{aligned} \tag{2.13}$$

as $n \to \infty$.

Proof. Integrating (2.3) from 0 to $x_{j,m}^n$ and letting $\theta(x_{j,m}^n, \lambda) = \frac{j\hat{\pi}}{\lambda_{n,m}^{1/p}}$, we have

$$x_{j,m}^n = \frac{j\hat{\pi}}{\lambda_{n,m}^{1/p}} + \int_0^{x_{j,m}^n} \frac{q_m(t)}{\lambda_{n,m}} S_p^p dt. \tag{2.14}$$

For $m = 1$, from (2.4), we deduce that

$$\frac{1}{\lambda_{n,1}^{1/p}} = \frac{1}{n\hat{\pi}} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p+2}{2}}} + \frac{1}{p(n\hat{\pi})^{p+1}} \int_0^1 q_1(x)dx + O\big(\frac{1}{n^p}\big), \tag{2.15}$$

and therefore, we obtain formula (2.11) by using (2.14) and (2.15).

For $m = 2$, from formula (2.5), the asymptotic estimate of eigenvalues $1/\lambda_{n,2}^{1/p}$ is considered as

$$\frac{1}{\lambda_{n,2}^{1/p}} = \frac{1}{n\hat{\pi}} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p+2}{2}} + a_2 (n\hat{\pi})^{p+1}} + \frac{1}{p(n\hat{\pi})^{p+1}} \int_0^1 q_2(x)dx + O\big(\frac{1}{n^{2p+1}}\big), \tag{2.16}$$

and, we conclude formula (2.12) by using (2.14) and (2.16).

For $m \geq 3$, from the formula (2.6), it can easily be shown that

$$\begin{aligned} \frac{1}{\lambda_{n,m}^{1/p}} &= \frac{1}{n\hat{\pi}} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p+2}{2}} + \dots + a_m (n\hat{\pi})^{\frac{mp+2}{2}}} + \frac{1}{p(n\hat{\pi})^{p+1}} \int_0^1 q_m(x)dx \\ &\quad + O\big(\frac{1}{n^{2p+1}}\big), \end{aligned} \tag{2.17}$$

and, we obtain formula (2.13) by using (2.14) and (2.17). $\square$

Theorem 2.4. *Asymptotic estimate of the nodal lengths for the problem* (1.1)-(1.2) *satisfies*

$$\begin{aligned} l_{j,1}^n &= \frac{1}{n} - \frac{1}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}}} + \frac{1}{pn^{p+1}\hat{\pi}^p} \int_0^1 q_1(t)dt \\ &\quad + \frac{1}{(n\hat{\pi})^p} \int_{x_{j,1}^n}^{x_{j+1,1}^n} q_1(t) S_p^p dt + O\big(\frac{1}{n^p}\big), \quad \text{for } m = 1, \end{aligned} \tag{2.18}$$

$$
\begin{aligned}
l_{j,2}^n &= \frac{1}{n} - \frac{1}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + a_2 n^{p+1} \hat{\pi}^p} + \frac{1}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_2(t) dt \\
&\quad + \frac{1}{(n\hat{\pi})^p} \int_{x_{j,2}^n}^{x_{j+1,2}^n} q_2(t) S_p^p dt + O\big(\frac{1}{n^{2p+1}}\big), \quad \textit{for } m = 2,
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
l_{j,m}^n &= \frac{1}{n} - \frac{1}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + \dots + a_m n^{\frac{mp+2}{2}} \hat{\pi}^{\frac{mp}{2}}} + \frac{1}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_m(t) dt \\
&\quad + \frac{1}{(n\hat{\pi})^p} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) S_p^p dt + O\big(\frac{1}{n^{2p+1}}\big), \quad \textit{for } m \geq 3.
\end{aligned} \tag{2.20}
$$

Proof. For a large $n \in \mathbb{N}$, integrating (2.3) on $[x_{j,m}^n, x_{j+1,m}^n]$ and using the definition of nodal lengths, we have

$$
\begin{aligned}
\frac{\hat{\pi}}{\lambda_{n,m}^{1/p}} &= x_{j+1,m}^n - x_{j,m}^n - \frac{1}{p\lambda_{n,m}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) S_p^p dt \\
&\quad - \frac{1}{\lambda_{n,m}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) \big(S_p^p - \frac{1}{p}\big) dt,
\end{aligned} \tag{2.21}
$$

or

$$
l_{j,m}^n = \frac{\hat{\pi}}{\lambda_{n,m}^{1/p}} + \frac{1}{p\lambda_{n,m}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) S_p^p dt + O\big(\frac{1}{\lambda_{n,m}}\big).
$$

□

For $m = 1$, $m = 2$ and $m \geq 3$, we can easily obtain (2.18), (2.19) and (2.20) by using the formulas (2.15), (2.16), (2.17) and (2.21), respectively.

3. Reconstruction of the potential function

In this section, we give an explicit formula for the potential function by using the nodal lengths. The method used in the proof of the theorem is similar to classical problems; p-Laplacian Sturm-Liouville eigenvalue problem and p-Laplacian energy-dependent Sturm-Liouville eigenvalue problem (see [11, 14, 20, 21]).

Theorem 3.1. *Let $q_m(x)$ be a real-valued $C[0,1]$-function on the interval $0 \leq x \leq 1$. Then*

$$
q_m(x) = \lim_{n\to\infty} p\lambda_{n,m} \Big(\frac{\lambda_{n,m}^{1/p} l_{j,m}^n}{\hat{\pi}} - 1 \Big), \tag{3.1}
$$

for $j = j_n(x) = \max\{j : x_{j,m}^n < x\}$ and $m \in \mathbb{Z}^+$.

Proof. We need to consider Theorem 2.3 for the proof. From (2.21), we have

$$
\frac{p\lambda_{n,m}^{1/p+1}}{\hat{\pi}} l_{j,m}^n = p\lambda_{n,m} + \frac{\lambda_{n,m}^{1/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) dt + \frac{p\lambda_{n,m}^{1/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) (S_p^p - \frac{1}{p}) dt.
$$

Then, we can use similar procedure as those in [14] for $j = j_n(x) = \max\{j : x_{j,m}^n < x\}$ to show

$$
\frac{\lambda_{n,m}^{1/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) dt \to q_m(x),
$$

and

$$
\frac{p\lambda_{n,m}^{1/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) \big(S_p^p - \frac{1}{p}\big) dt \to 0,
$$

pointwise almost everywhere. Hence, we obtain

$$q_m(x) = \lim_{n\to\infty} p\lambda_{n,m}\Big(\frac{\lambda_{n,m}^{1/p} l_{j,m}^n}{\hat{\pi}} - 1\Big).$$

$\square$

Theorem 3.2. *Let $\{l_{j,m}^{(n)} : j = 1, 2, \dots, n-1\}_{n=2}^{\infty}$ be a set of the nodal lengths of problem (1.1)-(1.2), where q_m is a real-valued $C[0,1]$-function. Let us define*

$$F_{n,1}(x) = p(n\hat{\pi})^p (nl_{j,1}^{(n)} - 1) - \frac{p}{a_1}(n\hat{\pi})^{p/2} + \int_0^1 q_1(t)dt, \quad \textit{for } m = 1. \tag{3.2}$$

$$F_{n,2}(x) = p(n\hat{\pi})^p \big(nl_{j,2}^{(n)} - 1\big) - \frac{p(n\hat{\pi})^{p/2}}{a_1 + a_2(n\hat{\pi})^{p/2}} + \int_0^1 q_2(t)dt, \quad \textit{for } m = 2. \tag{3.3}$$

$$F_{n,m}(x) = p(n\hat{\pi})^p \big(nl_{j,m}^{(n)} - 1\big) - \frac{p(n\hat{\pi})^{p/2}}{a_1 + \dots + a_m(n\hat{\pi})^{\frac{mp-p}{2}}} + \int_0^1 q_m(t)dt, \quad \textit{for } m \geq 3. \tag{3.4}$$

Then $\{F_{n,m}(x)\}$ converges to q_m pointwise almost everywhere in $L^1(0,1)$, for all cases.

Proof. We prove this theorem only for $m = 1$. Other cases can be shown similarly. For $m = 1$, by the asymptotic formulas of eigenvalues (2.4) and nodal lengths (2.18), we obtain

$$p\lambda_{n,1}\Big(\frac{\lambda_{n,1}^{1/p} l_{j,1}^n}{\hat{\pi}} - 1\Big) = p\lambda_{n,1}\big(nl_{j,1}^{(n)} - 1\big) - \frac{p}{a_1\pi}(n\hat{\pi})^{p/2+1} l_{j,1}^{(n)} + nl_{j,1}^{(n)} \int_0^1 q_1(t)dt + o(1).$$

Considering $nl_{j,1}^{(n)} = 1 + o(1)$, as $n \to \infty$, we have

$$p(n\hat{\pi})^p (nl_{j,1}^{(n)} - 1) - \frac{p}{a_1}(n\hat{\pi})^{p/2} \to q_1(x) - \int_0^1 q_1(t)dt,$$

pointwise almost everywhere in $L^1(0,1)$. $\square$

Conclusion. In this study, we give some asymptotic estimates for eigenvalues, nodal parameters and potential function of the p-Laplacian Sturm-Liouville eigenvalue problem (1.1)-(1.2). We show that the obtained results are the generalizations of the classical problem.

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