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# GENERIC REACTION-DIFFUSION MODEL WITH APPLICATION TO IMAGE RESTORATION AND ENHANCEMENT

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Abstract. This article provides the existence of a global solution to a generic reaction-diffusion system. The main result is a generalization of the work presented by [2, 5, 11] in the case of a reaction-diffusion equation. We show the existence of a global weak solution to the considered system in the case of quasi-positivity and a triangular structure condition on the nonlinearities [12]. An example of application of our result is demonstrated on a novel bio-inspired image restoration model [1].

## 1. Introduction

Nowadays, reaction-diffusion models play an important role in information processing. Non-linear reaction-diffusion models can describe many natural phenomena in a wide range of disciplines. Over the last few years, some amazing results were observed in engineering applications such as image processing. Among these applications, we cite Fitzhugh-Nagumo [6] model which allowed the detection of noisy image contours. We also cite the anisotropic diffusion described by Perona and Malik which includes local information to reduce noise and enhance contrast while preserving the edge. From where the idea of Catté et al. [5] to integrate directly the regularization into the equation by convolving the image with the Gaussian filter on the gradient of the noisy image to smooth the image first in order to avoid the dependence of the numerical scheme between the solution and the regularization procedure, this makes the problem well posed and the existence and uniqueness of the problem was proven by Catté et al. [5]. Other generalization of this work were made by Whitaker and Pizer, Li and Chen [8] and Weickert and Benhamouda [15]. In 2006, Morfu [10] proposed a model performing noise filtering and contrast enhancement where he combined the nonlinear diffusion process ruled by Fischer equation that was originally used to describe the spreading process of biological population without establishing any existence or consistency result. Until the work of Alaa et al. [2] combining the regularization procedure in Catté with Morfu model,

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the authors were able to demonstrate the existence and consistency of the their proposed model. We build up on their works by providing a generalization to the case of reaction-diffusion systems. In this paper, we tackle the global existence problem for a general reaction-diffusion system written in the form

$$\begin{gathered}
\partial_t u - \operatorname{div}(A(|\nabla u_\sigma|)\nabla u) = f(t,x,u,v) \quad \text{in } Q_T, \\
\partial_t v - d_v \Delta v = g(t,x,u,v) \quad \text{in } Q_T, \\
\partial_\nu u = 0 \quad \partial_\nu v = 0 \quad \text{on } \Sigma_T, \\
u(0,\cdot) = u_0(\cdot) \quad v(0,\cdot) = v_0(\cdot) \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$ and $T \in (0,\infty[$, $Q_T = ]0,T[\times\Omega$ and $\Sigma_T = ]0,T[\times\partial\Omega$ where $\partial\Omega$ denotes the boundary of $\Omega$. $\nu$ is the outward normal to the domain and $\partial_\nu$ is the normal derivative.

Let $\sigma > 0$, $\nabla u_\sigma$ be a regularization by convolution of $\nabla u$. It is defined as $\nabla u_\sigma = \nabla(G_\sigma * u)$ where $G_\sigma$ is the gaussian function. The anisotropic diffusitivity $A$ is a smooth non-increasing function such that $A(0) = 1$ and $\lim_{s\to\infty} A(s) = 0$.

The nonlinear functions $f, g : Q_T \times \mathbb{R}^2 \to \mathbb{R}$ are measurable and $f(t,x,.), g(t,x,.) : \mathbb{R}^2 \to \mathbb{R}$ are continuous. In addition the nonlinearities satisfy the positivity property

$$f(t,x,0,s) \geq 0 \quad \forall s \geq 0 \quad \text{and} \quad g(t,x,r,0) \geq 0 \quad \forall r \geq 0 \tag{1.2}$$

and a triangular structure

$$(f+g)(t,x,r,s) \leq L_1(r+s+1) \quad \text{and} \quad g(t,x,r,s) \leq L_2(r+s+1) \tag{1.3}$$

where $L_1$ and $L_2$ are positive constant. Furthermore,

$$\sup_{|r|+|s|\leq R} (|f(t,x,r,s)| + |g(t,x,r,s)|) \in L^1(Q_T) \tag{1.4}$$

for $R > 0$. However there is no further assumption on their growth. The initial conditions $u_0, v_0$ are only assumed to be square integrable.

Before tackling the main problem, we clearly state our definition of weak solution to the reaction-diffusion system.

**Definition 1.1.** We call $(u,v)$ a weak solution of the system (1.1) if

- $u, v \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$, $u(0,\cdot) = u_0$ and $v(0,\cdot) = v_0$
- $\forall \phi, \psi \in \mathcal{C}^1(Q_T)$ such that $\phi(\cdot,T) = 0$ and $\psi(\cdot,T) = 0$ we have

$$\begin{gathered}
\int_{Q_T} -u\partial_t\phi + A(|\nabla u_\sigma|)\nabla u \nabla\phi = \int_{Q_T} f(t,x,u,v)\phi + \int_\Omega u_0\phi(\cdot,0) \\
\int_{Q_T} -v\partial_t\psi + d_v\nabla v\nabla\psi = \int_{Q_T} g(t,x,u,v)\psi + \int_\Omega v_0\psi(\cdot,0),
\end{gathered} \tag{1.5}$$

where $f(t,x,u,v), g(t,x,u,v) \in L^1(Q_T)$.

Now, we enunciate the main result of the paper.

**Theorem 1.2.** *Under the assumptions (1.2-1.4) and for a continuous function $A$ as described above. The reaction-diffusion system* (1.1) *admits a global weak solution* $(u,v)$ *in the sense defined in* (1.5) *for all $u_0, v_0 \in L^2(\Omega)$ such that $u_0, v_0$ are positive.*

To prove our main result, we will proceed by steps. We truncate the problem and show that the approximate problem admits weak solutions using a Schauder fixed point. Afterward, we will provide some essential compactness and equi-integrablity results in order to pass to the limit and rigourously demonstrate the existence of global weak solution to the considered model. The layout of the paper is then as follows. First, the next section deals with an intermediate result where nonlinearities are bounded. In section two, we analyse the truncated problem, prove necessary estimations and show the convergence toward a global weak solution. Section three is a straightforward application of our result in a novel modified Fitz-Hugh-Nagumo model for image restoration. Lastly, a summary and conclusion are presented.

## 2. Existence result for truncated nonlinearities

In this presentation, we will first show the existence result for bounded source terms $f, g$. Then we will tackle in the next section the case of unbounded nonlinearities. For readability purposes, we denote by $\mathcal{V} = H^1(\Omega)$ and $\mathcal{H} = L^2(\Omega)$.

**Theorem 2.1.** *Under the above assumptions on the nonlinearities, if there exist $M_f, M_g \geq 0$, such that for almost every $(t,x) \in Q_T$,*

$$|f(t,x,r,s)| \leq M_f, \quad |g(t,x,r,s)| \leq M_g, \quad \forall (r,s) \in \mathbb{R}^2, \tag{2.1}$$

*then for every $u_0, v_0 \in L^2(\Omega)$, there exists a weak solution $(u,v)$ to the considered system* (1.1)*. Moreover there exists $C(M_f, M_g, \sigma, T, \|u_0\|_{L^2(\Omega)}, \|v_0\|_{L^2(\Omega)})$ such that*

$$\|(u,v)\|_{L^\infty(0,T;\mathcal{H})^2} + \|(u,v)\|_{L^2(0,T;\mathcal{V})^2} \leq C \tag{2.2}$$

*Furthermore if $u_0, v_0$ are positive and $f, g$ are quasi-positive then $u(t,x) \geq 0$ and $v(t,x) \geq 0$ for a.e. $(t,x) \in Q_T$*

**Remark 2.2.** Note that a proof of positivity relies on the quasipostivity of nonlinearities. This proof was presented in [2] in the case of reaction-diffusion equation. For the sake of simplicity we omit its proof here since it can be easily extended to the case of this system, we refer interested readers to the previously mentioned paper.

*Proof.* We will show the existence of a weak solution by the classical Schauder fixed point theorem. We introduce the space

$$\mathcal{W}(0,T) = \{u, v \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) : \partial_t u, \partial_t v \in L^2(0,T;\mathcal{V}')\} \tag{2.3}$$

Let $w = (w_1, w_2) \in \mathcal{W}(0,T)$ and let $(u,v)$ be the solution of a linearization of problem (1.1) given by

$$\begin{gathered}
(u,v) \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}(0,T;\mathcal{H}) \\
\forall \phi, \psi \in \mathcal{C}^1(Q_T) \text{ such that } \psi(\cdot,T) = 0 \text{ and } \phi(\cdot,T) = 0 \\
\int_{Q_T} -u\partial_t\phi + A(|\nabla(w_1)_\sigma|)\nabla u \nabla\phi = \int_{Q_T} f(t,x,w_1,w_2)\phi + \int_\Omega u_0\phi(\cdot,0) \\
\int_{Q_T} -v\partial_t\psi + d_v\nabla v\nabla\psi = \int_{Q_T} g(t,x,w_1,w_2)\psi + \int_\Omega v_0\psi(\cdot,0)
\end{gathered} \tag{2.4}$$

The application $w \in \mathcal{W}(0,T) \to (u,v) \in \mathcal{W}(0,T)$ is clearly well defined. In fact, $w_1$ is in $L^\infty(0,T;\mathcal{H})$, $G_\sigma$ is $\mathcal{C}^\infty(Q_T)$ therefore $A(|\nabla(w_1)_\sigma|)$ is $\mathcal{C}^\infty(Q_T)$ and since $A$ is non-increasing it satisfies

$$a \leq A(|\nabla w_\sigma|) \leq d \tag{2.5}$$

where $d > 0$ and $a$ is a positive constant that depends only on $\sigma$ and $A$. This last property coupled with the fact that nonlinearities are bounded implies that the differential operators in (2.4) are continous and coercive thus by application of the standard theory of Partial Differential Equations see [9, 3, 4] we obtain $(u, v)$ the solution of the linearized problem (2.4).

Now we establish some important estimates to construct the functional setting where Schauder fixed point theory is applicable. The following result holds for $0 \le t \le T$,

$$\begin{gathered}\frac{1}{2}\int_\Omega u^2(t) + \int_{Q_T} A(|\nabla(w_1)_\sigma|)\ |\nabla u|^2 = \frac{1}{2}\int_\Omega {u_0}^2 + \int_{Q_T} u\ f(t,x,w_1,w_2) \\ \frac{1}{2}\int_\Omega v^2(t) + d_v \int_{Q_T} |\nabla v|^2 = \frac{1}{2}\int_\Omega {v_0}^2 + \int_{Q_T} v\ g(t,x,w_1,w_2)\end{gathered} \tag{2.6}$$

Consequently,

$$\begin{gathered}\int_\Omega u^2(t) \le M_f + \int_{Q_T} u^2 + \int_\Omega {u_0}^2 \\ \int_\Omega v^2(t) \le M_g + \int_{Q_T} v^2 + \int_\Omega {v_0}^2\end{gathered} \tag{2.7}$$

Using Gronwall's inequality we obtain

$$\begin{gathered}\int_{Q_T} u^2 \le (\exp(T)-1)\Big(M_f + \int_\Omega {u_0}^2\Big) \\ \int_{Q_T} v^2 \le (\exp(T)-1)\Big(M_g + \int_\Omega {v_0}^2\Big)\end{gathered} \tag{2.8}$$

Substituting the expression above in (2.6), we obtain the desired result,

$$\begin{gathered}\sup_{0\le t\le T}\int_\Omega u^2(t) \le M_f + \big(\exp(T)-1\big)\Big(M_f + \int_\Omega {u_0}^2\Big) + \int_\Omega {u_0}^2 := C_u \\ \sup_{0\le t\le T}\int_\Omega v^2(t) \le M_g + (\exp(T)-1)\Big(M_g + \int_\Omega {v_0}^2\Big) + \int_\Omega {v_0}^2 := C_v\end{gathered} \tag{2.9}$$

Therefore by setting $C_1 = \max(C_u, C_v)$ we get

$$\|(u,v)\|_{L^\infty(0,T;\mathcal{H})^2} \le C_1 \tag{2.10}$$

Using (2.6) and (2.5) we deduce

$$\begin{gathered}\int_{Q_T} u^2 + |\nabla u|^2 \le \frac{M_f + \int_{Q_T} u^2 + \int_\Omega {u_0}^2}{\min(\frac{1}{2}, a)} \le C'_u \\ \int_{Q_T} v^2 + |\nabla v|^2 \le \frac{M_g + \int_{Q_T} v^2 + \int_\Omega {v_0}^2}{\min(\frac{1}{2}, d_v)} \le C'_v\end{gathered} \tag{2.11}$$

Setting $C_2 = \max(C'_u, C'_v)$, we conclude that

$$\|(u,v)\|_{L^2(0,T;\mathcal{V})^2} \le C_2 \tag{2.12}$$

Next we estimate the $\partial_t u$ and $\partial_t v$ in $L^2(0,T;\mathcal{V}')$. We know that

$$\begin{gathered}\partial_t u = \operatorname{div}(A(|\nabla u_\sigma|)\nabla u) + f(t,x,u,v) \\ \partial_t v = d_v\ \Delta v + g(t,x,u,v)\end{gathered} \tag{2.13}$$

It follows that
$$\begin{aligned} \|\partial_t u\|_{L^2(0,T;\mathcal{V}')} &\leq C\ \|\nabla u\|_{L^2(Q_T)} + M_f T \\ \|\partial_t v\|_{L^2(0,T;\mathcal{V}')} &\leq d_v \|\nabla v\|_{L^2(Q_T)} + M_g T \end{aligned} \tag{2.14}$$
Thereafter,
$$\begin{aligned} \|\partial_t u\|_{L^2(0,T;\mathcal{V}')} &\leq C\ C_1 + M_f T \\ \|\partial_t v\|_{L^2(0,T;\mathcal{V}')} &\leq d_v C_1 + M_g T \end{aligned} \tag{2.15}$$
Eventually,
$$\|(\partial_t u, \partial_t v)\|_{L^2(0,T;\mathcal{V}')^2} \leq \max(C\ C_1 + M_f\ T, d_v\ C_1 + M_g\ T) := C_3 \tag{2.16}$$
Now we are in a position to apply Schauder fixed point in the functional space
$$\begin{aligned} \mathcal{W}_0(0,T) = \big\{ & u, v \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) : \|(u,v)\|_{L^\infty(0,T;\mathcal{H})^2} \leq C_1, \\ & \|(u,v)\|_{L^2(0,T;\mathcal{V})^2} \leq C_2 \|(\partial_t u, \partial_t v)\|_{L^2(0,T;\mathcal{V}')^2} \leq C_3, \\ & u(\cdot,0) = u_0,\ v(\cdot,0) = v_0 \big\} \end{aligned} \tag{2.17}$$
We can easily verify that $\mathcal{W}_0(0,T)$ is a nonempty closed convex in $\mathcal{W}(0,T)$. To use Schauder's theorem we will show that the application
$$F : w \in \mathcal{W}_0(0,T) \to F(w) = (u,v) \in \mathcal{W}_0(0,T)$$
is weakly continuous.

Let us consider a sequence $w_n \in \mathcal{W}_0(0,T)$ such that $w_n$ converges weakly in $\mathcal{W}_0(0,T)$ toward $w$, and let $F(w_n) = (u_n, v_n)$. Thus,
$$\begin{aligned} \partial_t u_n &= \operatorname{div}(A(|\nabla w_{1n\sigma}|)\nabla u_n) + f(t,x,u_n,v_n) \\ \partial_t v_n &= d_v \Delta v_n + g(t,x,u_n,v_n) \end{aligned} \tag{2.18}$$
Based on the previous estimations, $(u_n, v_n)$ is bounded in $(L^2(0,T;\mathcal{V}))^2$ and $(\partial_t u_n, \partial_t v_n)$ is bounded in $(L^2(0,T;\mathcal{V}'))^2$ then by Aubin-Simon compactness [14] $(u_n, v_n)$ is relatively compact on $(L^2(Q_T))^2$; which means we can extract a subsequence denoted $w_n = (u_n, v_n)$ such that

- $u_n \rightharpoonup u$ in $L^2(0,T;\mathcal{V})$,
- $v_n \rightharpoonup v$ in $L^2(0,T;\mathcal{V})$,
- $f(t,x,w_n) \longrightarrow f(t,x,w)$ in $L^2(Q_T)$,
- $g(t,x,w_n) \longrightarrow g(t,x,w)$ in $L^2(Q_T)$,
- $u_n \longrightarrow u$ in $L^2(0,T;\mathcal{H})$ and a.e in $Q_T$,
- $v_n \longrightarrow v$ in $L^2(0,T;\mathcal{H})$ and a.e in $Q_T$,
- $\nabla u_n \rightharpoonup \nabla u$ in $L^2(0,T;\mathcal{H})$,
- $\nabla v_n \rightharpoonup \nabla v$ in $L^2(0,T;\mathcal{H})$,
- $w_n \longrightarrow w$ in $L^2(0,T;\mathcal{H})$ and a.e in $Q_T$,
- $A(|\nabla w_{1n\sigma}|) \longrightarrow A(|\nabla w_{1\sigma}|)$ in $L^2(0,T;\mathcal{V})$,
- $\partial_t u_n \rightharpoonup \partial_t u$ in $L^2(0,T;\mathcal{V}')$,
- $\partial_t v_n \rightharpoonup \partial_t v$ in $L^2(0,T;\mathcal{V}')$,

Using these convergences, we can pass to the limit in (2.19) and show that the limit $u$ and $v$ are solutions of the problem
$$\begin{aligned} \partial_t u &= \operatorname{div}(A(|\nabla w_{1\sigma}|)\nabla u) + f(t,x,w_1,w_2) \\ \partial_t v &= d_v\ \Delta v + g(t,x,w_1,w_2) \end{aligned} \tag{2.19}$$
Thus $F(w) = (u,v)$ then $F$ is weakly continuous which proves the desired results. $\square$

## 3. Existence result for unbounded nonlinearities

In this case, we truncate $f$ and $g$ using truncation function $\Psi_n \in \mathcal{C}_c^\infty(\mathbb{R})$, such that $0 \le \Psi_n \le 1$ and

$$\Psi_n(r) = \begin{cases} 1 & \text{if } |r| \le n \\ 0 & \text{if } |r| \ge n+1 \end{cases} \tag{3.1}$$

Thus, we can state that the approximate problem

$$\begin{gathered} \partial_t u_n = \operatorname{div}(A(|\nabla u_{n\sigma}|)\nabla u_n) + f_n(t,x,u_n,v_n) \\ \partial_t v_n = d_v\ \Delta v_n + g_n(t,x,u_n,v_n) \end{gathered} \tag{3.2}$$

where $f_n(t,x,u_n,v_n) = \Psi_n(|u_n|+|u_n|)\ f(t,x,u_n,v_n)$ and $g_n(t,x,u_n,v_n) = \Psi_n(|u_n|+|u_n|)\ g(t,x,u_n,v_n)$ admits a weak solution by means of theorem 2.1. In what follows, $C$ will often be reused to represent a constant independent of $n$. Now we show that up to a subsequence, $(u_n,v_n)$ converges to the weak solution $(u,v)$ of problem (1.1). For this we need to prove the following results.

**Lemma 3.1.** *Under the assumptions of the main result and for $(u_n,v_n)$ a weak solution of the truncated problem, there exists $C>0$ such that*

$$\|u_n+v_n\|_{L^2(Q_T)} \le C(1+\|v_n\|_{L^2(Q_T)}) \tag{3.3}$$

*Proof.* This estimate relies on the duality method see [12]. Let $\theta \in \mathcal{C}_c^\infty(Q_T)$ be such that $\theta \ge 0$ and let $\phi$ be a solution of

$$\begin{gathered} -\partial_t\phi - \operatorname{div}(A(|\nabla u_{n\sigma}|)\ u_n\nabla\phi) = \theta, \\ \partial_n\phi = 0, \\ \phi(T,\cdot) = 0 \end{gathered} \tag{3.4}$$

We know that there exists $C>0$ such that $\|\phi\|_{H^2(Q_T)} \le C\|\theta\|_{L^2(Q_T)}$ see [7, 13]. We set $W = \exp(-L_1 t)(u_n+v_n)$, by the mass control the following inequality holds,

$$\int_{Q_T} \partial_t W\phi + \int_{Q_T} \exp(-L_1\ t)(\operatorname{div}(A(|\nabla u_{n\sigma}|)\ u_n) + d_v\Delta v_n)\phi \le \int_{Q_T} L_1\exp(-L_1\ t)\phi$$

Integrating by parts and using (3.4) we get

$$\begin{aligned} \int_{Q_T} W\theta &\le \int_{Q_T} \exp(-L_1\ t)(d_v\Delta\phi - A(|\nabla u_{n\sigma}|)\Delta\phi - \nabla A(|\nabla u_{n\sigma}|)\ \nabla\phi)v_n \\ &\quad + \int_{Q_T} L_1\exp(-L_1\ t)\phi + \int_\Omega (u_0+v_0)\phi(0,\cdot), \end{aligned}$$

where $A(|\nabla u_{n\sigma}|)$ and $\nabla A(|\nabla u_{n\sigma}|)$ are bounded independently of $n$ in $L^\infty(Q_T)$; hence

$$\begin{aligned} \int_{Q_T} W\theta &\le C[1+\|u_0+v_0\|_{L^2(\Omega)} + \|v_n\|_{L^2(Q_T)}]\|\phi\|_{H^2(Q_T)} \\ &\le C(1+\|v_n\|_{L^2(Q_T)})\|\theta\|_{L^2(Q_T)} \end{aligned}$$

which by duality completes the proof. □

**Lemma 3.2.** *Let $(u_n,v_n)$ be the solution of the approximate problem* (3.2). *Then*

(1) *There exists a constant $M$ depending only on $\int_\Omega u_0, \int_\Omega v_0, L_1, T$ and $|\Omega|$ such that*

$$\int_{Q_T} (u_n+v_n) \le M \quad \forall t \in [0,T] \tag{3.5}$$

(2) *There exists $C_1 > 0$ such that*

$$\int_{Q_T} |\nabla u_n|^2 + |\nabla v_n|^2 \le C_1 \tag{3.6}$$

(3) *There exists $C_2 > 0$ such that*

$$\int_{Q_T} |f_n| + |g_n| \le C_2 \tag{3.7}$$

*Proof.* (1) The triangular structure of problem (1.1) implies that

$$(u_n + v_n)_t - \operatorname{div}(A(|\nabla u_{n\sigma}|)\nabla u_n) - d_v\ \Delta v_n \le L_1(u_n + v_n + 1) \tag{3.8}$$

integrating over $Q_t$, $0 < t \le T$ leads to

$$\int_\Omega (u_n + v_n)(t) \le \int_\Omega (u_0 + v_0) + L_1 \int_{Q_t} (u_n + v_n + 1) \tag{3.9}$$

using a standard Gronwall argument we get

$$\int_{Q_T} (u_n + v_n)(t) \le \Big[ \int_\Omega (u_0 + v_0) + L_1 |Q_T| \Big] \exp(L_1\ T) \tag{3.10}$$

and therefore the desired result is proven.

(2) We have $\partial_t v_n - d_v \Delta v_n = g_n \le L_2(1 + u_n + v_n)$,

$$\frac{1}{2} \int_{Q_T} (v_n^2)_t + d_v \int_{Q_T} |\nabla v_n|^2 \le L_2 \int_{Q_T} (1 + u_n + v_n)\ v_n \tag{3.11}$$

using Young's inequality and Lemma 3.1 we get

$$\begin{aligned} \frac{1}{2} \int_\Omega v_n^2 + d_v \int_{Q_T} |\nabla v_n|^2 &\le \frac{1}{2} \int_\Omega (v_0^2) + L_2 (C \int_{Q_T} v_n^2 + \int_{Q_T} (u_n + v_n)^2) \\ &\le \frac{1}{2} \int_\Omega (v_0^2) + C \int_{Q_T} v_n^2 \end{aligned}$$

and by Gronwall's lemma we deduce that

$$\int_{Q_T} v_n^2 \le C \tag{3.12}$$

which in return assures that $\int_{Q_T} |\nabla v_n|^2$ and $\int_{Q_T} u_n^2$ are bounded. Now let us show that $\int_{Q_T} |\nabla u_n|^2$ is bounded. We have $u_n + v_n$ satisfies

$$\partial_t (u_n + v_n) - \operatorname{div}(A(|\nabla (u_n)_\sigma|)) - d_v \Delta v_n = f_n + g_n \le L_1(1 + u_n + v_n) \tag{3.13}$$

Letting $w = \exp(-L_1\ t)(u_n + v_n)$,

$$\int_{Q_T} \partial_t w\ w + I + \int_{Q_T} \exp(-L_1\ t) d_v \nabla v_n\ \nabla(u_n + v_n) \le \int_{Q_T} \exp(-L_1\ t) L_1 w, \tag{3.14}$$

where

$$\begin{aligned} I &= \int_{Q_T} \exp(-L_1\ t) A(|\nabla (u_n)_\sigma|) \nabla u_n \nabla(u_n + v_n) \\ &= \int_{Q_T} \exp(-L_1\ t) A(|\nabla (u_n)_\sigma|) |\nabla(u_n + v_n)|^2 \\ &\quad - \int_{Q_T} \exp(-L_1\ t) A(|\nabla (u_n)_\sigma|) \nabla v_n \nabla(u_n + v_n) \end{aligned}$$

Since $A(|\nabla(u_n)_\sigma|) \geq a$, we have

$$I \geq a \int_{Q_T} |\nabla(u_n + v_n)|^2 - \int_{Q_T} \exp(-L_1\ t) A(|\nabla(u_n)_\sigma|) \nabla v_n \nabla(u_n + v_n) \tag{3.15}$$

Substituting in (3.14)

$$\begin{aligned} &\frac{1}{2} \int_\Omega w^2(T) + a \int_{Q_T} |\nabla(u_n + v_n)|^2 \\ &\leq C + \int_{Q_T} \exp(-L_1\ t)(d_v - A(|\nabla(u_n)_\sigma|)) \nabla v_n \nabla(u_n + v_n) \end{aligned}$$

Young's inequality on $|\nabla v_n\ \nabla(u_n + v_n)|$ implies

$$a \int_{Q_T} |\nabla(u_n + v_n)|^2 \leq C\Big(1 + C(\varepsilon) \int_{Q_T} |\nabla v_n|^2 + \varepsilon \int_{Q_T} |\nabla(u_n + v_n)|^2\Big)$$

Hence by choosing a suitable $\varepsilon$ we deduce that $\int_{Q_T} |\nabla(u_n + v_n)|^2$ is bounded and because $\int_{Q_T} |\nabla(v_n)|^2$ is bounded, $\int_{Q_T} |\nabla(u_n)|^2$ is bounded as well.

(3) For $v_n$ solution of

$$\partial_t v_n - d_v\ \Delta v_n = g_n \leq L_2(1 + u_n + v_n) \tag{3.16}$$

we can write

$$\partial_t v_n - d\Delta v_n + L_2(1 + u_n + v_n) - g_n = L_2(1 + u_n + v_n), \tag{3.17}$$

which implies

$$\int_{Q_T} \partial_t v_n + \int_{Q_T} (L_2(1 + u_n + v_n) - g_n) \leq \int_{Q_T} L_2(1 + u_n + v_n), \tag{3.18}$$

then

$$\int_\omega v_n(T) - \int_\omega v_n(0) + \int_{Q_T} (L_2(1 + u_n + v_n) - g_n) \leq \int_{Q_T} L_2(1 + u_n + v_n), \tag{3.19}$$

we know that $\int_{Q_T} L_2(1 + u_n + v_n)$ is bounded, which follows that

$$\|L_2(1 + u_n + v_n) - g_n\|_{L^1(Q_T)} \leq C, \tag{3.20}$$

therefore

$$\|g_n\|_{L^1(Q_T)} \leq C_g. \tag{3.21}$$

Since $L_1(1 + u_n + v_n) - f_n - g_n \geq 0$, we obtain the same for $f_n + g_n$, hence

$$\|f_n\|_{L^1(Q_T)} \leq C_f. \tag{3.22}$$

□

Now we deduce the result of the main theorem 1.2. According to lemma 3.2, $(u_n, v_n)$ is bounded in $(L^2(0, T, \mathcal{V}))^2$ and $(\partial_t u_n, \partial_t v_n)$ is bounded in $(L^2(0, T, \mathcal{V}') + L^1(Q_T))^2$. Therefore by Aubin-Simon, $(u_n, v_n)$ is relatively compact in $(L^2(Q_T))^2$, then we can extract a subsequence $(u_n, v_n)$ in $(L^2(Q_T))^2$ such that

- $u_n \rightharpoonup u$ in $L^2(Q_T)$ and a.e in $Q_T$,
- $v_n \rightharpoonup v$ in $L^2(Q_T)$ and a.e in $Q_T$,
- $\nabla G_\sigma * u_n \rightharpoonup \nabla G_\sigma * u$ in $L^2(Q_T)$ and a.e in $Q_T$,
- $A(|\nabla u_{1n\sigma}|) \longrightarrow A(|\nabla u_{1\sigma}|)$ in $L^2(Q_T)$,
- $f_n(t, x, u_n, v_n) \longrightarrow f(t, x, u, v)$ for a.e in $Q_T$,
- $g_n(t, x, u_n, v_n) \longrightarrow g(t, x, u, v)$ for a.e in $Q_T$.

To prove that $(u,v)$ is a weak solution of system (1.1), almost everywhere convergence is not sufficient. We actually need to prove that $f_n(t,x,u_n,v_n)$ converges strongly toward $f(t,x,u,v)$ in $L^1(Q_T)$ and this convergence is given by the following Lemma.

**Lemma 3.3.** *Under the additional assumption that, for $R>0$,*

$$\sup_{|r|+|s|\le R}(|f(t,x,r,s)|+|g(t,x,r,s)|)\in L^1(Q_T) \tag{3.23}$$

(1) *There exists $C>0$ such that*

$$\int_{Q_T}(u_n+2\ v_n)(|f_n|+|g_n|)\le C \tag{3.24}$$

(2) *$f_n$ and $g_n$ converges strongly toward $f$ and $g$ in $L^1(Q_T)$.*

*Proof.* We will present a sketch of the proof.

(1) Let $R_n=L_1(u_n+v_n+1)-f_n-g_n\ge 0$ and $S_n=L_2(u_n+v_n+1)-g_n\ge 0$. We have

$$(2v_n+u_n)-B_n=f_n+2g_n, \tag{3.25}$$

where $B_n=2d\Delta v_n+div(A(|\nabla(u_n)_\sigma|)\nabla u_n)$. Then

$$(2v_n+u_n)-B=-R_n+L_1(u_n+v_n+1)-S_n+L_2(u_n+v_n+1). \tag{3.26}$$

Multiplying (3.26) by $2v_n+u_n$ and integrating over $Q_T$, we obtain

$$\int_{Q_T}(2v_n+u_n)(R_n+S_n)\le C. \tag{3.27}$$

Since $\int_{Q_T}(v_n+u_n)^2$ is bounded, we obtain the inequality

$$\int_{Q_T}(2v_n+u_n)(|f_n|+|g_n|)\le C. \tag{3.28}$$

(2) We know that $f_n,g_n$ converge almost everywhere toward $f,g$. We will show that $f_n$ and $g_n$ are equi-integrable in $L^1(Q_T)$. The proof will be given for $f_n$, however the same result holds for $g_n$. For this, we let $\varepsilon>0$ and prove that there exists $\delta>0$ such that $|E|<\delta$ implies that $\int_E f_n<\varepsilon$. We have

$$\begin{aligned}\int_E|f_n|&=\int_{E\cap[u_n+2v_n\le k]}|f_n|+\int_{E\cap[u_n+2v_n>k]}|f_n|\\&\le\frac{1}{k}\int_E(u_n+2\ v_n)|f_n|+|E|\sup_{|u_n|+|v_n|\le k}|f_n(t,x,u_n,v_n)|\end{aligned} \tag{3.29}$$

and since (3.24) ensures that $\int_E(u_n+v_n)|f_n|$ is bounded. We can choose $\delta$ small enough and a larger $k$ such that $\int_E|f_n|\le\varepsilon$. The same thing holds for $g_n$ as well. ☐

## 4. Applications

An interesting example of application is the Modified Fitz-Hugh-Nagumo Model for image restoration [1] where the source terms have the form

$$\begin{gathered}f(u,v)=\frac{1}{\tau}u\ (u-a)(1-u)+\mu v\\ g(u,v)=u-bv\end{gathered} \tag{4.1}$$

**Remark 4.1.** When $\mu \geq 0$, the nonlinearities satisfy quasipostivity, mass control, and the triangular structure and therefore by direct application of the main result we can deduce global existence. It is also worth noting that there is no restriction on the growth of $f, g$ Consequently other types of non-polynomial nonlinearities can be handled.

If $\mu < 0$, the expression above do not satisfy the quasipostivity. However we can use the fact that

$$\begin{gathered} uf(u,v) \leq L_1(1+u^2+v^2) \\ uf(u,v)+vg(u,v) \leq L_2(1+u^2+v^2)\,. \end{gathered} \tag{4.2}$$

Multiplying each equation by its respective unknown in the truncated problem and summing up we directly obtain the following estimations:

$$\begin{gathered} \sup_{t\in[0,T]} \int_\Omega u_n^2 + v_n^2 \leq C \\ \int_{Q_T} |\nabla u_n|^2 + |\nabla v_n|^2 \leq C \\ \int_{Q_T} |u_n|^4 \leq C \end{gathered} \tag{4.3}$$

where $C$ depends only on $T$, $|\Omega|$, and initial conditions on $L^2(\Omega)$, which are sufficient to pass to the limit and obtain the result of the main theorem. In both cases, the modified Fitz-Hugh-Nagumo model admits a weak solution for initial conditions $u_0, v_0$ in $L^2(\Omega)$.

To illustrate the performance of the studied model we present in this paragraph some numerical results. The modified Fitz-Hugh-Nagumo can be approximate by the explicit scheme bellow:

$$\begin{gathered} \frac{u_{i,j}^{n+1}-u_{i,j}^n}{dt} - \operatorname{div}(A^n \nabla u_{i,j}^n) = \frac{1}{\tau} u_{i,j}^n (u_{i,j}^n - a)(1-u_{i,j}^n) + \mu\ v_{i,j}^n, \\ \frac{v_{i,j}^{n+1}-v_{i,j}^n}{dt} - d_v \Delta v_{i,j}^n = u_{i,j}^n + b\ v_{i,j}^n, \\ A^n = A(|\nabla(G_\sigma * u_{i,j}^n)|, \lambda^n), \\ \lambda^n = 1.4826\,\mathrm{median}(|\nabla u^n| - \mathrm{median}(|\nabla u^n|))/\sqrt{2} \end{gathered}$$

where median represents the median of an image over all its pixels and $A$ is a function that lowers the diffusion rate $d_u$ over regions of high gradients. An example of such function is given by $A(s,\lambda) = d_u/\sqrt{1+(s/\lambda)^2}$. Simulations done on a standard noisy image using the parameters $a = 0.5$, $\tau = 10^{-3}$, $d_u = 150$, $d_v = 250$, $dt = 1e^{-2}$ are represented in Figure 1. To quantitatively measure the performance of the model we illustrate in Table 1 two indicators:

(1) The measure of enhancement(EME) measure the quality improvement of the image. It is defined by : Let an image $u(N,M)$ be split into $k_1k_2$ blocks $w_{k,l}$ of sizes $l_1l_2$ then we define

$$EME = \frac{1}{k_1k_2} \sum_{l=1}^{k_1} \sum_{k=1}^{k_2} 20 \log\Big(\frac{u^w_{max;k,l}}{u^w_{min;k,l}}\Big)$$

where $u^w_{max;k,l}$ and $u^w_{min;k,l}$ are respectively maximum and minimum values of the image $u(N,M)$ inside the block $w_{k,l}$.

(2) The peak signal-to-noise ratio (PSNR) evaluates the performance of noise filtering. It is obtained by

$$PSNR = 10 \ \log_{10}(\frac{255^2}{SNR}) \tag{4.4}$$

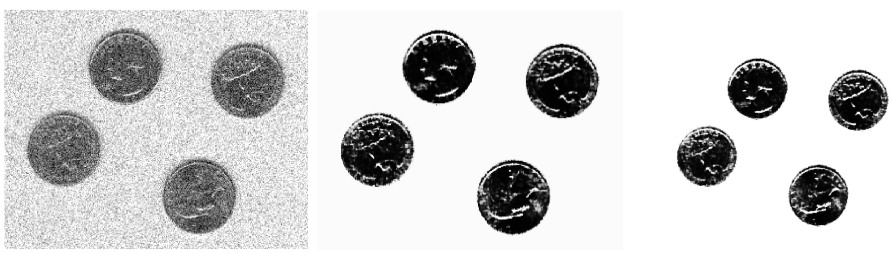
with

$$SNR = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}[u_{i,j} - u_{i,j}^{ref}]^2 \tag{4.5}$$

A higher value of EME and PSNR indicates that the image is well filtered and well enhanced.

Table 1. EME and PSNR values for the noisy image eight.tif for two different set of parameters

| Parameters | PSNR | EME |
|---|---|---|
| $b = 1$, $\mu = 1$ | 25.0153 | 12.7673 |
| $b = 1$, $\mu = -1$ | 25.0482 | 14.1897 |



(a) (b) (c)

Figure 1. Restoration of a noisy image using the modified Fitz-Hugh-Nagumo: (a) noisy image, (b) $b = -1$ and $\mu = -1$, (c) $b = 1$ and $\mu = 1$

**Conclusions.** As a summary, we demonstrated the existence of a global weak solution of the considered model. Also, we proved that the truncated problem admits a weak solution according to Schauder fixed point theorem. For unbounded nonlinearities satisfying suitable conditions, we established equi-integrablity and we derived a compactness results to be able to pass to the limit to get the desired result. To showcase the importance of the obtained result, a new application in the field of image restoration was given however its usefullness is not limited to this application and can be extended to resolve a range of problems in other fields.

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