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STEADY-STATE SOLUTIONS FOR SCHRÖDINGER EQUATIONS IN PHOTONIC LATTICE

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Abstract. In this article, we study a nonlinear Schrödinger equation arising in optics. Firstly we prove the existence of multiple solutions of this equation. Secondly, we consider a nonlinear Schrödinger system which is intimately related to the Schrödinger equation. We obtain the existence of nontrivial solutions to this system and we also get some results on its positive solutions. Finally, assuming Dirichlet or Neumann boundary condition, we show the existence and uniqueness of positive solution to the Schrödinger equation.

1. Introduction and statement of main results

1.1. **Schrödinger equation.** Schrödinger type equations have been studied extensively in the literature (see [2, 6, 10, 11, 13] for Schrödinger equation, [7, 12] for Schrödinger systems, i.e., two coupled equations, and comments at the end of sections 1.1 and 1.2).

In recent years, many exciting phenomena were found by careful experiments on light waves propagating in nonlinear periodic lattices. These phenomena are governed by the following Schrödinger equation (cf. [1, 25, 19] and the references therein)

$$i\frac{\partial\psi}{\partial z}+D\Delta\psi=g(x,|\psi|^2)\psi, \tag{1.1}$$

where $D>0$ means the beam diffraction coefficient and $\Delta\psi=\frac{\partial^2\psi}{\partial {x_1}^2}+\frac{\partial^2\psi}{\partial {x_2}^2}$. We take $D=1$ for convenience. Steady wave beam propagate along z-axis direction and transversely spread along $x=(x_1,x_2)\in\mathbb{R}^2$. The functions of concern are all periodic with respect to x, so equation (1.1) is viewed as defined over a periodic spatial domain Ω in $\mathbb{R}^2$. Steady state solution of equation (1.1) is a solution with the form $\psi(x,z)=e^{i\lambda z}u(x)$. Here $\lambda\in\mathbb{R}$ is a constant and $u(x)$ is a real-valued function. If we insert $\psi(x,z)$ into equation (1.1) and take $g(x,|\psi|^2)$ (cf. [3, 8, 9, 16, 25, 19]) as

$$g(x,|\psi|^2)=\frac{P}{1+V(x)+|\psi|^2},$$

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we obtain

$$\Delta u = \frac{Pu}{1+V+u^2} + \lambda u. \tag{1.2}$$

Here $V = V(x) \geq 0$ is potential function, which depends on the spatial variable x periodically (modulo Ω). In this paper, we set $V_0 = \max_{x\in\bar{\Omega}} V(x)$, $v_0 = \min_{x\in\bar{\Omega}} V(x)$.

A special case is when $V(x) \equiv 0$. In this case, equation (1.2) has the following simple form

$$\Delta u = \frac{Pu}{1+u^2} + \lambda u. \tag{1.3}$$

This equation will be heuristic when we study a system corresponding to equation (1.2), as we will see later. Yang and Zhang [25, 26] showed that

Theorem 1.1. (1) *For any P, equation* (1.2) *has a solution for some real value $\lambda = \lambda(P)$.*
(2) *If $P > 0$, $\lambda < 0$ and $|\lambda|$ is small enough, then equation* (1.2) *has a nontrivial solution.*
(3) *If $P < 0$, $0 < \lambda < |P|/(1+V_0)$, then equation* (1.2) *has a nontrivial solution.*
(4) *If $P < 0$, $|P|/(1+v_0) \leq \lambda$, then equation* (1.2) *has only the trivial solution.*

Schechter [19] studied equation (1.2) by the linking method. Before stating his results we need some notation. Let $\{\lambda_k; 0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots\}$ be eigenvalues of the operator $-\Delta$ on functions in $L^2(\Omega)$ having the same periods as Ω, then λ_k has finite multiplicity. The corresponding eigenfunctions belong to $L^\infty(\Omega)$. Schechter [19] proved the following result.

Theorem 1.2. (1) *If $P > 0$, $\lambda < 0$ and there is an $l \geq 0$ such that $\lambda_l + P/(1+v_0) \leq |\lambda| < \lambda_{l+1} + P/(1+V_0)$, $|\lambda| > \lambda_{l+1}$, then equation* (1.2) *has a nontrivial solution.*
(2) *If $P < 0$, $\lambda < 0$, and there is an $l \geq 0$ such that $\lambda_l + P/(1+V_0) < |\lambda| \leq \lambda_{l+1} + P/(1+v_0)$, $|\lambda| < \lambda_l$, then equation* (1.2) *has a nontrivial solution.*
(3) *If $P > 0$, $\lambda < 0$, and $0 < |\lambda| < P/(1+V_0)$, then equation* (1.2) *has a nontrivial solution.*
(4) *If $P < 0$, $\lambda > 0$, and $0 < \lambda < |P|/(1+V_0)$, then equation* (1.2) *has a nontrivial solution.*

Note that Schechter's results hold in arbitrary dimensions. In this paper, we consider the existence of multiple solutions of equation (1.2) and we obtain the following result.

Theorem 1.3. *If $P < 0$, $\lambda > 0$ and there is an integer $k > 0$ such that $\lambda + \lambda_k < \frac{|P|}{1+V_0}$, then equation* (1.2) *has at least k pairs of distinct solutions.*

The novelty of our result lies in establishing multiple solutions of equation (1.2) while in [25, 26, 19] the authors considered the existence of single solution under suitable conditions.

There are also lots of studies on equation (1.1) when the function g is defined in other forms. For instance, $g(x, |\psi|) = V(x) - \gamma|\psi|^{p-1}$ in [18, 2, 6, 13] (in fact, their setting was abstract and thus their results covered a wider range), $g(x, |\psi|) = l(l+1)/|x|^2$ in [11]. Other than steady-state solutions, blow-up solutions of Schrödinger equations have widely been investigated in the literature, too. In recent years, blow-up solutions of Schrödinger equation with defect have also attracted lots of attentions. We refer [10] for references in this direction.

1.2. **Schrödinger system: two coupled equations.** We also study the system

$$\begin{aligned} \Delta u &= \frac{Pu}{1+u^2+v^2} + \lambda u, \\ \Delta v &= \frac{Qv}{1+u^2+v^2} + \lambda v, \end{aligned} \tag{1.4}$$

which is intimately related to equations (1.2) and (1.3). In (1.4) P, Q and λ are parameters and the functions u, v are defined over a periodic bounded spatial domain $\Omega \subset \mathbb{R}^2$. System (1.4) is also a nonlinear photonic lattic model and has been studied by many researchers (cf. [4, 9, 15, 21, 23, 5, 24, 20] and the references therein). A solution of system (1.4) having the form (u, v) is called nontrivial if $u \not\equiv 0$ and $v \not\equiv 0$. If $u \equiv 0$ or $v \equiv 0$ but not both, it will be called a semi-trivial solution.

System (1.4) was studied in [4, 20, 14]. The authors proved that system (1.4) had semi-trivial solutions when P, Q, λ were suitably chosen. In particular, the authors of [20, 14] studied equation (1.3) instead of (1.4) after pointing out that the existence of nontrivial solutions of system (1.4) had not been proved.

In this article, we prove the existence of nontrivial solutions of system (1.4) in two special cases. We also study the positive solutions of system (1.4) in these two cases. Our results show that system (1.4) has nontrivial solutions and we hope they will be helpful for understanding the physics of the nonlinear Schrödinger system (1.4).

We give some comments on the recent work in [7, 12]. In [7] the author considered the system of coupled nonlinear Schrödinger-Korteweg-de Vries equations, where the parameter λ is different in each equation. We choose the same λ because our analysis on system (1.4) is based upon results of equations (1.2) and (1.3). In [12] the authors studied Schrödinger-Maxwell equations. They reduced this system to a single equation by "elimination method". More precisely, they represented one of the unknown functions by the other and then the system was equivalent to a single equation. Our system (1.4) is not solved by this method because we do not know how the unknown functions rely on each other.

1.3. **Schrödinger equation with zero Dirichlet or zero Neumann boundary condition.** Finally we show the existence and uniqueness of positive solution of equation (1.2) with Dirichlet boundary condition $u|_{\partial\Omega} = 0$ or Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$. Let $\varphi_1 \in W_0^{1,2}(\Omega)$ be an eigenfunction of $-\Delta$ (with zero Dirichlet boundary condition) corresponding to the first eigenvalue $\mu_1 = \mu_1(\Omega)$ with $\varphi_1 > 0$ in Ω and set $M_1 = \max_{\bar{\Omega}} \varphi_1$. Here we use μ_1 to distinguish from λ_1 which correspond to $-\Delta$ but without boundary condition. We obtain the following theorems.

Theorem 1.4. *If $\lambda \geq 0, P < 0$ and $\mu_1 + \lambda < \frac{|P|}{1+V_0+M_1}$, then equation* (1.2) *with Dirichlet boundary condition $u|_{\partial\Omega} = 0$ has a positive solution.*

Theorem 1.5. *If $P < 0$, then equation* (1.2) *with Dirichlet boundary condition $u|_{\partial\Omega} = 0$ has a unique positive solution.*

For Neumann boundary condition we obtain the uniqueness of positive solution for small $\lambda > 0$.

Theorem 1.6. *If $P < 0$, $\lambda > 0$ and $\lambda < \frac{|P|}{1+V_0}$, then equation* (1.2) *with Neumann boundary condition $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$ has a unique positive solution. When $V(x) \equiv 0$, under the above conditions, the unique positive solution of equation* (1.3) *is $u = \sqrt{-1 - \frac{P}{\lambda}}$.*

The rest of this article is organized as follows. In section 2, we prove Theorem 1.3 by verifying the conditions of Clark Theorem. In section 3, we investigate nontrivial solutions and positive solutions of system (1.4) in two special cases. Our discussions in this section are based on the existing results of equations (1.2) and (1.3). In section 4, we establish the existence and uniqueness of positive solution of equation (1.2) with zero Dirichlet (resp. Neumann) boundary condition, which proves Theorems 1.4-1.6.

2. Multiple solutions

In this section, we prove Theorem 1.3. We use the following result of Clark[17, p.53, Theorem 9.1].

Theorem 2.1. *Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$ with I even, bounded from below, and satisfying (PS) condition. Suppose $I(0) = 0$, there is a set $K \subset E$ such that K is homeomorphic to S^{j-1} by an odd map, and $\sup_K I < 0$. Then I possesses at least j distinct pairs of critical points.*

We will verify the conditions of Theorem 2.1. The corresponding energy functional of equation (1.2) is

$$I(u) = \frac{1}{2}\int_\Omega \Big\{|\nabla u|^2 + \lambda u^2 + P[\log(1+V+u^2) - \log(1+V)]\Big\}\,dx$$

for $u \in W^{1,2}(\Omega)$. It is easy to see that $I \in C^1(W^{1,2}(\Omega), \mathbb{R})$. I is even, i.e., $I(-u) = I(u)$ and $I(0) = 0$.

Lemma 2.2. *If $P < 0$ and $\lambda > 0$, then I is bounded from below.*

Proof. For any $\epsilon > 0$, we have

$$\log(1+V+u^2) - \log(1+V) \le |u| \le \epsilon|u|^2 + 1/(4\epsilon),$$

and then

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega \Big\{|\nabla u|^2 + \lambda u^2 + P[\log(1+V+u^2) - \log(1+V)]\Big\}\,dx \\ &\ge \frac{P}{8\epsilon}|\Omega| + \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + \lambda u^2 + P\epsilon u^2\Big)\,dx \qquad (2.1)\\ &= \frac{P}{8\epsilon}|\Omega| + \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 + \frac{\lambda u^2}{2}\Big)\,dx. \end{aligned}$$

To conclude Lemma 2.2, we take $\epsilon = -\lambda/(2P)$ in the last equality. □

Lemma 2.3. *If $P < 0$ and $\lambda > 0$, then I satisfies (PS) condition. In other words, if $\{u_n\}$ is a sequence in $W^{1,2}(\Omega)$ such that*

(i) *$I(u_n)$ is bounded,*
(ii) *$I'(u_n) \to 0$ as $n \to \infty$,*

then $\{u_n\}$ contains a subsequence which converges in $W^{1,2}(\Omega)$.

Proof. Suppose $\{u_n\} \subset W^{1,2}(\Omega)$ such that $|I(u_n)| \le M$ and $I'(u_n) \to 0$, where $M > 0$ is a fixed constant. We have
$$|\int_\Omega \Big\{|\nabla u_n|^2 + \lambda u_n^2 + P[\log(1+V+u_n^2) - \log(1+V)]\Big\}\,dx| \le 2M$$
and
$$|\langle I'(u_n), v\rangle| = |\int_\Omega \Big\{\nabla u_n \nabla v + \lambda u_n v + P\frac{u_n v}{1+V+u_n^2}\Big\}\,dx| \le \epsilon_n \|v\|_{W^{1,2}(\Omega)}, \quad (2.2)$$
where $\epsilon_n > 0$ approaches to 0 as $n \to \infty$. Note that by (2.1) it follows that
$$I(u_n) - \frac{P}{8\epsilon}|\Omega| \ge \frac{1}{2}\int_\Omega \Big\{|\nabla u_n|^2 + \frac{\lambda}{2}u_n^2\Big\}\,dx.$$
So $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$. Without loss of generality we set u_n convergence weakly to some u in $W^{1,2}(\Omega)$. By Rellich-Kondrachov Theorem, we know that $u_n \to u$ and $\frac{u_n}{1+V+u_n^2} \to \frac{u}{1+V+u^2}$ in $L^2(\Omega)$. Letting $n \to \infty$ in (2.2) we obtain
$$\int_\Omega \Big\{\nabla u \nabla v + \lambda uv + P\frac{uv}{1+V+u^2}\Big\}\,dx = 0. \quad (2.3)$$
Plugging (2.3) into (2.2) and taking $v = u_n - u$, we have
$$\Big|\int_\Omega \Big\{|\nabla(u_n-u)|^2 - P(u_n-u)\Big[\frac{u_n}{1+V+u_n^2} - \frac{u}{1+V+u^2}\Big] + \lambda(u_n-u)^2\Big\}\,dx\Big| \le \epsilon_n\|u_n-u\|_{W^{1,2}(\Omega)}.$$
Then
$$\begin{aligned}
&\int_\Omega |\nabla(u_n-u)|^2\,dx \\
&\le \epsilon_n\|u_n-u\|_{W^{1,2}(\Omega)} + |P|\int_\Omega |u_n-u|\Big|\frac{u_n}{1+V+u_n^2} - \frac{u}{1+V+u^2}\Big|\,dx \\
&+\lambda\int_\Omega |u_n-u|^2\,dx \to 0.
\end{aligned}$$
We obtain $u_n \to u$ in $W^{1,2}(\Omega)$. □

Lemma 2.4. *If $P < 0$, $\lambda > 0$ and $\lambda + \lambda_k < \frac{|P|}{1+V_0}$, then there is a set $K \subset W^{1,2}(\Omega)$ such that K is homeomorphic to S^{k-1} by an odd map and $\sup_K I < 0$.*

Proof. Indeed, let ϕ_k be one eigenfunction corresponding to λ_k. Without loss of generality, we can take $\|\phi_k\|_{W^{1,2}(\Omega)} = \|\nabla\phi_k\|_{L^2(\Omega)} + \|\phi_k\|_{L^2(\Omega)} = 1$. Then $(1+\lambda_k)\int \phi_k^2\,dx = 1$ since $-\Delta\phi_k + \phi_k = (\lambda_k+1)\phi_k$. Let
$$K = K(r) = \Big\{\sum_{i=1}^k \alpha_i\phi_i : \sum_{i=1}^k \alpha_i^2 = r^2\Big\}.$$
Clearly K is homeomorphic to S^{k-1} by an odd map for all $r > 0$.

We claim that $\sup_K I < 0$ for $r > 0$ small enough. In fact, we have
$$\log(1+V+u^2) - \log(1+V) \ge \frac{u^2}{1+V+u^2} \ge \frac{u^2}{1+V_0+u^2}$$
provided $0 \le V(x) \le V_0$. By Jensen's inequality
$$\varphi\Big(\frac{\int_\Omega f(x)p(x)dx}{\int_\Omega p(x)dx}\Big) \le \frac{\int_\Omega \varphi(f(x))p(x)dx}{\int_\Omega p(x)dx},$$

if we take $\varphi(t) = \frac{1}{1+V_0+t}$ and $f(x) = p(x) = u^2(x)$, we have

$$\int_\Omega \frac{u^2}{1+V_0+u^2}\,dx \geq \int_\Omega \frac{u^2}{1+V_0+\frac{\int_\Omega u^4\,dx}{\int_\Omega u^2\,dx}}\,dx.$$

Using Gagliardo-Nirenberg inequality, we obtain

$$\int_\Omega u^4\,dx \leq C\|u\|^2_{W^{1,2}(\Omega)}\int_\Omega u^2\,dx,$$

and then

$$\int_\Omega \frac{u^2}{1+V_0+u^2}\,dx \geq \frac{\int_\Omega u^2\,dx}{1+V_0+C\|u\|^2_{W^{1,2}(\Omega)}}.$$

For $u \in K$ we have $\|u\|^2_{W^{1,2}(\Omega)} = r^2$, so we have arrived at the energy upper bound

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega \Big\{|\nabla u|^2 + \lambda u^2 + P[\log(1+V+u^2) - \log(1+V)]\Big\}\,dx \\
&\leq \frac{1}{2}\int_\Omega \Big\{|\nabla u|^2 + \lambda u^2 + P\frac{u^2}{1+V_0+Cu^2}\Big\}\,dx \\
&\leq \frac{1}{2}\int_\Omega \Big\{|\nabla u|^2 + \lambda u^2 + P\frac{u^2}{1+V_0+Cr^2}\Big\}\,dx \\
&= \frac{1}{2}r^2 + \frac{1}{2}\Big(\lambda - 1 + \frac{P}{1+V_0+Cr^2}\Big)\int_\Omega u^2\,dx \\
&= \frac{1}{2}r^2 + \frac{1}{2}\Big(\lambda - 1 + \frac{P}{1+V_0+Cr^2}\Big)\sum_{i=1}^{k}\Big(\frac{\alpha_i^2}{\lambda_i+1}\Big).
\end{aligned}$$

But $\lambda - 1 + \frac{P}{1+V_0} < -1 - \lambda_k < 0$ provided $\lambda + \lambda_k < -\frac{P}{1+V_0}$. So there exists $r_1 > 0$, such that for all $r \in (0, r_1)$ we have $\lambda - 1 + \frac{P}{1+V_0+Cr^2} < -\lambda_k - 1$ and

$$\begin{aligned}
I(u) &\leq \frac{1}{2}r^2 + \frac{1}{2}\Big(\lambda - 1 + \frac{P}{1+V_0+Cr^2}\Big)\frac{r^2}{\lambda_k+1} \\
&= \frac{1}{2}r^2\Big[1 + \frac{1}{\lambda_k+1}(\lambda - 1 + \frac{P}{1+V_0+Cr^2})\Big].
\end{aligned}$$

Then there is $r_0 \in (0, r_1)$ such that $I(u) < 0$ for all $u \in K(r_0)$. □

Proof of Theorem 1.3. By Lemmas 2.2, 2.3, 2.4 and Theorem 2.1, the functional I has at least k pairs distinct critical points and then equation (1.2) has at least k pairs solutions. □

3. Nontrivial solutions and positive solution of the system

In this section, we focus on the existence of the nontrivial solutions of system (1.4). As pointed out by Schechter [20] (see also [14]), no one has showed the existence of nontrivial solutions of system (1.4). We consider two special cases in which system (1.4) has nontrivial solutions.

3.1. **Case $P = Q$.** System (1.4) becomes

$$\begin{aligned} \Delta u &= \frac{Pu}{1+u^2+v^2} + \lambda u, \\ \Delta v &= \frac{Pv}{1+u^2+v^2} + \lambda v. \end{aligned} \tag{3.1}$$

Observing this system is closely connected to equation (1.2), more precisely to equation (1.3), we have the following theorem.

Theorem 3.1. *If $\lambda > 0$, $P < 0$, and $\lambda < |P|$, then system* (3.1) *has infinitely many solutions.*

Proof. The following equation has a nontrivial solution, say w, by Theorem 1.1

$$\Delta w = \frac{Pw}{1+w^2} + \lambda w.$$

Then $(w\cos\theta, w\sin\theta)$, for all $\theta \in (-\pi, +\pi) \setminus \{0, \pm\frac{\pi}{2}\}$, will be nontrivial solutions of system (3.1). □

Remark 3.2. Combining our method and Theorems 1.1, 1.2 and results mentioned in section 1.2, one has similar results of Theorem 3.1.

Remark 3.3. If system (3.1) has positive solution (u, v) with $u > 0$ and $v > 0$, then (u, v) has the form (u, cu), i.e., v is proportional to u. In fact, $u > 0$ and $v > 0$ are solutions of linear equation

$$\Delta w + f(x)w = 0,$$

where $f(x) = |P|/[1 + u^2(x) + v^2(x)] - \lambda$. Then u and v are eigenfunctions corresponding to eigenvalue 0. So $v = cu$ for some constant $c > 0$ because 0 is the smallest eigenvalue which is simple. (If 0 is not smallest, then there is some eigenvalue, say $\lambda_0 < 0$. Denote φ_0 is a positive eigenfunction corresponding to λ_0. This will lead to a contradiction since different eigenfunction corresponding to different eigenvalue should be orthogonal in $L^2(\Omega)$. But u and φ_0 are positive, so they cannot be orthogonal in $L^2(\Omega)$.)

Remark 3.4. Under the assumption of Theorem 3.1, system (3.1) has infinitely positive solutions. In fact, our proof of Theorem 3.1 is based on Theorem 1.1 (3), which is proved by minimizing method. When w is a global minimizer so is $|w|$ because energy functional I satisfies $I(w) \geq I(|w|)$. Regularity theory and maximum principle ensure that $|w|$ is a positive solution. Thus $(w\cos\theta, w\sin\theta)$, for all $\theta \in (0, \pi/2)$, will be positive solutions of system (3.1).

3.2. **Case $Q = 0$.** System (1.4) will be "decoupled" into

$$\begin{aligned} \Delta u &= \frac{Pu}{1+u^2+v^2} + \lambda u, \\ \Delta v &= \lambda v. \end{aligned} \tag{3.2}$$

Theorem 3.5. (1) *If $-\lambda = \lambda_k > 0$, $P > 0$, and there is an integer $l \in (0, k)$ such that $\lambda_l + P/(1+\phi_k^0) \leq \lambda_k < \lambda_{l+1} + P/(1+\phi_k^1)$, then system* (3.2) *has a nontrivial solution.*

(2) *If $-\lambda = \lambda_k > 0$, $P < 0$, and there is an integer $l > k$ such that $\lambda_l + P/(1+\phi_k^1) < \lambda_k \leq \lambda_{l+1} + P/(1+\phi_k^0)$, then system* (3.2) *has a nontrivial solution.*

Where ϕ_k is the eigenfunction corresponding to λ_k and $\phi_k^0 = \min_{\bar\Omega}(\phi_k)^2$, $\phi_k^1 = \max_{\bar\Omega}(\phi_k)^2$.

Proof. Clearly $v = \phi_k$ is a nontrivial solution of the second equation of system (3.2). For this fixed function v, we conclude this theorem by Theorem 1.2. $\square$

Remark 3.6. If $P < 0$ and system (3.2) has positive solution (u, v) with $u > 0$ and $v > 0$, we find that λ must be 0 and then v is constant. We claim: all of the positive solutions of system (3.2) have the form (Cu_0^c, c), where C, c are positive constants and u_0^c is a positive solution of

$$\Delta u = \frac{(Pu)/1 + c^2}{1 + u^2} (=: \frac{P_c u}{1 + u^2}).$$

In fact, if (u, v) are positive solution of system (3.2), then $\lambda = 0$ and $v = c$ for some constant c. The first equation of system (3.2) becomes $\Delta u = \frac{Pu}{1+c^2+u^2}$, i.e., $\Delta w = \frac{P_c w}{1+w^2}$, where $w = u/\sqrt{1 + c^2}$. Using the same argument as in Remark 3.3 we obtain our claim.

We complete this section with a remark on more general cases.

Remark 3.7. If $\lambda > 0$, $P < 0$, $Q < 0$ and $\lambda \geq \min\{|P|, |Q|\}$, then equation (1.4) has no positive solution (u, v) with $u > 0$ and $v > 0$. This follows from Remark 4.6 of sec. 4 and $\lambda \geq \min\{|P|, |Q|\} > \min\{\frac{|P|}{1+v_0^2}, \frac{|Q|}{1+u_0^2}\}$, where $u_0 = \min_{x\in\Omega} u(x)$ and $v_0 = \min_{x\in\Omega} v(x)$. Please compare this remark with Remark 3.3.

4. Positive solution with zero Dirichlet or Neumann boundary condition

In this section, we prove Theorems 1.4, 1.5 and 1.6. We set the energy functional corresponding to equation (1.2) with zero Dirichlet boundary condition as

$$\tilde{I}(u) = \frac{1}{2} \int_\Omega \Big\{|\nabla u|^2 + \lambda u^2\Big\}\, dx + \frac{1}{2} \int_\Omega P \Big[\log(1 + V + u_+^2) - \log(1 + V)\Big]\, dx$$

for $u \in W_0^{1,2}(\Omega)$, where $u_+ = \max\{u, 0\}$. We first give some lemmas.

Lemma 4.1. *If $P < 0$ and $\lambda \geq 0$ then $\tilde{I}$ is bounded from below.*

Proof. Since

$$\log(1 + V + u_+^2) - \log(1 + V) \leq |u_+| \leq \epsilon|u_+|^2 + \frac{1}{4\epsilon},$$

we have

$$\begin{aligned}
\tilde{I}(u) &\geq \frac{1}{2} \int_\Omega \Big(|\nabla u|^2 + \lambda u^2 + P\epsilon|u_+|^2\Big)\, dx + \frac{P}{8\epsilon}|\Omega| \\
&\geq \frac{P}{8\epsilon}|\Omega| + \frac{1}{2} \int_\Omega \Big(\mu_1 + \lambda + P\epsilon\Big) u^2\, dx \\
&\geq \frac{P}{8\epsilon}|\Omega|,
\end{aligned}$$

where $\epsilon > 0$ is taken small enough such that $\mu_1 + \lambda + P\epsilon > 0$. $\square$

Lemma 4.2. *If $P < 0$ and $\lambda \geq 0$, then $\tilde{I}$ has a global minimizer, say w, in $W_0^{1,2}(\Omega)$.*

Proof. Let $\{u_n\}$ be a minimizing sequence satisfying $\tilde{I}(u_n) \to \inf_{W_0^{1,2}(\Omega)} \tilde{I}$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|\tilde{I}(u_n)| \leq |\inf_{W_0^{1,2}(\Omega)} \tilde{I}| + 1$. So for $\epsilon > 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Big(|\nabla u_n|^2 + \lambda u_n^2\Big)\,dx \\
&\leq |\inf_{W_0^{1,2}(\Omega)} \tilde{I}| + 1 - \frac{1}{2}P\int_\Omega \Big[\log(1+V+u_n^2) - \log(1+V)\Big]\,dx \\
&\leq |\inf_{W_0^{1,2}(\Omega)} \tilde{I}| + 1 - \frac{P}{8\epsilon}|\Omega| - \frac{P\epsilon}{2}\int_\Omega u_n^2\,dx.
\end{aligned}$$

Then $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Without loss of generality $u_n \to w$ weakly in $W_0^{1,2}(\Omega)$. Since $\tilde{I}$ is weakly lower continuity, $\tilde{I}$ has a global minimizer w in $W_0^{1,2}(\Omega)$. □

Lemma 4.3. *If $P < 0$, $\lambda > 0$ and $\mu_1 + \lambda < \frac{|P|}{1+V_0+M_1}$, then $\tilde{I}(w) < 0$, where w is defined in Lemma 4.2.*

Proof. It is well known that $\mu_1 > 0$. By the definition of w, we obtain $\tilde{I}(w) \leq \tilde{I}(\varphi_1)$. Noticing

$$\log(1+V+u_+^2) - \log(1+V) \geq \frac{u_+^2}{1+V+u_+^2},$$

we obtain

$$\begin{aligned}
\tilde{I}(\varphi_1) &= \frac{1}{2}\int_\Omega \Big(|\nabla\varphi_1|^2 + \lambda\varphi_1^2\Big)\,dx + \frac{1}{2}\int_\Omega P\Big[\log(1+V+\varphi_1^2) - \log(1+V)\Big]\,dx \\
&\leq \frac{1}{2}\int_\Omega \Big(\mu_1 + \lambda\Big)\varphi_1^2\,dx + \frac{1}{2}\int_\Omega \frac{P\varphi_1^2}{1+V+\varphi_1^2}\,dx \\
&\leq \Big[\frac{1}{2}(\mu_1+\lambda) + \frac{1}{2}\frac{P}{1+V_0+M_1}\Big]\int_\Omega \varphi_1^2\,dx < 0
\end{aligned}$$

provided that $\mu_1 + \lambda < \frac{-P}{1+V_0+M_1}$. □

Proof of Theorem 1.4. Firstly $w_+ \not\equiv 0$: if $w_+ \equiv 0$ then $\tilde{I}(w) = \frac{1}{2}\int_\Omega (|\nabla w|^2 + \lambda w^2)\,dx \geq 0$, which contradicts to Lemma 4.3.

Secondly $w \geq 0$. w is a weak solution of

$$\begin{gathered}
\Delta w = \frac{Pw_+}{1+w_+^2+V} + \lambda w \quad x \in \Omega, \\
w = 0 \quad x \in \partial\Omega.
\end{gathered} \tag{4.1}$$

Multiplying the first equation in (4.1) by w_- and integrating by parts on Ω, we obtain

$$\int_\Omega \big(|\nabla w_-|^2 + \lambda w_-^2\big)\,dx = 0.$$

So $w_- = 0$ and then $w \geq 0$, i.e., w is a weak solution of equation (1.2) with zero Dirichlet boundary condition. Note that w is a classical solution by elliptic regularity theory, and $w > 0$ follows from the strong maximum principle. □

To prove Theorem 1.5, we first prove the following lemma.

Lemma 4.4. *If $P < 0$, then equation* (1.2) *with Dirichlet boundary condition $u|_{\partial\Omega} = 0$ has at most one positive solution.*

Proof. If u, w are positive solutions of equation (1.2), then u, $w > 0$ in Ω and

$$\Delta u = \frac{Pu}{1+V+u^2} + \lambda u, \tag{4.2}$$

$$\Delta w = \frac{Pw}{1+V+w^2} + \lambda w. \tag{4.3}$$

Set $\Omega_+ = \{x \in \Omega; u(x) - w(x) > 0\}$ then Ω_+ is a piecewise C^1 smooth domain.

We claim that $\Omega_+ = \emptyset$. In fact, if $\Omega_+ \neq \emptyset$, multiplying equation (4.2) by w and subtracting equation (4.3) which is multiplied by u, we have

$$\begin{aligned} &\int_{\partial\Omega_+} \Big(w\frac{\partial u}{\partial \nu} - u\frac{\partial w}{\partial \nu}\Big)\, dx \\ &= \int_{\Omega_+} Puw\Big[\frac{1}{1+V+u^2} - \frac{1}{1+V+w^2}\Big]\, dx \\ &= \int_{\Omega_+} Puw\frac{w^2-u^2}{(1+V+w^2)(1+V+u^2)}\, dx > 0. \end{aligned}$$

However,

$$\int_{\partial\Omega_+} \Big(w\frac{\partial u}{\partial \nu} - u\frac{\partial w}{\partial \nu}\Big)\, dx = \int_{\partial\Omega_+\setminus\partial\Omega} \Big(w\frac{\partial u}{\partial \nu} - u\frac{\partial w}{\partial \nu}\Big)\, dx = \int_{\partial\Omega_+\setminus\partial\Omega} w\frac{\partial(u-w)}{\partial \nu}\, dx \le 0.$$

This contradiction shows that $\Omega_+ = \emptyset$. Similarly $\Omega_- = \{x \in \Omega; u(x) - w(x) < 0\} = \emptyset$. Therefore we obtain $u \equiv w$ in Ω. ☐

Remark 4.5. The method in the proof of Lemma 4.4 is modified from [22]. It is easy to see that the same conclusion holds if boundary condition $u|_{\partial\Omega} = 0$ is replaced by $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$.

Theorem 1.5 now follows from Theorem 1.4 and Lemma 4.4.

Proof of Theorem 1.6. We need only to prove that equation (1.2) has a positive solution because of Remark 4.5. But this is a direct consequence of Theorem 1.1 (3) and Remark 3.4. Notice that in Theorem 1.1 there is no boundary condition while we add Neumann boundary in our case, but we can use their proof since we can still integrate by parts in the proof. ☐

Remark 4.6. When $P < 0$, $\lambda > 0$ and $\lambda \geq |P|/(1 + v_0)$, equation (1.2) with zero Neumann boundary condition has no positive solution because the only solution is 0 as pointed out by Yang and Zhang in Theorem 1.1 (4). In particular this is a sharp conclusion for equation (1.3).

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