Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 136, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STABILITY OF AN N-COMPONENT TIMOSHENKO BEAM WITH LOCALIZED KELVIN-VOIGT AND FRICTIONAL DISSIPATION

TITA K. MARYATI, JAIME E. MUÑOZ RIVERA, AMELIE RAMBAUD, OCTAVIO VERA

Communicated by Marco Squassina

Abstract. We consider the transmission problem of a Timoshenko's beam composed by N components, each of them being either purely elastic, or a Kelvin-Voigt viscoelastic material, or an elastic material inserted with a frictional damping mechanism. Our main result is that the rate of decay depends on the position of each component. More precisely, we prove that the Timoshenko's model is exponentially stable if and only if all the elastic components are connected with one component with frictional damping. Otherwise, there is no exponential stability, but a polynomial decay of the energy as $1/t^2$. We introduce a new criterion to show the lack of exponential stability, Theorem 1.2. We also consider the semilinear problem.

1. Introduction

Here we study a transmission problem of a Timoshenko beam [14] of length ℓ composed by N components, each of them can be of three different types of materials: elastic, viscoelastic, or a material with a frictional damping mechanism as illustrated in Figure 1 below, for $N = 5$.

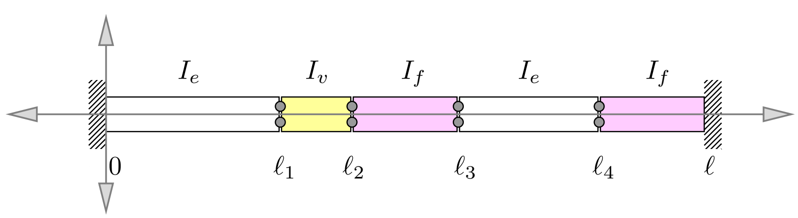


Figure 1. An example of five-components beam, where I_e is elastic, I_f is frictional, and I_v is viscoelastic component

2010 *Mathematics Subject Classification.* 35B40, 74K10, 35M33, 35Q74.
Key words and phrases. Timoshenko's model; beam equation; localized dissipation; viscoelaticity; lack of exponential stability; exponential and polynomial stability.

Submitted February 27, 2018. Published July 1, 2018.

Let us decompose the interval $I=[0,\ell]$ into N subintervals, $[0,\ell]=\cup_{i=1}^{n}\overline{I_i}$, such that $I_i=]\ell_{i-1},\ell_i[$ for $i=1,2,\dots,N$ with $\ell_0=0$ and $\ell_N=\ell$.

Over each interval I_i, one type of material is configured. We denote by I_v, I_e or I_f the subintervala where the viscoelastic component, elastic component, or the component with frictional mechanism is configured, respectively. In Figure 1 the intervals I_1 and I_4 are of type I_e, elastic components, I_2 is of viscoelastic type I_v, and so on. Let us denote the set

$$\widetilde{I}=\cup_{i=1}^{n}I_i=]0,\ell[\setminus\{\ell_0,\ell_1,\dots,\ell_N\}.$$

The set $\widetilde{I}$ is open and disconnected. The classical linear Timoshenko system given by

$$\varrho_1\varphi_{tt}-S_x=G_1,\quad \text{in}\quad \widetilde{I}\times\mathbb{R}_+, \tag{1.1}$$

$$\varrho_2\psi_{tt}-M_x+S=G_2,\quad \text{in}\quad \widetilde{I}\times\mathbb{R}_+, \tag{1.2}$$

Here we use the Dirichlet boundary conditions

$$\varphi(0,t)=\varphi(\ell,t)=\psi(0,t)=\psi(\ell,t)=0. \tag{1.3}$$

and the initial conditions

$$\varphi(x,0)=\varphi_0(x),\quad \psi(x,0)=\psi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\quad \psi_t(x,0)=\psi_1(x). \tag{1.4}$$

Here S and M stand for the shear force and the bending moment respectively, $\varrho_1=\varrho A$ and $\varrho_2=\varrho I^M$, where ϱ is the density of the material, A the cross-sectional area and I^M the second moment of the cross-section area. By φ we denote the transversal displacement and by ψ the shear angle displacement. The constitutive equations are given by

$$S(\varphi_x,\psi)=\kappa(x)\,(\varphi_x+\psi)+\kappa_0(x)\,(\varphi_{xt}+\psi_t),\quad M(\psi)=b(x)\psi_x+b_0(x)\psi_{xt}, \tag{1.5}$$

where $\kappa=k'\,G\,A$ and $b=E\,I^M$ are positive functions over $\widetilde{I}$. By E, G and k' we are denoting the Young's modulus, the modulus of rigidity and the transverse shear factor, respectively. We denote by b_0 and κ_0, positive functions which characterize the viscosity over I_v, vanishing over $I_e\cup I_f$. The localized frictional damping mechanism is described by the source terms

$$G_1(x,t)=-\gamma_1(x)\varphi_t,\quad G_2(x,t)=-\gamma_2(x)\psi_t, \tag{1.6}$$

where γ_1,γ_2 are positive only on the intervals I_f, vanishing over I_v and I_e.

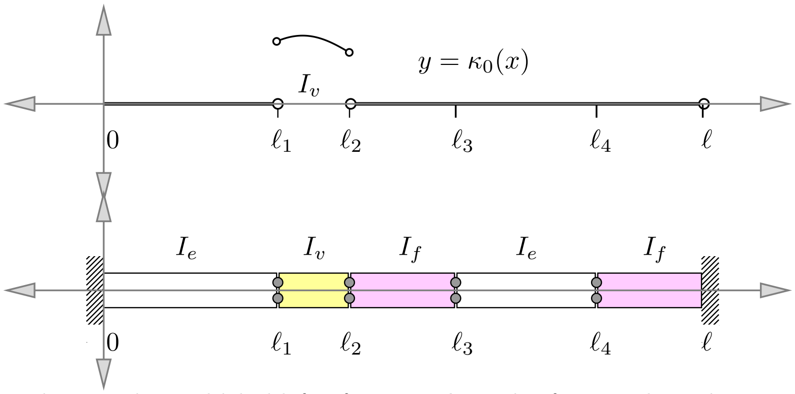
Therefore the elastic coefficients are discontinuous at the points where different materials are fitted. This characterizes the transmission problem. Hence the functions κ, κ_0, b, b_0, γ_1, $\gamma_2:[0,\ell]\to\mathbb{R}$ are such that its restrictions to I_i, $i=1,\dots,N$, are C^1 functions, with bounded discontinuities at the nodes ℓ_i, $i=1,\dots,N-1$; but even so, the stress as well as the bending moment must satisfy the laws of action and reaction at each point, therefore we have that any strong solutions of the problem must verify

$$\varphi,\psi,S,M\in H^1(0,\ell). \tag{1.7}$$

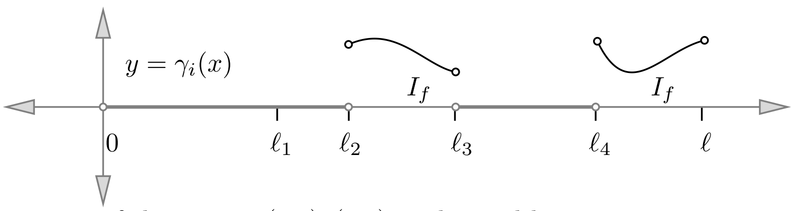
In particular (1.7) implies the transmission conditions at the interface points ℓ_i:

$$\varphi(\ell_i^-)=\varphi(\ell_i^+),\quad S(\ell_i^-)=S(\ell_i^+),\quad \psi(\ell_i^-)=\psi(\ell_i^+),\quad M(\ell_i^-)=M(\ell_i^+), \tag{1.8}$$

for $i=1,\dots N-1$. A typical example of a function $y=\kappa_0(x)$ is given in Figure 1:



A similar graph would hold for function b_0. The frictional mechanism is characterized by the functions $y = \gamma_i(x)$, $i = 1, 2$, for the same example is given as follows



The energy of the system (1.1)–(1.4), is denoted by

$$E(t) = \frac{1}{2}\int_0^\ell \varrho_1 |\varphi_t|^2 + \varrho_2 |\psi_t|^2 + \kappa |\varphi_x + \psi|^2 + b |\psi_x|^2 \, dx. \tag{1.9}$$

It is easy to see that

$$\frac{d}{dt}E(t) = -\int_0^l \kappa_0(x)|\varphi_{xt} + \psi_t|^2 \, dx + b_0(x)|\psi_{xt}|^2 + \gamma_1(x)|\varphi_t|^2 + \gamma_2(x)|\psi_t|^2 \, dx.$$

When $\kappa_0 = b_0 = \gamma_1 = \gamma_2 = 0$ the system is conservative. Regarding the novelty of our result, previous works on exponential stability consider only the effectiveness of the dissipative mechanism, whether or not it produces exponential stability, thus characterizing the dissipative mechanism as strong or weak respectively. For example to one-dimensional models was shown that the frictional dissipation exponentially stabilizes the model regardless of the position or region where the dissipative mechanism is concentrated, see for example [2, 5, 7, 4, 10, 13] to quote but a few. On the other hand, the dissipation produced by viscous materials, when effective over the whole domain, produces not only exponential stability but also analyticity of the corresponding semigroup. But when it concentrates in only a part of the domain, it loses effectiveness and produces neither exponential stability nor analyticity see [6, 8].

In this article we consider the two types of dissipative mechanisms, the frictional and the visco elastic dissipation both concentrated within the domain. Our main result is that the resulting dissipation will be strong or weak according to the position in which they are distributed over the domain. That is, we prove that if any elastic component (without dissipative mechanism) is next to a component with frictional dissipation, then the system is exponentially stable. Otherwise, when there is at least one component isolated between viscous components, then the system is no longer exponentially stable, but decays polynomially, that is we establish,

Theorem 1.1. *The transmission problem* (1.1)-(1.7) *($N \geq 2$) is exponentially stable if and only if any elastic part of the beam is connected with at least one component with frictional damping mechanisms. Otherwise the system is polynomially stable, with a rate of decay of the order t^{-2}.*

This type of result is closely related to the *optimal design problem*. The main tool we use to show the exponential stability is the Pruess' characterization of exponentially stable semigroups. We prove the lack of exponential stability using the following new criterion that we show in this article

Theorem 1.2. *Let $\mathcal{H}_0$ be a closed subspace of a Hilbert space $\mathcal{H}$. Let $\mathcal{T}_0(t)$ be a group on $\mathcal{H}_0$ such that $\|\mathcal{T}_0(t)\| = 1$ and $\mathcal{T}(t)$ be a contraction semigroup defined on $\mathcal{H}$. If the difference $\mathcal{T}(t) - \mathcal{T}_0(t)$ is compact from $\mathcal{H}_0$ to $\mathcal{H}$, then the semigroup $\mathcal{T}(t)$ is not exponentially stable.*

The remaining part of the paper is organized as follows. In Section 2 we show the well-posedness. In Section 3, we show the exponential stability. In Section 4 the lack of exponential stability and Theorem 1.2. In Section 5, we complete the proof of Theorem 1.1 by showing the polynomial decay. Finally, we show the same result to semilinear models.

2. Well-posedness

Let us introduce the phase space

$$\mathcal{H} = H_0^1(0,\ell) \times L^2(0,\ell) \times H_0^1(0,\ell) \times L^2(0,\ell).$$

This is a Hilbert space with the norm

$$\|\mathbf{U}\|_{\mathcal{H}}^2 = \int_0^\ell \varrho_1 |\Phi|^2 + \varrho_2 |\Psi|^2 + \kappa |\varphi_x + \psi|^2 + b |\psi_x|^2 \, dx, \tag{2.1}$$

for all $\mathbf{U} = (\varphi, \Phi, \psi, \Psi) \in \mathcal{H}$. Let $\mathcal{A}$ be the operator given by

$$\mathcal{A}\mathbf{U} = \begin{pmatrix} \Phi \\ \frac{1}{\varrho_1}[S_x - \gamma_1(x)\Phi] \\ \Psi \\ \frac{1}{\varrho_2}[M_x - S - \gamma_2(x)\Psi] \end{pmatrix}, \tag{2.2}$$

where S and M are given in (1.5). The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{\mathbf{U} \in \mathcal{H} : \Phi, \Psi \in H_0^1(0,\ell); S, M \in H^1(0,\ell)\}. \tag{2.3}$$

A straightforward calculation gives

$$\operatorname{Re}\langle \mathcal{A}\mathbf{U}, \mathbf{U}\rangle_{\mathcal{H}} = -\int_0^\ell \kappa_0 |\Phi_x + \Psi|^2 + b_0 |\Psi_x|^2 + \gamma_1 |\Phi|^2 + \gamma_2 |\Psi|^2 \, dx. \tag{2.4}$$

Therefore $\mathcal{A}$ is a dissipative operator. Under the above conditions the transmission problem (1.1)-(1.4) is equivalent to find $\mathbf{U} \in \mathcal{H}$, solution to

$$\mathbf{U}_t = \mathcal{A}\mathbf{U}, \quad \mathbf{U}(0) = \mathbf{U}_0. \tag{2.5}$$

where $\mathbf{U}_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H}$ is the initial datum, defined by (1.4). Under the above notations the well posedness is a matter of routine.

Theorem 2.1. *For any* $\mathbf{U}_0 \in \mathcal{H}$ *there exists a unique mild solution of* (2.5). *Moreover if* $\mathbf{U}_0 \in D(\mathcal{A})$, *then the solution is strong and* $\mathbf{U} \in C^1([0,\infty[;\mathcal{H}) \cap C([0,\infty[;D(\mathcal{A}))$.

Proof. It is sufficient to show that $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup. Note that $\mathcal{A}$ is dissipative, closed and densely defined on $\mathcal{H}$. It is straightforward to prove that $0 \in \varrho(\mathcal{A})$ (the resolvent set of $\mathcal{A}$). Our conclusion follows from Lummer Phillips's Theorem. □

We close this section by establishing the characterizations of the exponential and polynomial stabilization. due to Prüss [12]– Huang [9] and Borichev and Tomilov [1].

Theorem 2.2. *Let $S(t)$ be a contraction C_0-semigroup, generated by $\mathcal{A}$ over a Hilbert space $\mathcal{H}$. Then, Prüss* [12], *Huang* [9], *establish that there exists $C,\gamma > 0$ satisfying*

$$\|S(t)\| \le Ce^{-\gamma t} \Leftrightarrow i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\,\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le M,\ \forall \lambda \in \mathbb{R}. \tag{2.6}$$

For polynomial stability, Borichev and Tomilov [1] *established the existence of $C > 0$ such that*

$$\|\mathcal{S}(t)\mathcal{A}^{-1}\| \le \frac{C}{t^{1/\alpha}} \Leftrightarrow i\mathbb{R} \subset \varrho(\mathcal{A}) \text{ and } \|(i\,\lambda I - \mathcal{A})^{-1}\| \le M|\lambda|^{\alpha},\ \forall \lambda \in \mathbb{R} \tag{2.7}$$

3. Exponential stability

For simplicity, we assume that if I_{v_1} and I_{v_2} are two viscoelastic components, then

$$\overline{I_{v_1}} \cap \overline{I_{v_2}} = \emptyset. \tag{3.1}$$

This hypothesis is only to simplify arguments, the result remains valid even when (3.1) fails.

The resolvent equation $i\lambda\mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F}$, in terms of its coordinates is given by

$$i\lambda\varphi - \Phi = F_1, \tag{3.2}$$

$$i\lambda\varrho_1\Phi - S_x + \gamma_1\Phi = \varrho_1 F_2, \tag{3.3}$$

$$i\lambda\psi - \Psi = F_3, \tag{3.4}$$

$$i\lambda\varrho_2\Psi - M_x + S + \gamma_2\Psi = \varrho_2 F_4, \tag{3.5}$$

where $\mathbf{F} = (F_1,\dots,F_4) \in \mathcal{H}$ and φ and ψ verify Dirichlet boundary conditions (1.3).

Lemma 3.1. *The operator $\mathcal{A}$ defined by* (2.2) *and* (2.3) *satisfies $i\mathbb{R} \subset \varrho(\mathcal{A})$.*

Proof. We will reason by contradiction. Since $0 \in \varrho(\mathcal{A})$, the set

$$\mathcal{R} = \{\beta > 0 : [-i\beta, +i\beta] \subset \varrho(\mathcal{A})\} \neq \emptyset$$

Let $\overline{\lambda} := \sup \mathcal{R}$. If $\overline{\lambda} = \infty$, then there is nothing to prove. Let us suppose that $\overline{\lambda} < \infty$. Hence, there exists a sequence $\{\beta_n\}_n \subset \mathbb{R}$ such that $\beta_n \to \overline{\lambda}$ and $\|(i\beta_n I - \mathcal{A})^{-1}\| \to \infty$, that is there exists a sequence $\{\widetilde{F}_n\}_n$ of elements of $\mathcal{H}$ such that

$$\|\widetilde{F}_n\|_{\mathcal{H}} = 1, \quad \text{and} \quad \|\underbrace{(i\beta_n I - \mathcal{A})^{-1}\widetilde{F}_n}_{:=W_n}\|_{\mathcal{H}} \underset{n\to\infty}{\longrightarrow} +\infty.$$

Letting $X_n = W_n/\|W_n\|_{\mathcal{H}}$ and $F_n = \widetilde{F}_n/\|W_n\|_{\mathcal{H}}$, we have

$$\|X_n\|_{\mathcal{H}} = 1, \quad \text{and}(i\beta_n I - \mathcal{A})X_n = F_n \underset{n\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{H} \tag{3.6}$$

To arrive a contradiction it is enough to show $X_n \to 0$ as $n \to \infty$ strongly in $\mathcal{H}$. In fact, (2.4) and (3.6) yield

$$\begin{aligned} &\mathrm{Re}\langle i\beta_n X_n - \mathcal{A}X_n, X_n\rangle \\ &= \int_0^L \kappa_0|\Phi_x^n + \Psi^n|^2 + b_0|\Psi_x^n|^2 + \gamma_1\,|\Phi^n|^2 + \gamma_2\,|\Psi^n|^2\,dx \to 0. \end{aligned} \tag{3.7}$$

Since κ_0 and b_0 are positive over $\cup_{j=1}^m I_{v_j}$ we obtain

$$(\Phi_x^n + \Psi^n, \Psi_x^n) \to (0,0) \quad \text{strongly in } [L^2(\cup_{j=1}^m I_{v_j})]^2. \tag{3.8}$$

Where $\cup_{j=1}^m I_{v_j}$ is the union of all the intervals with viscoelastic component. Using (3.2)–(3.4) we obtain

$$(\varphi_x^n + \psi^n, \psi_x^n) = \frac{1}{i\beta_n}\left[(\Phi_x^n + \Psi^n, \Psi_x^n) + (F_{1,x}^n + F_3^n, F_{3,x}^n)\right] \to (0,0)$$

strongly in $[L^2(\cup_{j=1}^m I_{v_j})]^2$. Using (3.6) once more we obtain $\|\mathcal{A}X_n\| \le C$. Recalling the definition of $D(\mathcal{A})$ given in (2.2)–(2.3), we have

$$\int_0^\ell |\Phi_x^n|^2 + |\Psi_x^n|^2 + |S_x^n|^2 + |M_x^n|^2\,dx \le C \tag{3.9}$$

which in particular implies the estimate

$$\int_0^\ell |\Phi_x^n|^2 + |\Psi_x^n|^2\,dx + \int_{[0,\ell]\setminus\cup_{j=1}^m I_{v_j}} |S_x^n|^2 + |M_x^n|^2\,dx \le C. \tag{3.10}$$

Since $S_x^n = \kappa(\varphi_x^n + \psi^n)_x$ and $M_x^n = (b\psi_x^n)_x$ on $[0,\ell] \setminus \cup_{j=1}^m I_{v_j}$, there exists a subsequence of X_n, we still denote in the same way, such that

$$\begin{gathered} (\Phi^n, \Psi^n) \to (\Phi, \Psi) \quad \text{strongly in } [L^2(0,\ell)]^2, \\ (\varphi_x^n + \psi^n, \psi_x^n) \quad \to \quad (\varphi_x + \psi, \psi_x) \quad \text{strongly in } [L^2([0,\ell]\setminus\cup_{j=1}^m I_{v_j})]^2. \end{gathered}$$

The above convergence and (3.8) imply $X_n \to X$ strongly in $\mathcal{H}$. Since γ_1 and γ_2 are positive over $\cup_{i=1}^r I_{f_i}$, relation (3.7) implies

$$\varphi = \psi = \Phi = \Psi = 0, \quad \text{on } (\cup_{i=1}^r I_{f_i}) \cup (\cup_{j=1}^m I_{v_j})$$

Since any $I_e =]\alpha, \beta[$ is linked with I_v or I_f, without loss of generality we can assume that $\{\alpha\} = \overline{I_v} \cap \overline{I_e}$. Since $\varphi = \psi = 0$ in $I_v \cup I_f$, then system (3.2)–(3.5) over I_e can be written as

$$\begin{gathered} -\rho_1\lambda^2\varphi - (\kappa\varphi_x + \psi)_x = 0, \quad -\rho_2\lambda^2\psi - (b\psi_x)_x + \kappa(\varphi_x + \psi) = 0, \quad \text{in } [\alpha,\beta], \\ \varphi(\alpha) = \varphi_x(\alpha) = \psi(\alpha) = \psi_x(\alpha) = 0. \end{gathered}$$

By the uniqueness of ordinary differential equations we obtain $\overline{X} = 0$. The proof is now complete. $\square$

Let us introduce the notation

$$\begin{gathered} \mathcal{I}_\varphi(s) = \varrho_1\kappa|\Phi(s)|^2 + |S(s)|^2, \quad \mathcal{I}_\psi(s) = b\varrho_2|\Psi(s)|^2 + |M(s)|^2, \\ \mathcal{I}(s) = \mathcal{I}_\varphi(s) + \mathcal{I}_\psi(s). \end{gathered} \tag{3.11}$$

Lemma 3.2. *Let* $]\alpha,\beta[$ *any subinterval of* I_f*, then for* λ *large enough, we have*

$$\int_{I_v} \varrho_1|\Phi|^2+\varrho_2|\Psi|^2+\kappa|\varphi_x+\psi|^2+b|\psi_x|^2\,dx \le \frac{C}{|\lambda|}\left(\|U\|\|F\|+\|F\|^2_{\mathcal{H}}\right), \tag{3.12}$$

$$\begin{aligned}&\int_\alpha^\beta \varrho_1|\Phi|^2+\varrho_2|\Psi|^2+\kappa|\varphi_x+\psi|^2+b|\psi_x|^2\,dx\\ &\le C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+c\|F\|^2_{\mathcal{H}}+\frac{c}{|\lambda|}\left[\mathcal{I}(\alpha)+\mathcal{I}(\beta)\right]\end{aligned} \tag{3.13}$$

Proof. Multiplying the resolvent system by $\overline{\mathbf{U}}$, integrating over all the beam's length $(0,\ell)$, and using the dissipation (2.4) we obtain

$$\int_0^l \kappa_0\,|\Phi_x+\Psi|^2+b_0\,|\Psi_x|^2+\gamma_1\,|\Phi|^2+\gamma_2\,|\Psi|^2\;dx=\operatorname{Re}(\mathbf{F},\mathbf{U})_{\mathcal{H}} \tag{3.14}$$

The above relation implies

$$\int_{I_v}|\Phi_x+\Psi|^2+|\Psi_x|^2\,dx+\int_{I_f}|\Phi|^2+|\Psi|^2\;dx\le C\|\mathbf{F}\|_{\mathcal{H}}\|\mathbf{U}\|_{\mathcal{H}}. \tag{3.15}$$

From equation (3.5) we obtain

$$|\lambda|\|\Psi\|_{H^{-1}(I_v)}\le C\|M\|_{L^2(I_v)}+C\|S\|_{L^2(I_v)}+C\|F\|_{\mathcal{H}}$$

Therefore using (3.15), for λ large enough, we obtain

$$|\lambda|^2\|\Psi\|^2_{H^{-1}(I_v)}\le C\|U\|\|F\|+C\|F\|^2_{\mathcal{H}} \tag{3.16}$$

Then using interpolation and (3.15) and (3.16) we have

$$\begin{aligned}\|\Psi\|^2_{L^2(I_v)}&\le C\|\Psi\|_{H^{-1}(I_v)}\|\Psi\|_{H^1(I_v)}\\ &\le\frac{C}{|\lambda|}\left(\|U\|\|F\|+\|F\|^2_{\mathcal{H}}\right)^{1/2}\left(\|\Psi\|_{L^2(I_v)}+\|\Psi_x\|_{L^2(I_v)}\right)\\ &\le\frac{C}{|\lambda|}\left(\|U\|\|F\|+\|F\|^2_{\mathcal{H}}\right)+\frac12\|\Psi\|^2_{L^2(I_v)}.\end{aligned}$$

For λ large enough. Therefore

$$\|\Psi\|^2_{L^2(I_v)}\le\frac{C}{|\lambda|}\left(\|U\|\|F\|+\|F\|^2_{\mathcal{H}}\right). \tag{3.17}$$

Using (3.3), interpolation, and the above reasoning we obtain

$$\|\Phi\|^2_{L^2(I_v)}\le\frac{C}{|\lambda|}\left(\|U\|\|F\|+\|F\|^2_{\mathcal{H}}\right). \tag{3.18}$$

Using (3.2) and (3.15) we obtain

$$\int_{I_v}\kappa|\varphi_x+\psi|^2+b|\psi_x|^2\,dx\le\frac{C}{|\lambda|^2}\left(\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\|F\|^2_{\mathcal{H}}\right). \tag{3.19}$$

For λ large enough. From (3.17), (3.18), (3.19), the first part of the Lemma follows.

Now, let us consider the interval $I_f=]\alpha,\beta[$. multiplying (3.3) by $\overline{\varphi}$, (3.5) by $\overline{\psi}$, integrating over $]\alpha,\beta[$ and taking the real part we obtain

$$\int_{I_f}\kappa|\varphi_x+\psi|^2+b|\psi_x|^2\,dx=\left(S(s)\overline{\varphi}(s)+M(s)\overline{\psi}(s)\right)\Big|_\alpha^\beta+\int_{I_f}\varrho_1\,|\Phi|^2+\varrho_2\,|\Psi|^2\,dx+R,$$

with $|R| \leq C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}$. Using (3.2) and (3.4) we obtain

$$\Big| \big(S(s)\overline{\varphi}(s) + M(s)\overline{\psi}(s)\big)\Big|_{\alpha}^{\beta}\Big| \leq \frac{c}{|\lambda|}\mathcal{I}(\alpha) + \frac{c}{|\lambda|}\mathcal{I}(\beta) + c\|F\|_{\mathcal{H}}^2.$$

Therefore, thanks to (3.15) our conclusion follows. □

In what follows we will show the observability inequality. To do that, let us introduce the following notation.

$$\begin{aligned}\mathcal{L}(\alpha,\beta) = &\int_{\alpha}^{\beta} (b\varrho_2 q)_x|\Psi|^2 + q_x|M|^2 + (\kappa\varrho_1 q)_x|\Phi|^2 + q_x|S|^2\,dx \\ &- \int_{\alpha}^{\beta} q\varrho_1\kappa\Phi\overline{\Psi} - qS\overline{M}\,dx, + \int_{\alpha}^{\beta} q\left(\gamma_1\Phi\overline{S} + \gamma_2\Psi\overline{M}\right)\,dx\end{aligned}$$

where

$$q(x) = \frac{e^{nx} - e^{n\alpha}}{n}, \quad \text{or} \quad q(x) = \frac{e^{-n\beta} - e^{-nx}}{n}, \tag{3.20}$$

Note that $q'(x)$ is large in comparison to q for n large, hence there exists positive constants C_0 and C_1 such that

$$C_0\int_{\alpha}^{\beta} \mathcal{I}(s)\,dx \leq \mathcal{L}(\alpha,\beta) \leq C_1\int_{\alpha}^{\beta} \mathcal{I}(s)\,dx \tag{3.21}$$

Lemma 3.3. *Let* $\mathbf{U}$ *be solution to the resolvent system* (3.2)-(3.5)*. Let* $]\alpha,\beta[$ *any subinterval of* I_e*,* I_f *or* I_v*, then we have*

$$\Big|q(s)\mathcal{I}(s)\Big|_{\alpha}^{\beta} - \mathcal{L}(\alpha,\beta)\Big| \leq C\|\mathbf{U}\|\|\mathbf{F}\| + C\|\mathbf{F}\|^2, \quad]\alpha,\beta[\subset I_f \quad \text{or} \quad]\alpha,\beta[\subset I_e$$

and

$$\Big|q(s)\mathcal{I}(s)\Big|_{\alpha}^{\beta} - \mathcal{L}(\alpha,\beta)\Big| \leq C|\lambda|^{1/2}\|\mathbf{U}\|\|\mathbf{F}\| + C\|\mathbf{F}\|^2, \quad]\alpha,\beta[\subset I_v.$$

Proof. Multiply (3.3) by $q\overline{S}$ and integrating over $[\alpha,\beta]$ we obtain

$$i\lambda\int_{\alpha}^{\beta} \varrho_1 q\Phi\overline{S}\,dx - \int_{\alpha}^{\beta} qS_x\overline{S} - q\gamma_1\Phi\overline{S}\,dx = \int_{\alpha}^{\beta} \varrho_1 qF_2\overline{S}\,dx$$

Recalling the definition of S we obtain

$$\begin{aligned}&i\lambda\int_{\alpha}^{\beta} \varrho_1 q\Phi\kappa\overline{[\varphi_x + \psi]}\,dx - \int_{\alpha}^{\beta} qS_x\overline{S} - q\gamma_1\Phi\overline{S}\,dx \\ &= \int_{\alpha}^{\beta} \varrho_1 qF_2\overline{S}\,dx - i\lambda\int_{\alpha}^{\beta} \varrho_1 q\Phi\kappa_0\overline{[\Phi_x + \Psi]}\,dx\end{aligned}$$

Using (3.2) and recalling that $S = \kappa(\varphi_x + \psi) + \kappa_0(\Phi_x + \Psi)$ we obtain

$$-\frac{1}{2}\int_{\alpha}^{\beta} \kappa\varrho_1 q\frac{d}{dx}|\Phi|^2 + q\frac{d}{dx}|S|^2\,dx - \int_{\alpha}^{\beta} \varrho_1 q\kappa\Phi\overline{\Psi}\,dx + \int_{\alpha}^{\beta} q\gamma_1\Phi\overline{S}\,dx = \mathcal{G} \tag{3.22}$$

where

$$\mathcal{G} = \int_{\alpha}^{\beta} \varrho_1 q\kappa\Phi(\overline{F_{1,x} + F_3})\,dx - i\lambda\int_{\alpha}^{\beta} \varrho_1 q\Phi\kappa_0\overline{[\Phi_x + \Psi]}\,dx + \int_{\alpha}^{\beta} \varrho_1 qF_2\overline{S}\,dx$$

Integrating by parts (3.22) we obtain

$$\begin{aligned} &- q(s)\mathcal{I}_\varphi(s)\Big|_\alpha^\beta + \int_\alpha^\beta (\kappa\varrho_1 q)_x|\Phi|^2 + q_x|S|^2\,dx \\ &- \int_\alpha^\beta \varrho_1 q\kappa\Phi\overline{\Psi}\,dx + \int_\alpha^\beta q\gamma_1\Phi\overline{S}\,dx = 2\mathcal{G} \end{aligned} \tag{3.23}$$

Multiplying (3.5) by $q\overline{M}$, integrating over $[\alpha,\beta]$, and using the same above arguments we obtain

$$\begin{aligned} &- q(s)\mathcal{I}_\psi(s)\Big|_\alpha^\beta + \int_\alpha^\beta (b\varrho_2 q)_x|\Psi|^2 + q_x|M|^2\,dx \\ &+ \int_\alpha^\beta qS\overline{M}\,dx + \int_\alpha^\beta q\gamma_2\Psi\overline{M}\,dx = 2\mathcal{F} \end{aligned} \tag{3.24}$$

where

$$\mathcal{F} = -i\lambda\int_\alpha^\beta \varrho_2 q\Psi b_0\overline{\Psi_x}\,dx + \int_\alpha^\beta \varrho_2 qF_4\overline{M}\,dx.$$

Summing (3.23)–(3.24) and recalling the definition of $\mathcal{L}$ we obtain

$$- q(s)\mathcal{I}\Big|_\alpha^\beta + \mathcal{L}(\alpha,\beta) = 2\mathcal{G} + 2\mathcal{F} \tag{3.25}$$

Using (3.15) and (3.17) we obtain

$$|2\mathcal{G}| + |2\mathcal{F}| \le C\|\mathbf{U}\|\|\mathbf{F}\| + C\|\mathbf{F}\|^2, \quad \forall\,]\alpha,\beta[\subset I_e\cup I_f$$

Similarly, using (3.18) we obtain

$$|2\mathcal{G}| + |2\mathcal{F}| \le C|\lambda|^{1/2}\|\mathbf{U}\|\|\mathbf{F}\| + C\|\mathbf{F}\|^2, \quad \forall\,]\alpha,\beta[\subset I_v$$

Therefore our conclusion follows. □

Corollary 3.4. *Assume* (3.1) *holds. Then for any* $i = 1,\dots,N-1$*, there exists* $C>0$*, such that*

$$\mathcal{I}(\ell_i) \le C\left(\|\mathbf{U}\|_{\mathcal{H}}^2 + \|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}\right).$$

Proof. From (3.1) we can assume that any ℓ_i belongs to the border of some elastic or frictional component, since

$$S(\ell_i^-) = S(\ell_i^+), \quad M(\ell_i^-) = M(\ell_i^+).$$

Therefore we can apply Lemma 3.3 and inequalities (3.21) we obtain

$$\mathcal{I}(\ell_i) \le C\|\mathbf{U}\|_{\mathcal{H}}^2 + C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}$$

The conclusion follows. □

Now, we are in a position to prove the main result of this section.

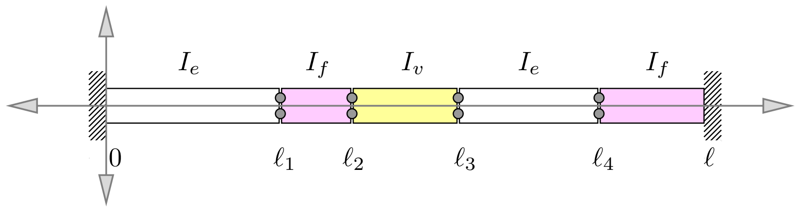


FIGURE 2. *A five-components beam, exponentially stable.*

Proof of the necessary condition of Theorem 1.1. From Lemma 3.2 we obtain for any interval I_v and I_f that

$$\begin{aligned}&\int_{I_v\cup I_f} \varrho_1|\Phi|^2+\varrho_2|\Psi|^2+\kappa|\varphi_x+\psi|^2+b|\psi_x|^2\,dx\\ &\le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+c\|\mathbf{F}\|^2_{\mathcal{H}}+\frac{c}{|\lambda|}\sum_{i=1}^{N-1}\mathcal{I}(\ell_i).\end{aligned}$$

Using Corollary 3.4 we obtain

$$\begin{aligned}&\int_{I_v\cup I_f} \varrho_1|\Phi|^2+\varrho_2|\Psi|^2+\kappa|\varphi_x+\psi|^2+b|\psi_x|^2\,dx\\ &\le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+c\|\mathbf{F}\|^2_{\mathcal{H}}+\epsilon\|\mathbf{U}\|^2_{\mathcal{H}}\end{aligned} \tag{3.26}$$

For $|\lambda|$ large enough. It remains to estimate the energy over intervals of type I_e. Let us denote $I_e=]\alpha,\beta[$. From hypothesis, this interval is linked with an interval of type I_f, for example at the point $\{\beta\}$. Using Lemma 3.3, over $I_e=]\alpha,\beta[$, we obtain

$$\int_{I_e}\mathcal{I}(s)\,ds\le c\mathcal{I}(\beta)+c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}. \tag{3.27}$$

Since $\beta\in I_f$, we apply the transmission conditions and the observability estimate, Lemma 3.3, for the frictional part

$$\mathcal{I}(\beta)\le c\int_{I_f}\mathcal{I}(s)\,ds+c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.$$

Hence, from (3.26) and (3.27), we obtain

$$\int_{I_e}\mathcal{I}(s)\,ds\le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+\frac{C}{|\lambda|^2}\left(\|\mathbf{U}\|^2_{\mathcal{H}}+\|\mathbf{F}\|^2_{\mathcal{H}}\right). \tag{3.28}$$

Therefore, adding all the energy over all interval I_e, I_f and I_v we obtain

$$\|\mathbf{U}\|^2\le C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+\epsilon\|\mathbf{U}\|^2_{\mathcal{H}}+C\|\mathbf{F}\|^2_{\mathcal{H}},$$

Which implies $\|\mathbf{U}\|\le C\|\mathbf{F}\|_{\mathcal{H}}$, the result follows thanks to part (2.6) of Theorem 2.2.

4. Lack of exponential stability

In this section we prove that system (1.1)–(1.4) does not decays exponentially to zero when hypotheses of Theorem 1.1 fails. The proof is based on Theorem 1.2. Before going into the details, we recall some results on the Calkin Algebra (see [3, pp. 248-250],).

4.1. **Calkin algebra.** Let $\mathcal{K}(\mathcal{H})$ be the set of all the compact operators over $\mathcal{H}$. It is a closed subspace and also a maximal ideal of $\mathcal{L}(\mathcal{H})$. The quotient space $\mathcal{C}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, called the Calkin algebra, is a complete space with the norm

$$\|S\|_{\rm ess} := \|\widetilde{S}\|_{\mathcal{C}(\mathcal{H})} := \inf\{\|S-K\|_{\mathcal{L}(\mathcal{H})};\ \ K \in \mathcal{K}(\mathcal{H})\}.$$

So any operator of $S \in \mathcal{L}(\mathcal{H})$ can be projected onto $\mathcal{C}(\mathcal{H})$ in the following way $\mathcal{M} : \mathcal{L}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$

$$\mathcal{M}(S) = \widetilde{S} = S + \mathcal{K}(\mathcal{H}).$$

Under the above notation we define the essential spectrum of S, $\sigma_{\rm ess}(S)$ as $\sigma(\widetilde{S})$ the spectrum $\widetilde{S} \in \mathcal{C}(\mathcal{H})$ and the essential spectral radius of an operator $S \in \mathcal{L}(\mathcal{H})$ as the spectral radios of $\widetilde{S}$, that is $r_{\rm ess}(S) := r(\widetilde{S})$. Note that from the definition of the essential norm, it holds:

$$\|S\|_{\rm ess} = \|S+K\|_{\rm ess}, \quad \forall K \in \mathcal{K}(\mathcal{H}).$$

This implies the following result, due to Weyl.

Theorem 4.1 (Weyl)**.** *The essential spectral radius is conserved under a relatively compact perturbation. That is to say, for any $S \in \mathcal{L}(X)$ and any $K \in \mathcal{K}(X)$, we have*

$$r_{\rm ess}(S) = r_{\rm ess}(S+K).$$

For an extension of this result, see [11, Theorem 5.35].

Let $S(t)$ be a semigroup. The type ω_0 (or growth bound) and the essential type $\omega_{\rm ess}$ of the semigroup are defined as

$$\omega_0(S) := \lim_{t\to\infty} \frac{\ln\|S(t)\|}{t}, \quad \omega_{\rm ess}(S) = \lim_{t\to\infty} \frac{1}{t} \ln\|S(t)\|_{\rm ess}, \tag{4.1}$$

Using the Gelfand Formula for the spectral radius of an operator,

$$r(S) = \lim_{n\to\infty} \|S^n\|^{1/n}.$$

Therefore, the spectral and the essential spectral radius of a semigroup $S(t)$ are given by

$$r(S(t)) = e^{\omega_0 t}, \quad r_{\rm ess}(S(t)) = r(\widetilde{S}(t)) = e^{\omega_{\rm ess} t}$$

Proposition 4.2. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup on the Banach X with generator A. Then*

$$\omega_0 = \max\{\omega_{\rm ess}, s(A)\},$$

where $s(A)$ is the spectral bound of A.

For a proof of this result see [3, Corollary 2.11]. We are now ready to prove our criterium for the lack of exponential stability of a C_0-semigroup.

4.2. **Proof of Theorem 1.2.** Since $\mathcal{T}_0(t)$, is a group satisfying $\|\mathcal{T}_0(t)\| = 1$, we have that for all $\lambda \in \sigma(\mathcal{T}_0(t))$, $|\lambda| = 1$. This implies that $r_{\rm ess}(\mathcal{T}_0(t)) = 1$. Let P be the orthogonal projection operator of $\mathcal{H}$ onto $\mathcal{H}_0$. Then $\mathcal{T}_0(t)P \in \mathcal{L}(\mathcal{H})$. Moreover, we have that

$$r_{\rm ess}(\mathcal{T}_0(t)P) \geq 1.$$

Otherwise, if $r_{\rm ess}(\mathcal{T}_0(t)P) < 1$, from the Gelfand formula we obtain

$$1 > \lim_{n\to\infty} \Big(\inf_{K\in\mathcal{K}(\mathcal{H})} \|[\mathcal{T}_0(t)P - K]^n\|_{\mathcal{H}} \Big)^{1/n}$$

$$\begin{aligned}
&\ge \lim_{n\to\infty}\Big(\inf_{K\in\mathcal{K}(\mathcal{H})}\|\mathcal{T}_0(t)^nP-K\|_{\mathcal{H}}\Big)^{1/n}\\
&\ge \lim_{n\to\infty}\Big(\inf_{K\in\mathcal{K}(\mathcal{H})}\|P\big(\mathcal{T}_0(t)^nP-K\big)\|_{\mathcal{H}}\Big)^{1/n}\\
&\ge \lim_{n\to\infty}\Big(\inf_{K\in\mathcal{K}(\mathcal{H}_0)}\|\mathcal{T}_0(t)^n-K\|_{\mathcal{H}_0}\Big)^{1/n}.
\end{aligned}$$

The last inequality holds because of the norm 1 of the projection operator. But this would imply $r_{\rm ess}(\mathcal{T}_0(t))<1$, which is a contradiction with the zero type of $\mathcal{T}_0(t)$ by Proposition 4.2. On the other hand, since $\mathcal{T}(t)-\mathcal{T}_0(t)$ is a compact operator from $\mathcal{H}_0$ to $\mathcal{H}$ the operator $[\mathcal{T}(t)-\mathcal{T}_0(t)]P$ is also compact operator over $\mathcal{H}$. Hence, from Theorem 4.1:

$$r_{\rm ess}(\mathcal{T}(t)P)=r_{\rm ess}(\mathcal{T}_0(t)P)\ge 1.$$

Using Gelfand's Formula once more, we have, for all $t>0$:

$$1\le r_{\rm ess}(\mathcal{T}(t)P)=\lim_{n\to\infty}\Big(\inf_{K\in\mathcal{K}(\mathcal{H})}\|[\mathcal{T}(t)-K]^n\|_{\mathcal{H}}\Big)^{1/n}\le\|\mathcal{T}(t)\|,$$

Therefore $\mathcal{T}(t)$ is not exponentially stable and the proof of Theorem 1.2 is complete.

4.3. **Lack of exponential stability.** Here we assume that the elastic part is not linked with a frictional component as in Figure 3, we claim the following result.

Proposition 4.3. *If there exists an elastic component not connected to a frictional component, then the transmission problem* (1.1)–(1.4) *with* $N\ge 2$ *is not exponentially stable.*

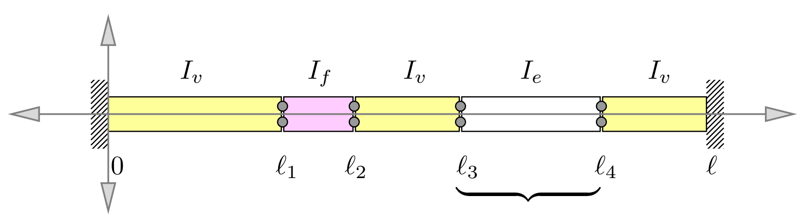


FIGURE 3. A five-components beam, non exponentially stable.

Proof. Let us denote by $I_e=]\alpha,\beta[$ the elastic interval that does not have any frictional neighbor. In Figure 3, the dissipative mechanisms are effective in all the components except in $I_4=]\ell_3,\ell_4[=]\alpha,\beta[$, this interval being isolated form the frictional ones. Let us define the space $\mathcal{H}_0$, as follows.

$$\mathcal{H}_0=\widetilde{H}^1_0(I_e)\times\widetilde{L}^2(I_e)\times\widetilde{H}^1_0(I_e)\times\widetilde{L}^2(I_e),$$

where

$$\begin{gathered}
\widetilde{L}^2(I_e)=\{g\in L^2(0,\ell):g(x)=0,\ \forall x\in]0,\ell[\backslash I_e\},\\
\widetilde{H}^1_0(I_e)=\{g\in H^1_0(0,\ell):g,g'\in L^2(I_e)\}.
\end{gathered}$$

Note that $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$, Denoting $\widehat{\mathbf{U}} = (\widehat{\varphi}, \widehat{\varphi}_t, \widehat{\psi}, \widehat{\psi}_t)$,

$$\begin{gathered}
\varrho_1 \widehat{\varphi}_{tt} - [\kappa(\widehat{\varphi}_x + \widehat{\psi})]_x = 0, \quad \text{in }]\alpha, \beta[\times \mathbb{R}_+, \\
\varrho_2 \widehat{\psi}_{tt} - [b\widehat{\psi}_x]_x + \kappa(\widehat{\varphi}_x + \widehat{\psi}) = 0, \quad \text{in }]\alpha, \beta[\times \mathbb{R}_+, \\
\widehat{\varphi}(\alpha, t) = \widehat{\varphi}(\beta, t) = 0, \quad \widehat{\psi}(\alpha, t) = \widehat{\psi}(\beta, t) = 0, \\
\widehat{\varphi}(x, 0) = \varphi_0, \quad \widehat{\varphi}_t(x, 0) = \varphi_1, \quad \widehat{\psi}(x, 0) = \psi_0, \quad \widehat{\psi}_t(x, 0) = \psi_1.
\end{gathered} \tag{4.2}$$

The elastic part being isolated from the rest of the components, this system is conservative, so it defines a group of isometries, with type 0. Now we extend the solution to $]0, \ell[$ as

$$\widetilde{\varphi}(x,t) = \begin{cases} \widehat{\varphi}(x,t), & x \in I_e =]\alpha, \beta[, \\ 0, & x \in]0, \ell[\setminus I_e, \end{cases} \quad \widetilde{\psi}(x,t) = \begin{cases} \widehat{\psi}(x,t), & x \in I_e =]\alpha, \beta[, \\ 0, & x \in]0, \ell[\setminus I_e. \end{cases}$$

Under these conditions, for any $\mathbf{U}_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in \mathcal{H}_0$ we define the semigroup $\mathcal{T}_0(t)$ as

$$\mathcal{T}_0(t)\, \mathbf{U}_0 = (\widetilde{\varphi}, \widetilde{\varphi}_t, \widetilde{\psi}, \widetilde{\psi}_t).$$

Thus we have $\omega_0(\mathcal{T}_0(t)) = 0$ on $\mathcal{H}_0$. To apply Theorem 1.2, it remains to show that $\mathcal{T}(t) - \mathcal{T}_0(t)$ is compact. Let $\mathbf{U}_0^n = (\varphi_0^n, \varphi_1^n, \psi_0^n, \psi_1^n) \in \mathcal{H}_0$ be a bounded sequence of $\mathcal{H}_0$. Denoting by

$$\mathbf{U}^n = (\varphi^n, \varphi_t^n, \psi^n, \psi_t^n) = \mathcal{T}(t)\mathbf{U}_0^n,$$

the solution to the original transmission problem with initial condition $\mathbf{U}_0^n$, and

$$\widetilde{\mathbf{U}}^n = (\widetilde{\varphi}^n, \widetilde{\varphi}_t^n, \widetilde{\psi}^n, \widetilde{\psi}_t^n) = \mathcal{T}_0(t)\, \mathbf{U}_0^n,$$

the solution to the modified problem. Let

$$\mathbf{Z}^n(t) := \mathbf{U}^n - \widetilde{\mathbf{U}}^n = (\varphi^n, \varphi_t^n, \psi^n, \psi_t^n) - (\widetilde{\varphi}^n, \widetilde{\varphi}_t^n, \widetilde{\psi}^n, \widetilde{\psi}_t^n) = (W^n, W_t^n, V^n, V_t^n).$$

Recalling that $\widetilde{I} = \cup_{k=1}^N I_k$, the sequence $\mathbf{Z}^n$ satisfies

$$\varrho_1 W_{tt} - [\kappa(W_x + V)]_x - [\kappa_0(W_{xt} + V_t)]_x + \gamma_1 W_t = 0 \quad \text{in } \widetilde{I} \times \mathbb{R}^+, \tag{4.3}$$

$$\varrho_2 V_{tt} - [bV]_x - [b_0 V_{xt}]_x + \kappa(W_x + V) + \gamma_2 V_t = 0 \quad \text{in } \widetilde{I} \times \mathbb{R}^+. \tag{4.4}$$

Let us introduce the energy of this problem,

$$E_{\mathbf{Z}^n}(t) := \frac{1}{2} \int_0^l \varrho_1 |W_t^n|^2 + \varrho_2 |V_t^n|^2 + \kappa |W_x^{n,i} + V^n|^2 + b|V_x^n|^2 \; dx.$$

Since we are in a Hilbert space, it suffices to show that there exists a subsequence of $\{\mathbf{Z}^n\}$ that converges in norm (or in energy). Multiplying equation (4.3) by W_t^n, (4.4) by V_t^n, and integrating on $\widetilde{I}$ we have

$$\begin{aligned}
&\frac{d}{dt} E_{\mathbf{Z}^n}(t) + \int_0^l \kappa_0 |W_{xt}^n + V_t^n|^2 + b_0 |V_{xt}^n|^2 + \gamma_1 |W_t^n|^2 + \gamma_2 |V_t^n|^2 \, dx \\
&= \kappa(W_x^n + V^n) W_t^n \big|_\alpha^\beta + \kappa_0 (W_{xt}^n + V_t^n) W_t^n \big|_\alpha^\beta + b V_x^n V_t^n \big|_\alpha^\beta + b_0 V_{xt}^n V_t^n \big|_\alpha^\beta \\
&= \kappa \widetilde{\varphi}_x^n \varphi_t^n \big|_\alpha^\beta + b \widetilde{\psi}_x^n \psi_t^n \big|_\alpha^\beta.
\end{aligned}$$

Note that $\widetilde{\varphi}_x^n(\alpha,t)$ and $\widetilde{\varphi}_x^n(\beta,t)$ are bounded in $L^2(0,T)$. Since $E_{\mathbf{Z}^n}(0)=0$, it follows that

$$\begin{aligned}
&E_{\mathbf{Z}^n}(t)+\int_0^T\int_0^l \kappa_0|W_{xt}^n+V_t^n|^2+b_0|V_{xt}^n|^2+\gamma_1|W_t^n|^2+\gamma_2|V_t^n|^2\,dx\\
&=\kappa\int_0^T \widehat{\varphi}_x^n\varphi_t^n\Big|_\alpha^\beta\,dt+b\int_0^T \widehat{\psi}_x^n\psi_t^n\Big|_\alpha^\beta\,dt.
\end{aligned}\tag{4.5}$$

In the viscoelastic intervals I_v, the sequences φ_t^n, ψ_t^n, are bounded in the space $L^2(0,T;\ H^1(I_v))$ (from the energy dissipation estimate). Moreover, φ_{tt}^n, ψ_{tt}^n are bounded in $L^2(0,T;\ H^{-1}(I_v))$. Hence, from compactness criterion of Aubin-Lions, we have, up to a subsequence,

$$(\varphi_t^n,\psi_t^n)\to(\varphi_t,\psi_t)\quad\text{strongly in } L^2(0,T;H^{1-\epsilon}(I_v)\times H^{1-\epsilon}(I_v)),$$

for all $0<\epsilon<1$. It yields

$$(\varphi_t^n(s,\cdot),\psi_t^n(s,\cdot))\to(\varphi_t(s,\cdot),\psi_t(s,\cdot))\quad\text{strongly in } L^2(0,T)\times L^2(0,T),$$

for $s=\alpha$ and $s=\beta$. Therefore we obtain, up to a subsequence, the strong convergence $E_{\mathbf{Z}^n}(t)\to E_{\mathbf{Z}}(t)$, where $\mathbf{Z}=\mathbf{U}-\widetilde{\mathbf{U}}$ is the difference of the weak limits. Therefore, since in a Hilbert space, the weak convergence and the convergence in norm imply the strong convergence, we conclude that $\mathcal{T}(t)-\mathcal{T}_0(t)$ is compact from $\mathcal{H}_0$ to $\mathcal{H}$. From Theorem 1.2, the semigroup $\mathcal{T}(t)$ is not exponentially stable and the proof of Proposition 4.3 is complete. □

5. Polynomial decay

To complete the proof of Theorem 1.1, it remains to show the polynomial decay, under a non exponential configuration (as in Figure 3 for example).

Proposition 5.1. *If there exists an elastic component not connected to a frictional component, then the semigroup $\mathcal{T}(t)$ defined by problem* (1.1)–(1.4) *with $N\geq 2$ decays polynomially as*

$$\|\mathcal{T}(t)\mathbf{U}_0\|_{\mathcal{H}}\leq\frac{c}{t^2}\|\mathbf{U}_0\|_{\mathcal{H}}.$$

Proof. As in the proof of the exponential stability we have

$$\sum_{i=1}^N\int_{I_{v_i}\cup I_{f_i}}\mathcal{I}(s)\,ds\leq c\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C\|\mathbf{F}\|_{\mathcal{H}}^2,\tag{5.1}$$

for $|\lambda|$ large enough. It remains to estimate the energy over the interval $I_e$we denote as $I_e=(\alpha,\beta)$. By the hypotheses, $\alpha\in I_v$ or $\beta\in I_v$. Using Lemma 3.3 over I_e we obtain

$$\int_{I_e}\mathcal{I}(s)\,ds\leq C\mathcal{I}(\beta)+C\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}.\tag{5.2}$$

Using Lemma 3.3 over I_v, we have

$$\mathcal{I}(\beta)\leq C|\lambda|^{1/2}\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C|\lambda|^{1/2}\|\mathbf{F}\|_{\mathcal{H}}^2.\tag{5.3}$$

From inequality (3.12) of Lemma 3.2 and Lemma 3.3 we have

$$\int_{I_e}\mathcal{I}(s)\,ds\leq C|\lambda|^{1/2}\|\mathbf{U}\|_{\mathcal{H}}\|\mathbf{F}\|_{\mathcal{H}}+C|\lambda|^{1/2}\|\mathbf{F}\|_{\mathcal{H}}^2.\tag{5.4}$$

From where it follows, with the Young inequality, that $\|\mathbf{U}\|_{\mathcal{H}}^2\leq c|\lambda|\|\mathbf{F}\|_{\mathcal{H}}^2$. Our conclusion follows thanks part 2.7 of Theorem 2.2. □

6. Semi linear problem

Here we prove the exponential and polynomial stability for a long class of locally Lipschitz $\mathcal{F}$ functions over a Hilbert space $\mathcal{H}$. We consider are the following hypotheses: For any ball $B_R = \{W \in \mathcal{H} : \|W\|_{\mathcal{H}} \le R\}$, there exists a function $\widetilde{\mathcal{F}_R}$ globally of Lipschitz such that

$$\mathcal{F}(0) = 0, \quad \mathcal{F}(U) = \widetilde{\mathcal{F}_R}(U), \quad \forall U \in B_R; \tag{6.1}$$

additionally, that there exists a positive constant κ_0 such that

$$\int_0^t \widetilde{\mathcal{F}_R}(U(s))U(s)\,ds \le \kappa_0 \|U(0)\|^2_{\mathcal{H}}, \quad \forall U \in C([0,T];\mathcal{H})\,. \tag{6.2}$$

Under these condition, we present the following result.

Theorem 6.1. *Let $\{S(t)\}_{t\ge 0}$ be a contraction, exponentially or polynomially stable semigroup with infinitesimal generator $\mathbb{A}$ over the phase space $\mathcal{H}$. Let $\mathcal{F}$ locally Lipschitz on $\mathcal{H}$ satisfying conditions* (6.1) *and* (6.2)*. If there exists a global solution to*

$$U_t - \mathbb{A}U = \mathcal{F}(U), \quad U(0) = U_0 \in \mathcal{H}, \tag{6.3}$$

then the solution decays exponentially or polynomially respectively.

Proof. By hypotheses, there exist positive constants c_0 and γ such that $\|S(t)\| \le c_0 e^{-\gamma t}$, and $\widetilde{\mathcal{F}_R}$ is globally Lipschitz with Lipschitz constant K_0 satisfying (6.1) and (6.2). Let us consider the space

$$E_\mu = \big\{V \in L^\infty(0,\infty;\mathcal{H}) : t \mapsto e^{-\mu t}\|V(s)\| \in L^\infty(\mathbb{R})\big\}$$

Using standard fixed point arguments we can show that there exists only one global solution to

$$U^R_t - \mathbb{A}U^R = \widetilde{\mathcal{F}_R}(U^R), \quad U^R(0) = U_0 \in \mathcal{H}, \tag{6.4}$$

Multiplying the above equation by U^R we obtain that

$$\frac{1}{2}\frac{d}{dt}\|U^R(t)\|^2_{\mathcal{H}} - (\mathbb{A}U^R, U^R)_{\mathcal{H}} = (\widetilde{\mathcal{F}_R}(U^R), U^R)_{\mathcal{H}}$$

Since the semigroup is contractive, its infinitesimal generator is dissipative, therefore

$$\|U^R(t)\|^2_{\mathcal{H}} \le \|U_0\|^2_{\mathcal{H}} + 2\int_0^t (\widetilde{\mathcal{F}_R}(U^R), U^R)_{\mathcal{H}}\,dt$$

Using (6.2) we obtain

$$\|U^R(t)\|^2_{\mathcal{H}} \le (1+k_0)\|U_0\|^2_{\mathcal{H}}$$

Nota that for $R > (1+k_0)\|U_0\|^2_{\mathcal{H}}$, we have that

$$\widetilde{\mathcal{F}_R}(V) = \mathcal{F}(V), \quad \forall \|V\|_{\mathcal{H}} \le R$$

In particular we have

$$\widetilde{\mathcal{F}_R}(U^R(t)) = \mathcal{F}(U^R(t)).$$

This means that U^R is also solution of system (6.3) and because of the uniqueness we conclude that $U^R = U$. Therefore to show the exponential stability to system (6.3), it is sufficient to show the exponential decay to system (6.4). To do that, we use fixed points arguments.

$$\mathcal{T}(V) = S(t)U_0 + \int_0^t S(t-s)\widetilde{\mathcal{F}_R}(V(s))\,ds,$$

Note that $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$ for δ small, $(\gamma-\delta>0)$. In fact, for any $V \in E_{\gamma-\delta}$ we have

$$\begin{aligned}
\|\mathcal{T}(V)\|_{\mathcal{H}} &\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \int_0^t \|\widetilde{\mathcal{F}_R}(V(s))\|_{\mathcal{H}} e^{-\gamma(t-s)}\,ds \\
&\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 \int_0^t \|V(s)\|_{\mathcal{H}} e^{-\gamma(t-s)}\,ds \\
&\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + K_0 e^{-\gamma t} \int_0^t e^{\delta s}\,ds \sup_{s\in[0,t]} \{e^{(\gamma-\delta)s}\|V(s)\|_{\mathcal{H}}\} \\
&\le \|U_0\|_{\mathcal{H}} e^{-\gamma t} + \frac{K_0 C}{\delta} e^{-(\gamma-\delta)t}.
\end{aligned}$$

Therefore, $\mathcal{T}(V) \in E_{\gamma-\delta}$. Using standard arguments we can show that $\mathcal{T}^n$ satisfies

$$\|\mathcal{T}^n(W_1) - \mathcal{T}^n(W_2)\| \le \frac{(k_1 t)^n}{n!} \|W_1 - W_2\|_{\mathcal{H}}$$

Therefore we have a unique fixed point satisfying

$$\mathcal{T}^n(U) = U = S(t)U_0 + \int_0^t S(t-s)\widetilde{\mathcal{F}_R}(U(s))\,ds,$$

That is U is a solution of (6.4), and since $\mathcal{T}$ is invariant over $E_{\gamma-\delta}$, then the solution decays exponentially. To show the polynomial stability we consider the space

$$E_p = \{V \in L^\infty(0,\infty;\mathcal{H}) : t \mapsto (1+t)^p \|V(s)\| \in L^\infty(\mathbb{R})\}$$

To show the invariance we use

$$\sup_{t>0} (1+t)^p \int_0^t (1+t-s)^{-p}(1+s)^{-p}\,ds < C$$

and use the same above reasoning. □

We finish this section with an application to the semilinear the Timoshenko model

$$\begin{gathered}
\rho_1 \varphi_{tt} - S_x + \gamma_1 \varphi_t + \mu_1 \varphi |\varphi|^{\alpha_1} = 0 \quad \text{in } \widetilde{I} \times (0,\infty), \\
\rho_2 \psi_{tt} - M_x + S + \gamma_2 \psi_t + \mu_2 \psi |\psi|^{\alpha_2} = 0 \quad \text{in } \widetilde{I} \times (0,\infty),
\end{gathered} \tag{6.5}$$

satisfying conditions (1.3) and (1.4). Here μ_1 and μ_2 are positive constants.

Theorem 6.2. *With the same hypotheses as in Theorem 1.1 there exists only one global solution to system* (6.5) *that decays exponentially to zero when any elastic componentes is linked to a frictional component. Otherwise the solution decays polynomially with rate* t^{-2}.

Proof. For $U = (\varphi, \varphi_t, \psi, \psi_t)^t$, the nonlinear function $\mathcal{F}$ can be written as

$$\mathcal{F}(U) = -(0,\ \mu_1 \varphi|\varphi|^{\alpha_1}, 0, \mu_2 \psi |\psi|^{\alpha_2})^t$$

Therefore for $V_i = (\varphi_i, \varphi_{i,t}, \psi_i, \psi_{i,t})^t$ with $i = 1, 2$, we obtain

$$[\mathcal{F}(V_1) - \mathcal{F}(V_2)] = (0, \varphi_1|\varphi_1|^{\alpha_1} - \varphi_2|\varphi_2|^{\alpha_1}, 0, \psi_1|\psi_1|^{\alpha_2} - \psi_2|\psi_2|^{\alpha_2})$$

Using the mean value theorem to $g(s) = |s|^\alpha s$ we obtain the inequality

$$\big|s|s|^\alpha - \tau|\tau|^\alpha\big| \le (|s|^\alpha + |\tau|^\alpha)|s-\tau|$$

Taking the norm in $\mathcal{H}$ and since φ_i and ψ_i belong to $H^1(0,\ell) \subset L^\infty(0,\ell)$ then we have

$$\|\mathcal{F}(V_1) - \mathcal{F}(V_2)\|_{\mathcal{H}}^2 \le \rho_1 |cR|^{2\alpha_1} \int_0^\ell |\varphi_1 - \varphi_2|^2\,dx + \rho_1 |cR|^{2\alpha_2} \int_0^\ell |\psi_1 - \psi_2|^2\,dx$$

where we used

$$\|\phi_1\|_{L^\infty} \le c\|\psi_1\|_{H^1}, \quad \text{and} \quad V_1, V_2 \in B_R$$

Therefore,

$$\|\mathcal{F}(V_1) - \mathcal{F}(V_2)\|_{\mathcal{H}}^2 \le K\|V_1 - V_2\|_{\mathcal{H}}^2$$

Where $K = \max\{\rho_1|cR|^{2\alpha_1}, \rho_2|cR|^{2\alpha_2}\}$. Therefore $\mathcal{F}$ is locally Lipschitz. Since

$$(\mathcal{F}(U), U)_{\mathcal{H}} = -\frac{d}{dt}\int_0^\ell \frac{\mu_1}{1+\alpha_1}|\varphi|^{2+\alpha_1} + \frac{\mu_2}{1+\alpha_2}|\psi|^{2+\alpha_2}\,dx$$

Therefore,

$$\int_0^t (\mathcal{F}(U), U)_{\mathcal{H}}\,dt \le \int_0^\ell \frac{\mu_1}{1+\alpha_1}|\varphi(0)|^{2+\alpha_1} + \frac{\mu_2}{1+\alpha_2}|\psi(0)|^{2+\alpha_2}\,dx$$

This implies that there exists a positive constant

$$\kappa_0 = \max\{\frac{\mu_1}{1+\alpha_1}|cR|^{2\alpha_1}, \frac{\mu_2}{1+\alpha_2}|cR|^{2\alpha_2}\}$$

such that

$$\int_0^t (\mathcal{F}(U), U)_{\mathcal{H}}\,dt \le \kappa_0\|U_0\|_{\mathcal{H}}^2$$

Note that for this function, there exists the cut-off function

$$f_{1,R_2}(x) = \begin{cases} \mu_1 x|x|^{\alpha_1} & x \le R_2, \\ \mu_1 x|R_2|^{\alpha_1} & |x| \ge R_2, \end{cases} \qquad f_{2,R_2}(x) = \begin{cases} \mu_2 x|x|^{\alpha_2} & x \le R_2, \\ \mu_2 x|R_2|^{\alpha_2} & |x| \ge R_2. \end{cases}$$

It is not difficult to check that

$$\widetilde{\mathcal{F}_{R_2}} = (0, f_{1,R_2}, 0, f_{2,R_2})^t$$

satisfies conditions (6.1)–(6.2) and is globally Lipschtiz. Then the result follows. □

Acknowledgements. The authors want to thank the Bío-Bío University project GI 171608/VC for their economic support.

References

[1] A. Borichev, Y. Tomilov; Optimal polynomial decay of functions and operator semigroups. *Mathematische Annalen*, 347: 455–478, 2009.

[2] G. Chen, S. A. Fulling, F. J. Narcowich, S. Sun; Exponential decay of energy of evolution equation with locally distributed damping. *SIAM Journal of Applied Mathematics*, 51(1): 266–301, 1991.

[3] K.J. Engel, R. Nagel; *One parameter Semigroups for Linear Evolution Equations*. Springer. New York, 1991.

[4] Ho. Lop Fat; Exact controllability of the one-dimensional wave equation with locally distributed control. *SIAM Journal on Control and Optimization*, 28: 733–748, 1990.

[5] V. Komornik; Rapid boundary stabilization of the wave equation. *SIAM Journal on Control and Optimization*, 29: 197–208, 1991.

[6] K. Liu, Z. Liu; Exponential decay of the energy of the euler bernoulli beam with locally distributed kelvin-voigt damping. *SIAM Journal of Control and Optimization*, 36: 1086–1098, 1998.

[7] W. Liu, G. Williams. The exponential stability of the problem of transmission of the wave equation. *Bulletin of the Australian Mathematical Society*, pages 305–327, 1998.

[8] Z. Liu, S. Zheng; Semigroups associated with dissipative systems. *CRC Reseach Notes in Mathematics 398*, 1999.
[9] Huang Fa Lun; Characteristic conditions for exponential stability of linear dynamical systems in hilbert spaces. *Annals of Differential Equations*, 1: 43–56, 1985.
[10] M. Nakao; Decay of solutions of the wave equation with a local nonlinear dissipation. *Mathematische Annalen*, 305: 403–417, 1996.
[11] A. Pazy; *Perturbation theory for linear operators*. Springer-Verlag. Berlin, 1980.
[12] J. Prüss; On the spectrum of c_0-semigroups. *Trans. AMS*, 284: 847–857, 1984.
[13] R. Quintanilla; Exponential decay in mixtures with localized dissipative term. *Applied Mathematical Letters*, 18: 1381–1388, 2005.
[14] S. Timoshenko; *Vibration Problems in Engineering*. Van Nostrand Inc., 1928.

Tita K. Maryati
Islamic State University (UIN) Syarif Hidayatullah Jakarta, Indonesia
E-mail address: `tita.khalis@uinjkt.ac.id`

Jaime E. Muñoz Rivera
Department of Mathematics, University of Bío-Bío, Concepción, Chile
E-mail address: `jemunozrivera@gmail.com`

Amelie Rambaud
Department of Mathematics, University of Bío-Bío, Concepción, Chile
E-mail address: `amelie.rambaud@yahoo.fr`

Octavio Vera
Department of Mathematics, University of Bío-Bío, Concepción, Chile
E-mail address: `octaviovera49@gmail.com`