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OPTIMAL DISTRIBUTED CONTROL PROBLEM FOR THE MODIFIED SWIFT-HOHENBERG EQUATIONS

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Abstract. This article concerns the optimal distributed control for the modified Swift-Hohenberg equation. Using the Dubovitskii and Milyutin functional analytical approach, we prove the Pontryagin maximum principle of the controlled modified Swift-Hohenberg equation. A necessary optimality condition is established for the problem in fixed final time horizon case. Also we indicate how to utilize the obtained results.

1. Introduction

In 1977, Jack B. Swift and Pierre Hohenberg [23] derived a partial differential equation from the equations for thermal convection, named as the Swift-Hohenberg equation (SH equation) thereafter, when they considered the effects of thermal fluctuations on the convective instability. It takes the form

$$u_t = -(1+\Delta)^2 u - au + N(u), \tag{1.1}$$

where $u = u(x,t)$ or $u = u(x,y,t) \in \mathbb{R}$, $t \geq 0$, $x, y \in \mathbb{R}$, a is a real constant, Δ is the Laplace operator, and $N(u)$ is some smooth nonlinearity.

This equation is noted for its pattern-forming behavior and has been widely used as a model for the study of various issues in pattern formation. These include the effects of noise on bifurcations, pattern selection, spatiotemporal chaos and the dynamics of defects. It also has been used to model patterns in simple fluids (e.g. Rayleigh-Bénard (RB) convection) and in a variety of complex fluids and biological materials, such as neural tissues. The motivation for the study which led to the SH equation was the analogy between bifurcations in the hydrodynamic behavior of fluids and the associated partial differential equations on the one hand, and continuous phase transitions in thermodynamic systems on the other hand [22]. As one of the universal equations used in the description of pattern formation in spatially extended dissipative systems, the SH equation can also be found in the study of convective hydrodynamics, plasma confinement in toroidal devices, viscous film flow and bifurcating solutions of the Navier-Stokes equation [4, 12, 16].

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Because of its wide applications in science, the SH equation has drawn lots of interest among physicists and mathematicians since it is presented [15, 18]. Moreover, to model specific experimental effects, on the SH equation, there are lots of modifications introduced. Specifically, the spontaneous formation of patterns in spatially extended systems has attracted much attention over the past few decades. Many beautiful patterns such as spatially periodic rolls, hexagonal cell structures, and spiral waves have been observed in experiments [3]. While at this very point, the SH equation receives its unusual popularity and attains the second life [22].

As a phenomenological model for pattern-forming systems near the onset of instability, the paper [3] presents a modified SH equation

$$u_t = -(1+\Delta)^2 u - au - b|\nabla u|^2 - u^3, \tag{1.2}$$

in which b is a real constant, ∇u is the gradient of u, and $u = u(x,y,t) \in \mathbb{R}$. Note that we recover the usual SH equation if we set $b = 0$. The additional term $b|\nabla u|^2$, reminiscent of the Kuramoto-Sivashinsky equation ([20]), breaks the symmetry $u \to -u$. It is needed to obtain stable hexagonal patterns. In fact, people can obtain stripes and hexagonal patterns from (1.2) by adopting proper parameters in equation. By this modified SH equation, [3] investigates some of the interfaces between competing spatially-periodic patterns. These interesting interfaces can be modelled as modulated fronts, i.e. as waves that are time-periodic in an appropriate co-moving coordinate frame. Both speed and shape of these interfaces therefore vary periodically in time. Furthermore, they prove the existence of modulated fronts that connect stable with unstable patterns. These modulated fronts respectively describe (i) stable hexagons that invade the unstable rest state at $u = 0$, (ii) stable hexagons that invade unstable roll solutions, (iii) stable hexagons that invade unstable hexagons, and lastly, (iv) stable roll solutions that invade unstable hexagons.

In this article, we are interested in the optimal control investigations of the modified SH equation (1.2) in one spatial dimension. At this stage, the present state of the research is entirely different with those investigations in other directions for this said equation. To the best of our knowledge, few results are known even on the control problem investigations of the SH equation, let alone its optimal control. Optimal control problems for the SH equation are largely unexplored and need more attention. Here we try to give all related references. Duan and Gao [4] prove the existence and uniqueness of weak solution to modified SH equation and the existence of optimal solution to an optimal distributed control problem of modified SH equation with a specific linear quadratic cost functional.

Taking $N(u) = \gamma u - \beta u^3$ in equation (1.1), Stanton and Golovin [19] investigate the following SH equation

$$u_t = -(1+\Delta)^2 u - au + \gamma u^2 - \beta u^3,$$

which is used to model the nonlinear dynamics of RB and Marangoni convection, and many other systems, and has been extensively studied. Feedback control of systems described by a supercritical SH equation in one dimension is first considered in [9]. It was shown that applying localized feedback at a few spatial locations can stabilize both uniform and pattern states. In [19], they investigate the possibility of applying a global feedback control to a pattern-forming system whose dynamics is described by a SH equation in 2D by means of stability analysis and numerical simulations.

The Pontryagin maximum principle unifies calculus of variations and control theory of ordinary differential equations ([6]) and establishes the theoretical basis of the modern optimal control theory along with the Bellman dynamic programming principle. In this paper, we commit ourselves to infinite dimensional generalizations of the maximum principle and aim at the optimal control theory of partial differential equations, a subject of much theoretical and practical interest [13]. In contrast to the finite dimensional setting, the maximum principle for the infinite dimensional system does not generally hold as a necessary condition for optimal control. Paying particular attention to the time optimal and norm optimal problems, Fattorini [5] has found some optimal controls which either do not satisfy the Pontryagin maximum principle or satisfy it in a certain weak form, which justified this argument.

This paper is concerned with the optimal distributed control problem of the modified SH equation. By the Dubovitskii and Milyutin functional analytical approach, the cone of directions of decrease, the cone of feasible directions and the cone of tangent directions as well as their dual cones are, respectively, derived. Then the Pontryagin maximum principle of the optimal distributed control problem is proven. The necessary optimality condition is established for the problem in fixed final time horizon case. In the end, a remark on how to use obtained results is also made as an illustration.

It is true that the feedback control of dynamical systems has many merits comparing to the open-loop control [21]. However, an undeniable fact is that the latter, the open-loop control, has its own advantages in the investigation of the infinite dimensional systems, such as the efficiency and accuracy of the open-loop control algorithms as well as the robustness aspect of investigational systems [17]. Just as Ho and Pepyne [10] said in "The No Free Lunch Theorem of Optimization (NFLT)", a general-purpose universal optimization strategy is impossible. Therefore the open-loop control investigation to the modified SH equation is both necessary and interesting.

The rest of this article is organized as follows. In section 2, we will present the main results of this paper. An optimal distributed control problem is formulated and the weak solution of the controlled state system is recalled. The Pontryagin maximum principle of the optimal control problem is established in fixed final time horizon case. The proof of the main results is given in section 3. In Section 4, we address the numerical solution and make a remark on how to use obtained necessary optimality condition. Section 5 concludes the paper with remarks.

2. Main results

Let $T > 0$ and Ω be an open connected bounded domain in $\mathbb{R}$. We investigate the partial differential equation

$$\begin{aligned} &u_t(x,t) + ku_{xxxx}(x,t) + 2u_{xx}(x,t) + au(x,t) + b|u_x(x,t)|^2 + u^3(x,t) \\ &= \tilde{f}(x,t), \quad x \in \Omega, t \in (0,T), \end{aligned} \tag{2.1}$$

which is the 1-D version of modified SH equation (1.2). As mentioned above, instead of considering the full RB problem or general reaction-diffusion systems, [3] studies the modified SH equation (1.2) as a phenomenological model for pattern-forming systems near the onset of instability (see a discussion in [14] about the validity of the SH model for the RB problem). In (2.1), k is an arbitrary constant. We supplement the equation with the initial data

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.2}$$

and the boundary condition

$$u(x,t) = u_{xx}(x,t) = 0, \quad x \in \partial\Omega. \tag{2.3}$$

In this article, we take $\Omega = (0,1)$, $V = H_0^2(0,1)$, $U = H_0^1(0,1)$ and $H = L^2(0,1)$. Take the Hilbert space

$$W(0,T;V) = \{\varphi \in L^2(0,T;V) : \varphi_t \in L^2(0,T;V^*)\}$$

equipped with the norm

$$\|\varphi\|_{W(0,T;V)} = \Big(\|\varphi\|^2_{L^2(0,T;V)} + \|\varphi_t\|^2_{L^2(0,T;V^*)}\Big)^{1/2}$$

where $V^* = H^{-2}(0,1)$ is the dual space of V. It is supposed that V is dense in U and U is dense in H such that, by identifying V^*, U^* and H^*, we have

$$V \hookrightarrow U \hookrightarrow H = H^* \hookrightarrow U^* \hookrightarrow V^*,$$

each embedding being dense, in which $H^* = L^2(0,1)$ being the dual space of H, $U^* = H^{-1}(0,1)$ the dual space of U [2].

Furthermore, we introduce a definition of weak solution to the modified SH equation (2.1) with (2.2), (2.3). A function $u(x,t) \in W(0,T;V)$ is a weak solution to (2.1), (2.2), (2.3), if

$$\begin{aligned}
&\langle \frac{\partial}{\partial t} u(\cdot,t), \varpi(\cdot)\rangle_H + k\langle u_{xx}(\cdot,t), \varpi_{xx}(\cdot)\rangle_H - 2\langle u_x(\cdot,t), \varpi_x(\cdot)\rangle_H \\
&+ a\langle u(\cdot,t), \varpi(\cdot)\rangle_H + b\langle |u_x(\cdot,t)|^2, \varpi(\cdot)\rangle_H + \langle u^3(\cdot,t), \varpi(\cdot)\rangle_H \\
&= \langle \tilde{f}(\cdot,t), \varpi(\cdot)\rangle_H
\end{aligned} \tag{2.4}$$

for all $\varpi(\cdot) \in V$, $t \in [0,T]$ a.e. and $u(\cdot,0) = u_0(\cdot) \in V$. Here and thereafter, $\langle \cdot,\cdot\rangle_{\mathcal{X}}$ is the inner product of Hilbert space $\mathcal{X}$.

Under the definition of such weak solution, suppose that k is sufficiently large, $u_0 \in V$ and $\tilde{f} \in L^2(0,T;H)$, then the equations (2.1), (2.2), (2.3) admit a unique weak solution $u(x,t) \in W(0,T;V)$, which is the result has been proven in [4] by Galerkin method. In this paper, unless otherwise stated, in what follows when we speak of a solution of (2.1), (2.2), (2.3), we shall always mean the weak solution in the sense of (2.4).

Now we consider the optimal distributed control of investigated system in fixed final time horizon case. For $T > 0$, take $\tilde{f}(x,t) = f(x,t) + \alpha(t)$, in which $f(x,t) \in L^2(0,T;H)$. And $\alpha(t) \in L^2(0,T)$ plays the role of control. Let U_{ad} be a non-empty closed convex set of $L^2(0,T)$ and then take it as the admissible control set. Consider an optimal control problem for the system (2.1), (2.2), (2.3) with the general cost functional

$$\min_{\alpha(\cdot)\in U_{ad}} J(u,\alpha) = \min_{\alpha(\cdot)\in U_{ad}} \int_0^T \int_0^1 L(u(x,t),\alpha(t),x,t)\,dx\,dt. \tag{2.5}$$

Here, the cost function J is quite general in the sense that it contains most practically concerned ones like quadratic cost functional of the following form

$$J(u,\alpha) = \int_0^T \int_0^1 \rho_1 \left|u(x,t) - u^\dagger(x,t)\right|^2 dx\,dt + \int_0^T \rho_2 \alpha^2(t)\,dt, \tag{2.6}$$

where $\rho_i > 0$, $i = 1, 2$ are constants, and given $u^\dagger$ is the desired optimal state. The last quadratic term reflects the cost of control. The control target is to drive the state variable u to match the given desired state $u^\dagger$ by adjusting the control function α with minimal energy and work, which is exactly the main object of interest in [4].

The objective of this paper is to study the optimal distributed control problem (2.5) in fixed final time horizon case, for the modified SH equation (2.1), (2.2), (2.3). Take $u(\cdot, t) \in W(0, T; V)$. The control space is $L^2(0, T)$ and the control function satisfies a convex constraint $\alpha(\cdot) \in U_{ad}$. Here, we assume that the set U_{ad} of admissible controls has the non-empty interior with respect to $L^2(0, T)$ topology, i.e., $\mathrm{int}_{L^2(0,T)} U_{ad} \neq \emptyset$.

The following two assumptions for the cost functional in (2.5) are assumed:

(A1) L is a functional defined on $V \times U_{ad} \times [0, 1] \times [0, T]$ and
$$\frac{\partial L(u(x,t), \alpha(t), x, t)}{\partial u}, \quad \frac{\partial L(u(x,t), \alpha(t), x, t)}{\partial \alpha}$$
exist for every $(u, \alpha) \in V \times U_{ad}$ and L is continuous in its variables.

(A2
$$\int_0^1 \Big| \frac{\partial L(u(x,t), \alpha(t), x, t)}{\partial u} \Big| \, dx, \quad \int_0^1 \Big| \frac{\partial L(u(x,t), \alpha(t), x, t)}{\partial \alpha} \Big| \, dx$$
are bounded for $t \in [0, T]$.

Define $X_T = W(0, T; V) \times L^2(0, T)$. Let (u^*, α^*) be the solution to optimal control problem (2.5) subject to the equation (2.1), (2.2), (2.3). Set
$$\Omega_1 = \{(u, \alpha) \in X_T : \alpha(t) \in U_{ad},\ t \in [0, T] \text{ a.e.}\},$$
$$\begin{aligned}\Omega_2 = \Big\{&(u, \alpha) \in X_T : u_t(x,t) + k u_{xxxx}(x,t) + 2u_{xx}(x,t) + a u(x,t) \\ &+ b|u_x(x,t)|^2 + u^3(x,t) = f(x,t) + \alpha(t), \\ &u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ &u(x,0) = u_0(x), \quad u(x,T) = u^*(x,T)\Big\}.\end{aligned}$$
Then problem (2.5) is equivalent to questing for $(u^*, \alpha^*) \in \Omega = \Omega_1 \cap \Omega_2$ such that
$$J(u^*, \alpha^*) = \min_{(u,\alpha) \in \Omega} J(u, \alpha). \tag{2.7}$$

Thus far, we have seen that problem (2.7) is an extremum problem on the constraint Ω_1 and the equality constraint Ω_2. In this situation, the Dubovitskii and Milyutin functional analytical approach has been turned out to be very powerful to solve such kind of extremum problems (see e.g. [1, 8, 20]). The general Dubovitskii and Milyutin theorem for the extremum problem (2.7) can be stated as follows.

Theorem 2.1 (Dubovitskii-Milyutin). *Suppose the functional $J(u, \alpha)$ assumes a minimum at the point (u^*, α^*) in Ω. Assume that $J(u, \alpha)$ is regularly decreasing at (u^*, α^*) with the cone of directions of decrease K_0 and the constraint Ω_1 is regular at (u^*, α^*) with the cone of feasible directions K_1; and that the equality constraint Ω_2 is also regular at (u^*, α^*) with the cone of tangent directions K_2. Then there exist continuous linear functionals f_0, f_1, f_2, not all identically zero, such that $f_i \in K_i^*$, the dual cone of K_i, $i = 0, 1, 2$, which satisfy the condition*
$$f_0 + f_1 + f_2 = 0. \tag{2.8}$$

In this article, using Theorem 2.1, we establish the necessary optimality condition of optimal control problem (2.5) for the modified SH equation (2.1), (2.2), (2.3). The main results of this paper are formulated as Theorem 2.2 below.

Theorem 2.2. *Suppose (u^*, α^*) is a solution to the optimal control problem (2.5). Then there exist $\kappa_0 \geq 0$ and $v(x,t)$, not identically zero, such that the following maximum principle holds:*

$$\begin{gathered} \Big\{ \int_0^1 \Big[\kappa_0 \frac{\partial L(u^*, \alpha^*, x, t)}{\partial \alpha} - v(x,t)\Big] dx \Big\} [\alpha(t) - \alpha^*(t)] \geq 0, \\ \forall \alpha(t) \in U_{ad},\ t \in [0,T] \ a.e., \end{gathered} \tag{2.9}$$

where the function $v(x,t)$ satisfies the adjoint equation

$$\begin{gathered} v_t(x,t) - k v_{xxxx}(x,t) - 2v_{xx}(x,t) - a v(x,t) + 2b\,[u^*_{xx}(x,t) v(x,t) \\ + u^*_x(x,t) v_x(x,t)] - 3(u^*(x,t))^2 v(x,t) \\ = \kappa_0 \frac{\partial L(u^*, \alpha^*, x, t)}{\partial u}, \\ v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, \quad v(x,T) = \psi(x). \end{gathered} \tag{2.10}$$

3. Proof of Theorem 2.2

To prove Theorem 2.2, in the direction of Theorem 2.1, we need to determine the cone of directions of decrease K_0, the cone of feasible directions K_1, the cone of tangent directions K_2 and their respective dual cones K_i, $i = 0,1,2$. Moreover, in these dual cones, to derive the continuous linear functionals f_i, $i = 0,1,2$. Then, step by step, by the equation (2.8), to derive the final result, which is exactly the Pontryagin maximum principle.

Firstly, find the cone of directions of decrease K_0. By assumption, $J(u,\alpha)$ is differentiable at any point (u^0, α^0) in any direction (u, α) and its directional derivative is

$$\begin{aligned} &J'(u^0, \alpha^0; u, \alpha) \\ &= \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} [J(u^0 + \varepsilon u, \alpha^0 + \varepsilon \alpha) - J(u^0, \alpha^0)] \\ &= \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \Big\{ \int_0^T \int_0^1 [L(u^0 + \varepsilon u, \alpha^0 + \varepsilon \alpha, x, t) - L(u^0, \alpha^0, x, t)]\, dx\, dt \Big\} \\ &= \int_0^T \int_0^1 \Big[\frac{\partial L(u^0, \alpha^0, x, t)}{\partial u} u + \frac{\partial L(u^0, \alpha^0, x, t)}{\partial \alpha} \alpha \Big] dx\, dt. \end{aligned}$$

The cone of directions of decrease of the functional $J(u,\alpha)$ at point (u^*, α^*) is thereupon determined by

$$\begin{aligned} K_0 &= \big\{ (u, \alpha) \in X_T : J'(u^*, \alpha^*; u, \alpha) < 0 \big\} \\ &= \Big\{ (u, \alpha) \in X_T : \int_0^T \int_0^1 \Big[\frac{\partial L(u^*, \alpha^*, x, t)}{\partial u} u + \frac{\partial L(u^*, \alpha^*, x, t)}{\partial \alpha} \alpha \Big] dx\, dt < 0 \Big\}. \end{aligned}$$

If $K_0 \neq \emptyset$, then for any $f_0 \in K_0^*$, there exists a $\kappa_0 \geq 0$ such that

$$f_0(u, \alpha) = -\kappa_0 \int_0^T \int_0^1 \Big[\frac{\partial L(u^*, \alpha^*, x, t)}{\partial u} u + \frac{\partial L(u^*, \alpha^*, x, t)}{\partial \alpha} \alpha \Big] dx\, dt.$$

Secondly, for the cone of feasible directions K_1, since $\Omega_1 = W(0,T;V) \times U_{ad}$, in which $\mathrm{int}_{L^2(0,T)} U_{ad} \neq \emptyset$ by assumption, so the interior of Ω_1 is not empty, i.e. , $\mathring{\Omega}_1 \neq \emptyset$. And at point (u^*, α^*), the cone of feasible directions K_1 of Ω_1 is determined by

$$\begin{aligned} K_1 &= \big\{\kappa\big(\mathring{\Omega}_1 - (u^*,\alpha^*)\big) : \kappa > 0\big\} \\ &= \big\{h : h = \kappa(u-u^*, \alpha-\alpha^*),\ (u,\alpha) \in \mathring{\Omega}_1,\ \kappa > 0\big\}. \end{aligned}$$

As a result, for an arbitrary $f_1 \in K_1^*$, if there is an $\bar{a}(t) \in L^2(0,T)$, such that the linear functional defined by

$$f_1(u,\alpha) = \int_0^T \bar{a}(t)\alpha(t)\,dt \tag{3.1}$$

is a support to $\tilde{\Omega}_1$ at point α^*, then

$$\bar{a}(t)[\alpha(t) - \alpha^*(t)] \geq 0, \quad \forall\, \alpha(t) \in U_{ad},\ t \in [0,T] \text{ a.e.} \tag{3.2}$$

We proceed in the next step to derive the cone of tangent directions K_2. Define the operator $G : X_T \to L^2(0,T;H) \times (L^2(0,T))^4 \times (V)^2$ by

$$\begin{aligned} G(u,\alpha) = \Big(&\vartheta(x,t), u(0,t), u(1,t), u_{xx}(0,t), u_{xx}(1,t), u(x,0) - \phi(x), \\ &u(x,T) - u^*(x,T)\Big), \end{aligned}$$

in which $\vartheta(x,t) = u_t(x,t) + ku_{xxxx}(x,t) + 2u_{xx}(x,t) + au(x,t) + b|u_x(x,t)|^2 + u^3(x,t) - f(x,t) - \alpha(t)$. Then

$$\Omega_2 = \{(u,\alpha) \in X_T : G(u(x,t),\alpha(t)) = 0\}.$$

The Fréchet-derivative of the operator $G(u,\alpha)$ is

$$G'(u,\alpha)(\hat{u},\hat{\alpha}) = \Big(\hat{\vartheta}(x,t), \hat{u}(0,t), \hat{u}(1,t), \hat{u}_{xx}(0,t), \hat{u}_{xx}(1,t), \hat{u}(x,0), \hat{u}(x,T)\Big)$$

in which we define

$$\begin{aligned} \hat{\vartheta}(x,t) = {}& \hat{u}_t(x,t) + k\hat{u}_{xxxx}(x,t) + 2\hat{u}_{xx}(x,t) + a\hat{u}(x,t) + 2bu_x(x,t)\hat{u}_x(x,t) \\ &+ 3u^2(x,t)\hat{u}(x,t) - \hat{\alpha}(t). \end{aligned}$$

Since (u^*, α^*) is the solution to the problem (2.5), it follows that $G(u^*,\alpha^*) = 0$. Choosing arbitrary

$$(g_1, g_2, g_3, g_4, g_5, g_6, g_7) \in L^2(0,T;H) \times (L^2(0,T))^4 \times (V)^2$$

and solving the equation

$$G'(u^*,\alpha^*)(\hat{u},\hat{\alpha}) = (g_1(x,t), g_2(t), g_3(t), g_4(t), g_5(t), g_6(x), g_7(x)),$$

we obtain

$$\begin{gathered} \hat{u}_t(x,t) + k\hat{u}_{xxxx}(x,t) + 2\hat{u}_{xx}(x,t) + a\hat{u}(x,t) \\ + 2bu_x^*(x,t)\hat{u}_x(x,t) + 3(u^*(x,t))^2\hat{u}(x,t) - \hat{\alpha}(t) = g_1(x,t), \\ \hat{u}(0,t) = g_2(t),\ \hat{u}(1,t) = g_3(t), \quad \hat{u}_{xx}(0,t) = g_4(t), \\ \hat{u}_{xx}(1,t) = g_5(t), \quad \hat{u}(x,0) = g_6(x),\ \hat{u}(x,T) = g_7(x). \end{gathered} \tag{3.3}$$

Next, we assume that the linearized system

$$
\begin{aligned}
&u_t(x,t)+ku_{xxxx}(x,t)+2u_{xx}(x,t)+au(x,t)\\
&+2bu_x^*(x,t)u_x(x,t)+3(u^*(x,t))^2u(x,t)=\alpha(t),\\
&u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0,\quad u(x,0)=0,
\end{aligned}
\tag{3.4}
$$

is controllable. Then choose $\alpha(t)=\hat\alpha(t)\in L^2(0,T)$ such that $u(x,T)=g_7(x)-\eta(x,T)$ and let u be the solution to the linearized system (3.4). Choose $\hat u(x,t)=u(x,t)+\eta(x,t)$, where $\eta(x,t)$ satisfies the following equations

$$
\begin{aligned}
&\eta_t(x,t)+k\eta_{xxxx}(x,t)+2\eta_{xx}(x,t)+a\eta(x,t)\\
&+2bu_x^*(x,t)\eta_x(x,t)+3(u^*(x,t))^2\eta(x,t)=g_1(x,t),\\
&\eta(0,t)=g_2(t),\quad \eta(1,t)=g_3(t),\quad \eta_{xx}(0,t)=g_4(t),\\
&\eta_{xx}(1,t)=g_5(t),\quad \eta(x,0)=g_6(x).
\end{aligned}
$$

In this way, it suffices for $(\hat u,\hat\alpha)$ satisfying (3.3). Therefore $G'(u^*,\alpha^*)$ maps the space X_T onto $L^2(0,T;H)\times(L^2(0,T))^4\times(V)^2$. Moreover, the cone of the tangent directions K_2 to the constraint Ω_2 at point (u^*,α^*) consists of the kernel of $G'(u^*,\alpha^*)$, i.e., (u,α) satisfies the following equations in X_T,

$$
\begin{aligned}
&u_t(x,t)+ku_{xxxx}(x,t)+2u_{xx}(x,t)+au(x,t)\\
&+2bu_x^*(x,t)u_x(x,t)+3(u^*(x,t))^2u(x,t)=\alpha(t),\\
&u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0,\quad u(x,0)=0,
\end{aligned}
\tag{3.5}
$$

and

$$
u(x,T)=0. \tag{3.6}
$$

Let

$$
\begin{aligned}
K_{21}&=\{(u,\alpha)\in X_T:(u(x,t),\alpha(t))\text{ satisfies (3.5)}\},\\
K_{22}&=\{(u,\alpha)\in X_T:(u(x,t),\alpha(t))\text{ satisfies (3.6)}\}.
\end{aligned}
$$

Then the cone of tangent directions $K_2=K_{21}\cap K_{22}$. Consequently,

$$
K_2^*=K_{21}^*+K_{22}^*.
$$

For any $f_2\in K_2^*$, decompose $f_2=f_{21}+f_{22}$, $f_{2i}\in K_{2i}^*$, the dual cone of K_{2i}, $i=1,2$. Then $f_{21}(u,\alpha)=0$ and for all $u(x,t)\in W(0,T;V)$ satisfying $u(x,T)=0$, there exists a $\psi(x)\in V^*$ such that

$$
f_{22}(u,\alpha)=\int_0^1 u(x,T)\psi(x)\,dx.
$$

It then follows from Theorem 2.1 that there exist continuous linear functionals, not all identically zero, such that

$$
f_0+f_1+f_{21}+f_{22}=0.
$$

Therefore, when selecting (u,α) satisfies (3.5), $f_{21}(u,\alpha)=0$. Moreover,

$$
\begin{aligned}
f_1(u,\alpha)&=-f_0(u,\alpha)-f_{22}(u,\alpha)\\
&=\kappa_0\int_0^T\int_0^1\Big[\frac{\partial L(u^*,\alpha^*,x,t)}{\partial u}u(x,t)+\frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha}\alpha(t)\Big]\,dx\,dt\\
&\quad-\int_0^1 u(x,T)\psi(x)\,dx.
\end{aligned}
$$

Now it is only one step away from obtaining the necessary optimality condition and establishing the Pontryagin maximum principle for problem (2.5). For this purpose, we need to formulate the adjoint system of (3.4). Here, define the adjoint system as (2.10). As with (2.1), (2.2), (2.3), the existence and uniqueness of solution to the adjoint system can be obtained similarly.

Theorem 3.1. *The solution of system* (3.4) *and that of its adjoint system* (2.10) *have the following relationship*

$$\begin{aligned}&\int_0^1 u(x,T)\psi(x)\,dx - \kappa_0\int_0^T\int_0^1 \frac{\partial L(u^*,\alpha^*,x,t)}{\partial u}u(x,t)\,dx\,dt\\&= \int_0^T\int_0^1 \alpha(t)v(x,t)\,dx\,dt.\end{aligned}$$

Proof. Multiply equation (2.10) by $v(x,t)$ and integrate the product by parts over $[0,T]\times[0,1]$ with respect to t and x respectively. The proof then follows. □

Next we give the proof of the main results in this paper.

Proof of Theorem 2.2. By Theorem 3.1, we can rewrite $f_1(u,\alpha)$ as

$$f_1(u,\alpha) = \int_0^T\Big\{\int_0^1\Big[\kappa_0\frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha} - v(x,t)\Big]dx\Big\}\alpha(t)\,dt.$$

In view of (3.1),

$$\bar{a}(t) = \int_0^1\Big[\kappa_0\frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha} - v(x,t)\Big]\,dx\,.$$

Then (3.2) reads

$$\begin{gathered}\Big\{\int_0^1\Big[\kappa_0\frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha} - v(x,t)\Big]dx\Big\}[\alpha(t)-\alpha^*(t)]\ge 0,\\ \alpha(t)\in U_{ad},\quad t\in[0,T]\text{ a.e.,}\end{gathered}\tag{3.7}$$

where κ_0 and $v(x,t)$ are not identical to zero simultaneously. Since otherwise, there are definitely $f_0=0, f_1=0, f_{22}=0$ and $f_{21}=0$, which contradict with the fact in Theorem 2.1 that these continuous linear functionals are not all identically zero.

On the other hand, if K_0 is a null set, then

$$\int_0^T\int_0^1\Big[\frac{\partial L(u^*,\alpha^*,x,t)}{\partial u}u(x,t) + \frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha}\alpha(t)\Big]\,dx\,dt = 0,$$

for all $(u,\alpha)\in X_T$. In particular, if we choose $\kappa_0=1$ and $\psi(x)=0$, then from Theorem 3.1 it follows that

$$\int_0^T\int_0^1\frac{\partial L(u^*,\alpha^*,x,t)}{\partial u}u(x,t)\,dx\,dt = -\int_0^T\int_0^1 v(x,t)\alpha(t)\,dx\,dt$$

so

$$\int_0^T\Big\{\int_0^1\Big[\frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha} - v(x,t)\Big]\,dx\Big\}\alpha(t)\,dt = 0,\quad \forall\alpha(t)\in L^2(0,T),$$

from which we obtain

$$\int_0^1\Big[\frac{\partial L(u^*,\alpha^*,x,t)}{\partial\alpha} - v(x,t)\Big]\,dx = 0,\quad \forall t\in[0,T]\text{ a.e.}$$

Therefore (3.7) still holds.

In addition, if there is a nonzero solution $\hat{v}(x,t)$ to the adjoint system

$$\begin{aligned}
&\hat{v}_t(x,t) - k\hat{v}_{xxxx}(x,t) - 2\hat{v}_{xx}(x,t) - a\hat{v}(x,t) + 2b[u^*_{xx}(x,t)\hat{v}(x,t) \\
&+ u^*_x(x,t)\hat{v}_x(x,t)] - 3(u^*(x,t))^2\hat{v}(x,t) \\
&= \kappa_0 \frac{\partial L(u^*,\alpha^*,x,t)}{\partial u}, \\
&\ \hat{v}(0,t) = \hat{v}(1,t) = \hat{v}_{xx}(0,t) = \hat{v}_{xx}(1,t) = 0, \quad \hat{v}(x,T) = \psi(x)
\end{aligned} \tag{3.8}$$

such that the following equality holds

$$\int_0^1 \hat{v}(x,t)\,dx = 0, \quad \forall t \in [0,T] \text{ a.e.},$$

then when we choose $\kappa_0 = 0$ and $\psi(x) = \hat{v}(x,T)$, (3.7) is still valid. Since otherwise, if for any nonzero solution $\hat{v}$ of (3.8), it has

$$\int_0^1 \hat{v}(x,t)\,dx \not\equiv 0,$$

in this case we say the situation is non-degenerate. Then the linearized system (3.4) is controllable. In fact, if (3.4) is not controllable, then there exist a $\psi(x) \in V^*$ such that

$$\int_0^1 u(x,T)\psi(x)\,dx = 0, \quad \psi(x) \not\equiv 0.$$

Choose $\kappa_0 = 0$, $\hat{v}$ to be the solution of (3.8). Then it follows from Theorem 3.1 that

$$\int_0^T \Big[\int_0^1 \hat{v}(x,t)\,dx\Big]\alpha(t)\,dt = 0, \quad \forall \alpha(t) \in L^2(0,T).$$

Thus

$$\int_0^1 \hat{v}(x,t)\,dx = 0, \quad \forall t \in [0,T] \text{ a.e.}$$

This is a contradiction. Under the case of (3.8), the system (3.4) is consequently controllable.

Combining the results above, we have obtained the Pontryagin maximum principle (2.9) for the problem (2.5) subject to the system (2.1), (2.2), (2.3). This completes the proof of main results. □

4. An iterative algorithm

In this section, we show how to use the results obtained above for the numerical solutions to the investigational optimal control problem. That is to say, we will give, by the Pontryagin maximum principle along with an iterative algorithm, the profile for numerically solving the optimal distributed control problem of the modified SH equation in fixed final time horizon case, i.e., the problem (2.5).

Essentially speaking, by the necessary optimality condition of optimal control, such as the Pontryagin maximum principle, a two-point boundary-value problem solution is an effective numerical method for solving optimal control problems. Through necessary conditions for numerically solving optimal control problems, there are two approaches available for now. It is commonly believed that the indirect method, which is mainly the multiple shooting method, is the most powerful numerical method. By the Pontryagin maximum principle, one can construct a two-point boundary-value problem. The optimal control of the lumped parameter systems can be obtained by solving this two-point boundary-value problem. Of course, except for the complexity when the original problem involves inequality constraints of both state variables and controls, the difficulty for shooting method additionally includes the "guess" for the initial data to start the iterative numerical process. It demands that the user understands the essential of the problem well in physics. Unfortunately, in all likelihood it is no easy job. The gradient method is developed to overcome this difficulty; and then the "min-H" approach, which corrected from the gradient method ([7, 24]), comes with the higher convergence rate. In the following, we show how to utilize the min-H iterative method to solve the extremum problem.

To this end, rewrite the Pontryagin maximum principle (2.9) as follows:

$$\alpha^*(t)H_\alpha(u^*,\alpha^*) = \max_{\alpha(\cdot)\in U_{ad}} \alpha(t)H_\alpha(u^*,\alpha^*), \tag{4.1}$$

where

$$H(u,\alpha) = \int_0^1 [\alpha(t)v(x,t) - \kappa_0 L(u,\alpha,x,t)]\,dt.$$

Therefore, the so-called "min-H" iterative algorithm is formulated below.

First, give $\alpha^0(t)$ and solve the state equation (2.1), (2.2), (2.3) to get $u^0(x,t)$.

(I) By $\alpha^0(t)$, $u^0(x,t)$, solve the adjoint equation (2.10) to get $v^0(x,t)$.

(II) In view of $u^0(x,t)$, $v^0(x,t)$ and the Pontryagin maximum principle (4.1), to determine $\alpha^1(t)$.

(III) Give $\alpha^1(t)$ and solve the state equation (2.1), (2.2), (2.3) to get $u^1(x,t)$.

(IV) Calculate $J(u^1,\alpha^1)$. If it does not reach the minimum, replace (u^0,α^0) with (u^1,α^1) and redo the steps above until we get the proper $J(u^1,\alpha^1)$.

Subsequently, we can proceed the numerical computation using the algorithm above after setting some parameters such as k, a, b, $u_0(x)$, T, $f(x,t)$, $L(u,\alpha,x,t)$ and so on. Furthermore, for the convenience, the quadratic cost functional (2.6) is a good choice. Noting that it is an optimal control problem of distributed parameter system governed by nonlinear partial differential equations, to get the numerical solutions for the optimal control-trajectory pair is not an easy job. Here, although we do not give the detailed numerical simulation, the algorithm does give the concrete steps so that people can follow and finish this nontrivial work.

5. Conclusions

For the infinite dimensional system, the maximum principle does not generally hold as a necessary condition for optimal control. Thus, in the optimal control theory of partial differential equation, an important and interesting problem is the infinite dimensional generalization of the maximum principle. The SH equation is a partial differential equation for a scalar field which has been widely used as a model for the study of various issues in pattern formation. Optimal control problems for the SH equation are largely unexplored and need more attention. This paper investigates an optimal distributed control problem of modified SH equation and in the fixed final time horizon case, establishes the necessary optimality condition, the Pontryagin maximum principle. Furthermore, to show the application of obtained results, a remark on how to use the obtained results is made and numerically solving the investigated problem is briefly discussed. We remark that an important goal of this paper is to provide a framework for using functional analysis and control techniques to analyze and optimize the distributed parameter systems. This result may be applied to other much complex nonlinear partial differential equations.

As a direct continuation of the present paper, the future work can include the investigations of optimal distributed control problem of the modified SH equation in free final time horizon case under weaker additional conditions and derives further new results of current interests. Moreover, in the free final time horizon case, we can cancel the assumptions imposed on the preceding fixed final time horizon problem. Namely, the admissible control set neither needs be convex nor contains interior points as well as the cost functional needs not be differentiable with respect to the control variable. Therefore, the admissible control set can be any set. An interesting case is that it is allowed to contain only finite many points. We can also consider the optimal boundary control problem of the SH equation in these two cases. People can refer to [8, 11, 20] for the general information.

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