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EXISTENCE OF SOLUTIONS TO NONLINEAR p-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS WITH HIGHER-ORDER DERIVATIVE TERMS

YOU-HUI SU, YONGZHEN YUN, DONGDONG WANG, WEIMIN HU

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Abstract. In this article, we discuss the existence of positive solution to a nonlinear p-Laplacian fractional differential equation whose nonlinearity contains a higher-order derivative

$$D_{0+}^{\beta}\phi_p\big(D_{0+}^{\alpha}u(t)\big)+f\big(t,u(t),u'(t),\dots,u^{(n-2)}(t)\big)=0,\quad t\in(0,1),$$
$$u(0)=u'(0)=\dots=u^{(n-2)}(0)=0,$$
$$u^{(n-2)}(1)=au^{(n-2)}(\xi)=0,\quad D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha}u(1)=0,$$

where $n-1<\alpha\le n$, $n\ge 2$, $1<\beta\le 2$, $0<\xi<1$, $0\le a\le 1$ and $0\le a\xi^{\alpha-n}\le 1$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$. D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives, and $f\in C((0,1)\times[0,+\infty)^{n-1},[0,+\infty))$. The Green's function of the fractional differential equation mentioned above and its relevant properties are presented, and some novel results on the existence of positive solution are established by using the mixed monotone fixed point theorem and the upper and lower solution method. The interesting of this paper is that the nonlinearity involves the higher-order derivative, and also, two examples are given in this paper to illustrate our main results from the perspective of application.

1. Introduction

In the past decades, there has been a growing interest in the study of the fractional differential equations due to the intensive development of the fractional calculus theory itself and its applications in various sciences such as engineering, control theory, blood flow phenomena, bode analysis of feedback amplifiers, electroanalytical chemistry, and aerodynamics, etc., for details, see [1, 7, 5, 14, 15, 12] and references therein. For example, in studying a transfer process in porous material, Mehaute [20] discussed the following fractional differential equations

$${}_0D_t^{1/d-1}J(t)=LX(t),$$

where $J(t)$ is the macroscopic flow across the material interface, $X(t)$ is the local driving force, L is a constant, and d is the fractal dimension of the material. In

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the meantime, the existence theory of solutions to the fractional boundary-value problems has attracted the attention of many researchers quite recently, see [2, 3, 6, 18, 19, 21, 23, 26, 28, 29, 34] and their references.

We find that p-Laplacian differential equation has been widely applied in analyzing mechanics, physics, dynamic systems and other related fields of mathematical modeling. Hence, there have been many published papers which are devoted to the existence of solutions to the differential equations with p-Laplacian operator, see [8, 9, 10, 17, 24, 25, 27, 30, 31, 32] and their references. For example, in studying the turbulent flow in a porous medium, Leibenson introduced the p-Laplacian equation in [13] as follows

$$(\phi_p(x'(t)))' = f(t, x(t), x'(t)),$$

where $\phi_p(s)$ is p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$ and $(\phi_p)^{-1} = \phi_q$, and $1/p + 1/q = 1$.

So, based on the above illustration, it is of significance to make the study of the nonlinear p-Laplacian fractional differential equation. In order to better explore the existence of positive solution to the nonlinear p-Laplacian fractional differential equation, here we briefly review some related results in the existing literature [31, 11, 16, 4].

Tian and Li [31] investigated the existence of positive solution to the following fractional differential equations with p-Laplacian operator

$$\begin{gathered} D^{\alpha}_{0+}\phi_p\big(D^{\beta}_{0+}u(t)\big) + f\big(t, u(t)\big) = 0, \quad t \in (0,1), \\ u(0) = 0, D^{\gamma}_{0+}u(1) = \lambda D^{\gamma}_{0+}u(\xi) = 0, D^{\beta}_{0+}u(0) = 0, \end{gathered} \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta, \gamma \in \mathbb{R}$, $0 < \alpha < 1$, $1 < \beta \le 2$, $0 < \gamma \le 1$ and $1 + \gamma \le \beta$, $0 < \xi < 1$, $\lambda \in [0, +\infty)$ and $\lambda\xi^{\beta-\gamma-1} < 1$. D^{α}_{0+}, D^{β}_{0+} are the standard Riemann-Liouville fractional derivatives, and $f \in C([0,1] \times [0,+\infty), [0,+\infty))$. The existence results on positive solution to fractional differential equations (1.1) are obtained by using some fixed point theorems in a cone.

There are very few publications concerning the existence of positive solutions to fractional differential equations with nonlinear terms involving the derivative [4, 11, 16]. Cheng et al. [4] investigated the positive solutions to the following fractional differential equations whose nonlinearity contains the one-order derivative as the form

$$\begin{gathered} D^{\alpha}_{0+}u(t) + f\big(t, u(t), u'(t)\big) = 0, \quad t \in (0,1),\ n-1 < \alpha \le n, \\ u^{(i)}(0) = 0,\ i = 0, 1, 2, \dots, n-2, \quad [D^{\beta}_{0+}u(t)]_{t=1} = 0,\ 2 \le \beta \le n-2, \end{gathered} \tag{1.2}$$

where $u^{(i)}$ represents the ith derivative of u, $n > 4$ $(n \in \mathbb{N})$, D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order $n - 1 < \alpha \le n$ and $f(t, u, u')$: $[0,1] \times [0,\infty) \times (-\infty,+\infty) \to [0,\infty)$ satisfies Carathéodory type conditions. Some sufficient conditions for the existence of positive solutions to boundary-value problem (1.2) are established by using fixed-point theorem.

It is notable that the nonlinear term $f(t, u(t))$ in equation (1.1) does not involve the derivative. In [4, 11, 16], attention was mainly focused on the existence of fractional differential equations with nonlinear terms involving the first-order derivative and the p-Laplacian operator is not involved. Apparently, the nonlinear term which is to be studied in this paper

$$f(t,u(t),u'(t),\dots,u^{(n-2)}(t)),n=1,2,\dots,$$

contains the higher-derivative, and we believe the study in this paper is theoretically and practically significant because it will represent a more general case. Naturally, it is interesting and necessary to study the existence of positive solutions to p-Laplacian fractional differential equations with nonlinear terms involving the higher-derivative.

In this paper, we mainly study the existence of positive solutions to the following p-Laplacian fractional differential equations with nonlinear terms involving the higher-derivative:

$$\begin{gathered} D_{0+}^{\beta}\phi_p(D_{0+}^{\alpha}u(t))+f\big(t,u(t),u'(t),\dots,u^{(n-2)}(t)\big)=0,\quad t\in(0,1),\\ u(0)=u'(0)=\dots=u^{(n-2)}(0)=0,\\ u^{(n-2)}(1)=au^{(n-2)}(\xi)=0,\quad D_{0+}^{\alpha}u(0)=D_{0+}^{\alpha}u(1)=0, \end{gathered}\tag{1.3}$$

where $n-1<\alpha\le n$, $n\ge 2$, $1<\beta\le 2$, $0<\xi<1$, $0\le a\le 1$ and $0\le a\xi^{\alpha-n}\le 1$, $\phi_p(s)=|s|^{p-2}s$, $p>1$, $\phi_p^{-1}=\phi_q$, $\frac{1}{p}+\frac{1}{q}=1$. D_{0+}^{α}, D_{0+}^{β} are the standard Riemann-Liouville fractional derivatives, and $f\in C((0,1)\times[0,+\infty)^{n-1},[0,+\infty))$.

The Green's function of the boundary-value problem (1.3) and the relevant properties are to be presented later, and because of the nonlinear terms involving the higher-derivative in fractional differential equations (1.3), it's very difficult or even impossible to obtain the existence of positive solution of it by using some fixed point theorem in a cone, such as nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed point theorem, and it is the same for the methods listed in [4, 11, 16, 31]. The reason for that is the nonlinearity is in a high dimensional space and is not controlled in a cone because of the nonlinear terms involving the higher-derivative, and so we establish some novel results on the existence of positive solution by using the mixed monotone fixed point theorem and the upper and lower solution method.

The first special feature and innovative contribution of our work is that we present in this paper the Green's function of the differential equation and its relevant properties, which is very difficult because the differential equation relates to the standard Riemann-Liouville fractional derivatives and p-Laplacian operator. The second special feature and innovative contribution of our work is that the nonlinearity involves the higher-order derivative, which is also not so easy for the nonlinearity is not controlled in a cone because of the nonlinear terms involving the higher-derivative. Therefore, we try to deal with this problem by using a new method which is different from many other works [4, 11, 16, 31]. In addition, two examples are also given in this paper to illustrate our main results from the viewpoint of applications.

The structure of our paper is as follows. Section 1 is the introduction of the paper. In Section 2, some necessary definitions and lemmas which are cited in our paper are presented. In Section 3, we construct an equivalent fractional differential equation. The Green's function of the equivalent fractional differential equation is constructed, and its properties are presented in Section 4. The existence results on unique positive solution to the fractional boundary-value problem (1.3) are obtained in Section 5. The existence theorem of at least single positive solution to the fractional boundary-value problem (1.3) is proved in Section 6. In Section 7, we give two examples to illustrate our main results.

2. Preliminaries

To prove our main results, in this section we present some basic definitions and technical lemmas which can help us to better understand our main results and proofs. For the basic terminologies, we refer the reader to references [7, 11, 22, 33].

Definition 2.1 ([11])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function y: $(0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([11])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} y(s) ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0, \infty)$.

Let P be a normal cone of a Banach space E, and $e \in P$ with $\|e\| \leq 1$, $e \neq \theta$ (θ is zero element of E). Define

$$Q_e = \{x \in P : \text{ there exist constants } m, M > 0 \text{ such that } me \leq x \leq Me\}.$$

Definition 2.3 ([33])**.** Let T be a operator satisfies $T : Q_e \times Q_e \to Q_e$. T is said to be mixed monotone if $T(x, y)$ is nondecreasing in x and nonincreasing in y, i.e., if $x_1 \leq x_2$ ($x_1, x_2 \in Q_e$) implies $T(x_1, y) \leq T(x_2, y)$ for any $y \in Q_e$, and $y_1 \geq y_2$ ($y_1, y_2 \in Q_e$) implies $T(x, y_1) \leq T(x, y_2)$ for any $x \in Q_e$. Element $x^* \in Q_e$ is called a fixed point of T if $T(x^*, x^*) = x^*$.

Next we give some Lemmas which are used in our main results.

Lemma 2.4 ([11])**.** *The equality* $I_{0+}^{\gamma} I_{0+}^{\delta} y(t) = I_{0+}^{\gamma+\delta} y(t)$, $\gamma > 0$, $\delta > 0$ *holds for* $y \in C(0,1) \cap L(0,1)$.

Lemma 2.5 ([11])**.** *The equality* $D_{0+}^{\gamma} I_{0+}^{\gamma} y(t) = y(t)$, $\gamma > 0$ *holds for* $y \in C(0,1) \cap L(0,1)$.

Lemma 2.6 ([11])**.** *Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then the fractional differential equation*

$$D_{0+}^{\alpha} y(t) = 0,$$

has a unique solution $y(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n}$, *where* $C_i \in R$, $i = 1, 2, \dots, n$, n *is the smallest integer greater than or equal to* α.

Lemma 2.7 ([11])**.** *Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} y(t) = y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n},$$

for some $C_i \in R$, $i = 1, 2, \dots, n$, *where* n *is the smallest integer greater than or equal to* α.

Lemma 2.8 ([33]). *Assume that the operator $T : Q_e \times Q_e \to Q_e$ is mixed monotone operator and there exists a constant δ $(0 < \delta < 1)$, such that*

$$T(tx, \frac{1}{t}y) \geq t^{\delta} T(x, y), x, y \in Q_e, 0 < t < 1.$$

Then the operator T has a unique fixed point $x^*(x^* \in Q_e)$.

3. Equivalence of fractional differential equation

In this section, we construct an equivalent fractional differential equation, and prove that search for the solution of fractional differential equation (1.3) is equivalent to finding the solution of it.

Lemma 3.1. *Let $u(t) = I_{0^+}^{n-2} v(t)$, $v \in C[0,1]$, then the fractional boundary-value problem (1.3) is equivalent to the following fractional differential equation*

$$\begin{gathered} D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha-n+2} v(t)) + f(t, I_{0^+}^{n-2} v(t), I_{0^+}^{n-3} v(t), \dots, I_{0^+}^{1} v(t), v(t)) = 0, \\ t \in (0,1) \\ v(0) = 0, \quad v(1) = a v(\xi), \quad D_{0^+}^{\alpha-n+2} v(0) = D_{0^+}^{\alpha-n+2} v(1) = 0, \end{gathered} \tag{3.1}$$

where $n-1 < \alpha \leq n$, $n \geq 2$, $1 < \beta \leq 2$, $0 < \xi < 1$, $0 \leq a \leq 1$ and $0 \leq a\xi^{\alpha-n} \leq 1$, $\phi_p = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in C((0,1) \times [0,+\infty)^{n-1}, [0,+\infty))$. Moreover, $v \in C([0,1],[0,+\infty))$ is a positive solution of the differential equation (3.1) means that $u(t) = I_{0^+}^{n-2} v(t)$ is a positive solution of the differential equation (1.3).

Proof. Let $u(t) = I_{0^+}^{n-2} v(t)$, it follow from the definition of Riemann-Liouville fractional derivative, Lemma 2.4 and Lemma 2.5 that

$$\begin{gathered} D_{0^+}^{n} I_{0^+}^{n-\alpha} u(t) = D_{0^+}^{n} I_{0^+}^{n-\alpha} I_{0^+}^{n-2} v(t) = D_{0^+}^{n} I_{0^+}^{2n-\alpha-2} v(t) = D_{0^+}^{\alpha-n+2} v(t), \\ u'(t) = D_{0^+}^{1} I_{0^+}^{n-2} v(t) = D_{0^+}^{1} I_{0^+}^{1} I_{0^+}^{n-3} v(t) = I_{0^+}^{n-3} v(t), \\ u''(t) = D_{0^+}^{2} I_{0^+}^{n-2} v(t) = D_{0^+}^{2} I_{0^+}^{2} I_{0^+}^{n-4} v(t) = I_{0^+}^{n-4} v(t), \\ \dots \\ u^{(n-3)}(t) = D_{0^+}^{n-3} I_{0^+}^{n-2} v(t) = D_{0^+}^{n-3} I_{0^+}^{n-3} I_{0^+}^{1} v(t) = I_{0^+}^{1} v(t), \\ u^{(n-2)}(t) = D_{0^+}^{n-2} I_{0^+}^{n-2} v(t) = v(t). \end{gathered}$$

Therefore

$$\begin{gathered} D_{0^+}^{\beta} (\phi_p D_{0^+}^{\alpha-n+2} v(t)) + f(t, I_{0^+}^{n-2} v(t), I_{0^+}^{n-3} v(t), \dots, I_{0^+}^{1} v(t), v(t)) = 0, \\ v(0) = u^{n-2}(0) = u^{n-3}(0) = \dots = u(0) = 0, \\ v(1) = a v(\xi), D_{0^+}^{\alpha-n+2} v(0) = D_{0^+}^{\alpha-n+2} v(1) = 0. \end{gathered}$$

From above discussions, let $u(t) = I_{0^+}^{n-2} v(t)$, then the differential equation (1.3) is equivalent to the differential equation (3.1).

Now, let $v \in C([0,1],[0,+\infty))$ is a positive solution of the differential equation (3.1). Then

$$\begin{aligned} D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha-n+2} v(t)) &= D_{0^+}^{\beta} \phi_p(D_{0^+}^{n} I_{0^+}^{2n-\alpha-2} v(t)) \\ &= D_{0^+}^{\beta} \phi_p(D_{0^+}^{n} I_{0^+}^{n-\alpha} I_{0^+}^{n-2} v(t)) \\ &= D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} I_{0^+}^{n-2} v(t)) \end{aligned}$$

$$= D^{\beta}_{0^+}\phi_p(D^{\alpha}_{0^+}u(t)),$$

and

$$f(t, I^{n-2}_{0^+}v(t), I^{n-3}_{0^+}v(t), \dots, I^{1}_{0^+}v(t), v(t)) = f(t, u(t), u'(t), \dots, u^{(n-2)}(t)),$$

which implies that $u(t) = I^{n-2}_{0^+}v(t)$ is a positive solution of the differential equation (1.3). The proof is complete. $\square$

4. Properties of Green's Function

In this section, we obtain the Green's function of fractional boundary-value problem (3.1) and its some properties.

Lemma 4.1. *Assume that $y \in C[0,1]$ and $n-1 < \alpha \le n$, then the following fractional boundary-value problem*

$$\begin{gathered} D^{\alpha-n+2}_{0^+}v(t) + y(t) = 0, \quad t \in (0,1), \\ v(0) = 0, \quad v(1) = av(\xi), \end{gathered} \tag{4.1}$$

has a unique solution

$$v(t) = \int_0^1 G(t,s)y(s)d(s),$$

where

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-n+1} - a[t(\xi-s)]^{\alpha-n+1} - (1-a\xi^{\alpha-n+1})(t-s)^{\alpha-n+1}}{(1-a\xi^{\alpha-n+1})\Gamma(\alpha-n+2)}, & 0 \le s \le t \le 1, s \le \xi, \\ \frac{[t(1-s)]^{\alpha-n+1} - (1-a\xi^{\alpha-n+1})(t-s)^{\alpha-n+1}}{(1-a\xi^{\alpha-n+1})\Gamma(\alpha-n+2)}, & 0 < \xi \le s \le t \le 1, \\ \frac{[t(1-s)]^{\alpha-n+1} - a[t(\xi-s)]^{\alpha-n+1}}{(1-a\xi^{\alpha-n+1})\Gamma(\alpha-n+2)}, & 0 \le t \le s \le \xi < 1, \\ \frac{[t(1-s)]^{\alpha-n+1}}{(1-a\xi^{\alpha-n+1})\Gamma(\alpha-n+2)}, & 0 \le t \le s \le 1, \xi \le s. \end{cases} \tag{4.2}$$

Lemma 4.2. *Let $n-1 < \alpha \le n, 0 < \xi < 1, 0 \le a \le 1$. If $y(t) \in C[0,1]$ and $y(t) \ge 0$ hold, then the fractional differential equation (4.1) has a unique solution $v(t) \ge 0$, $t \in [0,1]$.*

Lemma 4.3. *Assume that $y \in C[0,1]$ and $n-1 < \alpha \le n$, $0 < \xi < 1$, $1 < \beta \le 2$, $0 \le a \le 1$, then the following fractional differential equation*

$$\begin{gathered} D^{\beta}_{0^+}\phi_p(D^{\alpha-n+2}_{0^+}v(t)) = y(t), \quad t \in (0,1), \\ v(0) = 0, v(1) = av(\xi), D^{\alpha-n+2}_{0^+}v(0) = D^{\alpha-n+2}_{0^+}v(1) = 0, \end{gathered} \tag{4.3}$$

has a unique solution

$$v(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds,$$

where

$$H(s,\tau) = \begin{cases} \frac{s^{\beta-1}(1-\tau)^{\beta-1} - (s-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le \tau \le s \le 1, \\ \frac{s^{\beta-1}(1-\tau)^{\beta-1}}{\Gamma(\beta)}, & 0 \le s \le \tau \le 1, \end{cases} \tag{4.4}$$

and $G(t,s)$ is defined in Lemma 4.1.

Proof. It follows from Lemma 2.7 that

$$\phi_p(D_{0^+}^{\alpha-n+2}v(t)) = I_{0^+}^{\beta}y(t) + C_1t^{\beta-1} + C_2t^{\beta-2}, \quad t \in (0,1),$$

where $C_1, C_2 \in \mathbb{R}$. According to the boundary condition $D_{0^+}^{\alpha-n+2}v(0) = 0$ and $D_{0^+}^{\alpha-n+2}v(1) = 0$, one has

$$C_1 = -I_{0^+}^{\beta}y(t)|_{t=1} = -\frac{1}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}y(\tau)d\tau,\ C_2 = 0,$$

this implies

$$\begin{aligned}
\phi_p(D_{0^+}^{\alpha-n+2}v(t)) &= I_{0^+}^{\beta}y(t) - t^{\beta-1}I_{0^+}^{\beta}y(1) \\
&= \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}y(\tau)d\tau - \frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}y(\tau)d\tau \\
&= -\int_0^1 H(t,\tau)y(\tau)d\tau,
\end{aligned}$$

i.e.,

$$D_{0^+}^{\alpha-n+2}v(t) + \phi_q\Big(\int_0^1 H(t,\tau)y(\tau)d\tau\Big) = 0.$$

Therefore, the fractional boundary-value problems (4.3) is equivalent to the following fractional boundary-value problems

$$\begin{gathered}
D_{0^+}^{\alpha-n+2}v(t) + \phi_q\Big(\int_0^1 H(t,\tau)y(\tau)d\tau\Big) = 0, \quad t \in (0,1), \\
v(0) = 0, \quad v(1) = av(\xi).
\end{gathered} \tag{4.5}$$

It follows from Lemma 4.1 that the fractional boundary-value problems (4.5) exists a unique solution

$$v(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)y(\tau)d\tau\Big)ds.$$

The proof is complete. □

Lemma 4.4. *Assume that $0 \le a\xi^{\alpha-n} \le 1$ holds, there exist the functions $G(t,s)$ and $H(t,s)$ be defined by (4.2) and (4.4) such that*

(i) *$G(t,s)$ and $H(t,s)$ are continuous functions on $[0,1]\times[0,1]$;*

(ii) *$G(t,s) \le \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}$ for $(t,s) \in [0,1]\times[0,1]$, $H(t,s) \le \frac{t^{\beta-1}}{\Gamma(\beta)}$ for $(t,s) \in [0,1]\times[0,1]$;*

(iii) *$G(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$, $H(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$;*

(iv) *$G(t,s) \le G(s,s)$ for $(t,s) \in [0,1]\times[0,1]$, $H(t,s) \le H(s,s)$ for $(t,s) \in [0,1]\times[0,1]$;*

(v) *there exist positive functions $\gamma(s) \in C[0,1]$ and $\rho(s) \in C[0,1]$ such that*

$$\min_{t\in[\xi,1]} G(t,s) \ge \gamma(s)\max_{t\in[0,1]} G(t,s) = \gamma(s)G(s,s) \quad \text{for } 0 < s < 1,$$

and

$$\min_{t\in[\xi,1]} H(t,s) \ge \rho(s)\max_{t\in[0,1]} H(t,s) = \rho(s)H(s,s) \quad \text{for } 0 < s < 1.$$

Proof. From the definition of $G(t,s)$ and $H(t,s)$, it is easy to check that (i) and (ii) be satisfied. We shall prove that (iii) holds, set

$$\begin{gathered}
g_1(t,s)=[t(1-s)]^{\alpha-n+1}-a[t(\xi-s)]^{\alpha-n+1}-(1-a\xi^{\alpha-n+1})(t-s)^{\alpha-n+1},\\
\text{for } 0\le s\le t\le 1, s\le \xi;\\
g_2(t,s)=[t(1-s)]^{\alpha-n+1}-(1-a\xi^{\alpha-n+1})(t-s)^{\alpha-n+1},\\
\text{for } 0<\xi\le s\le t\le 1;\\
g_3(t,s)=[t(1-s)]^{\alpha-n+1}-a[t(\xi-s)]^{\alpha-n+1},\quad 0\le t\le s\le \xi<1;;\\
g_4(t,s)=[t(1-s)]^{\alpha-n+1},\quad 0\le t\le s\le 1,\xi\le s.
\end{gathered}$$

To prove that (iii) is true, we need to show that $g_i\ge 0$ for $i=1,2,3,4$.

(1) If $t<\xi$, since $0\le a\xi^{\alpha-n+1}\le 1$ and $0<\xi<1$, we have

$$\begin{aligned}
&g_1(t,s)\\
&=(t-ts)^{\alpha-n+1}-a(t\xi-ts)^{\alpha-n+1}-(t-s)^{\alpha-n+1}+a\xi^{\alpha-n+1}(t-s)^{\alpha-n+1}\\
&=[(t-ts)^{\alpha-n+1}-(t-s)^{\alpha-n+1}]-a\xi^{\alpha-n+1}\big[\big(t-\frac{ts}{\xi}\big)^{\alpha-n+1}-(t-s)^{\alpha-n+1}\big]\\
&\ge (t-ts)^{\alpha-n+1}-\big(t-\frac{ts}{\xi}\big)^{\alpha-n+1}\ge 0.
\end{aligned}$$

Moreover, if $t\ge\xi$, then

$$\begin{aligned}
g_1(t,s)&=t^{\alpha-n+1}\big[(1-s)^{\alpha-n+1}-a(\xi-s)^{\alpha-n+1}-(1-a\xi^{\alpha-n+1})(1-\frac{s}{t})^{\alpha-n+1}\big]\\
&=t^{\alpha-n+1}\big\{[(1-s)^{\alpha-n+1}-(1-\frac{s}{t})^{\alpha-n+1}]\\
&\quad +a\xi^{\alpha-n+1}[(1-\frac{s}{t})^{\alpha-n+1}-(1-\frac{s}{\xi})^{\alpha-n+1}]\big\}\ge 0.
\end{aligned}$$

(2) If $0<\xi\le s\le t\le 1$, according to $0\le a\xi^{\alpha-n}\le 1$, there is

$$\begin{aligned}
g_2(t,s)&\ge (1-s)^{\alpha-n+1}t^{\alpha-n+1}-(t-s)^{\alpha-n+1}\\
&=t^{\alpha-n+1}[(1-s)^{\alpha-n+1}-(1-\frac{s}{t})^{\alpha-n+1}]\ge 0.
\end{aligned}$$

(3) If $0\le t\le s\le\xi<1$, we obtain

$$\begin{aligned}
g_3(t,s)&=t^{\alpha-n+1}\big[(1-s)^{\alpha-n+1}-a\xi^{\alpha-n+1}(1-\frac{s}{\xi})^{\alpha-n+1}\big]\\
&\ge t^{\alpha-n+1}[(1-s)^{\alpha-n+1}-(1-\frac{s}{\xi})^{\alpha-n+1}]\\
&\ge 0,\quad \text{for } 0\le a\xi^{\alpha-n}\le 1.
\end{aligned}$$

(4) It is obvious that $g_4(t,s)\ge 0$ for $0\le t\le s\le 1$, $\xi\le s$.

Similarly, $H(t,s)\ge 0$ for $t,s\in(0,1)$. From above discussions, we conclude that $G(t,s)\ge 0$ and $H(t,s)\ge 0$ for any $t,s\in(0,1)$. So property (iii) holds.

Now we prove that (iv) holds. Firstly, we check that $g_1(t,s)$ and $g_2(t,s)$ are nonincreasing with respect to $t\in[s,1]$.

$$\begin{aligned}
\frac{\partial g_1(t,s)}{\partial(t)}&=(\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n+1}-a(\xi-s)^{\alpha-n+1}(\alpha-n+1)t^{\alpha-n}\\
&\quad -(1-a\xi^{\alpha-n+1})(\alpha-n+1)(t-s)^{\alpha-n}\\
&=(\alpha-n+1)[t^{\alpha-n}(1-s)^{\alpha-n+1}-a(\xi-s)^{\alpha-n+1}t^{\alpha-n}
\end{aligned}$$

$$
\begin{aligned}
&\quad - (1-a\xi^{\alpha-n+1})(t-s)^{\alpha-n}]\\
&= (\alpha-n+1)t^{\alpha-n}[(1-s)^{\alpha-n+1} - a\xi^{\alpha-n+1}(1-\frac{s}{\xi})^{\alpha-n+1}\\
&\quad - (1-a\xi^{\alpha-n+1})(1-\frac{s}{t})^{\alpha-n}]\\
&\le (\alpha-n+1)t^{\alpha-n}\Big[(1-s)^{\alpha-n+1} - a\xi^{\alpha-n+1}(1-\frac{s}{\xi})(1-s)^{\alpha-n+1}\\
&\quad - (1-a\xi^{\alpha-n+1})(1-s)^{\alpha-n}\Big]\\
&= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}[(1-s) - a\xi^{\alpha-n+1}(1-\frac{s}{\xi}) - (1-a\xi^{\alpha-n+1})]\\
&= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}[s(a\xi^{\alpha-n}-1)]\\
&\le 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial g_2(t,s)}{\partial(t)} &= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n+1} - (1-a\xi^{\alpha-n+1})(\alpha-n+1)(t-s)^{\alpha-n}\\
&= (\alpha-n+1)[t^{\alpha-n}(1-s)^{\alpha-n+1} - (1-a\xi^{\alpha-n+1})(t-s)^{\alpha-n}]\\
&= (\alpha-n+1)t^{\alpha-n}[(1-s)^{\alpha-n+1} - (1-\frac{s}{t})^{\alpha-n}(1-a\xi^{\alpha-n+1})]\\
&\le (\alpha-n+1)t^{\alpha-n}[(1-s)^{\alpha-n+1} - (1-s)^{\alpha-n}(1-a\xi^{\alpha-n+1})]\\
&= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}[(1-s) - (1-a\xi^{\alpha-n+1})]\\
&= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}(a\xi^{\alpha-n+1}-s)\\
&\le (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}(a\xi^{\alpha-n+1}-\xi)\\
&= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}\xi(a\xi^{\alpha-n}-1)\\
&\le 0.
\end{aligned}
$$

Then, $g_1(t,s)$ and $g_2(t,s)$ is non-increasing with respect to $t \in [s,1]$.

Secondly, we show that $g_3(t,s)$ and $g_4(t,s)$ are nondecreasing with respect to $t \in [0,s]$.

$$
\begin{aligned}
\frac{\partial g_3(t,s)}{\partial(t)} &= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n+1} - a(\xi-s)^{\alpha-n+1}(\alpha-n+1)t^{\alpha-n}\\
&= (\alpha-n+1)t^{\alpha-n}[(1-s)^{\alpha-n+1} - a(\xi-s)^{\alpha-n+1}]\\
&\ge (\alpha-n+1)t^{\alpha-n}[(1-s)^{\alpha-n+1} - a(1-s)^{\alpha-n+1}]\\
&= (\alpha-n+1)t^{\alpha-n}(1-s)^{\alpha-n}(1-a) \ge 0,
\end{aligned}
$$

which implies that $g_3(t,s)$ is nondecreasing with respect to t on $[0,s]$. It is obvious that $g_4(t,s)$ is nondecreasing with respect to t on $[0,s]$. Therefore,

$$
\begin{aligned}
G(t,s) &\le G(s,s) \quad \text{for } 0 \le s \le t \le 1,\\
G(t,s) &\le G(s,s) \quad \text{for } 0 \le t \le s \le 1.
\end{aligned}
$$

In conclusion

$$
G(t,s) \le G(s,s) \quad \text{for } (t,s) \in [0,1]\times[0,1].
$$

Thirdly, setting

$$
h_1(t,s) = t^{\beta-1}(1-s)^{\beta-1} - (t-s)^{\beta-1}, \quad 0 \le s \le t \le 1,
$$

$$h_2(t,s)=t^{\beta-1}(1-s)^{\beta-1},\quad 0\le t\le s\le 1,$$

we have

$$\begin{aligned}\frac{\partial h_1(t,s)}{\partial(t)}&=(\beta-1)t^{\beta-2}(1-s)^{\beta-1}-(\beta-1)(t-s)^{\beta-2}\\&=(\beta-1)[t^{\beta-2}(1-s)^{\beta-1}-(t-s)^{\beta-2}]\\&=(\beta-1)t^{\beta-2}[(1-s)^{\beta-1}-(1-\frac{s}{t})^{\beta-2}]\le 0,\end{aligned}$$

which means that $h_1(t,s)$ is nonincreasing with respect to t for $0\le s\le t\le 1$. It is easily to see that $h_2(t,s)$ is nondecreasing with respect to t for $0\le t\le s\le 1$. Thus

$$H(t,s)\le H(s,s)\text{for } 0\le s\le t\le 1$$

and

$$H(t,s)\le H(s,s)\quad\text{for } 0\le t\le s\le 1.$$

From the above discussion,

$$H(t,s)\le H(s,s)\quad\text{for } (t,s)\in[0,1]\times[0,1].$$

So property (iv) holds.

Let's now show that (v) is true. First, $g_1(t,s)$, $g_2(t,s)$ are nonincreasing with respect to $t\in[s,1]$, and $g_3(t,s)$, $g_4(t,s)$ are nondecreasing with respect to $t\in[0,s]$, so there is

$$\begin{aligned}\min_{\xi\le t\le 1}G(t,s)&=\begin{cases}\min_{\xi\le t\le 1}\{g_1(t,s),g_3(t,s)\}, & 0\le s<\xi,\\ \min_{\xi\le t\le 1}\{g_2(t,s),g_4(t,s)\}, & \xi\le s<1,\end{cases}\\&=\begin{cases}g_1(t,s), & 0\le s<\xi,\\ \lambda_1(s), & \xi\le s<1,\end{cases}\end{aligned}$$

where $\lambda_1(s)=\min\{g_2(1,s),g_4(\xi,s)\}$, $\lambda_1(s)\in C(\xi,1)$ and $\lambda_1(s)>0$. Let

$$\gamma(s)=\begin{cases}\frac{g_1(t,s)}{G(s,s)}, & 0\le s<\xi,\\ \frac{\lambda_1(s)}{G(s,s)}, & \xi\le s<1,\end{cases}$$

where

$$G(s,s)=\begin{cases}\frac{[s(1-s)]^{\alpha-n+1}-a[s(\xi-s)]^{\alpha-n+1}}{(1-a\xi^{\alpha-n+1})\Gamma(\alpha-n+2)}, & 0\le s<\xi,\\ \frac{[s(1-s)]^{\alpha-n+1}}{(1-a\xi^{\alpha-n+1})\Gamma(\alpha-n+2)}, & \xi\le s<1,\end{cases}$$

From above discussions,

$$\min_{t\in[\xi,1]}G(t,s)\ge\gamma(s)\max_{t\in[0,1]}G(t,s)=\gamma(s)G(s,s),\quad 0<s<1.$$

Second, $h_1(t,s)$ is nonincreasing with respect to t on $[s,1]$, and $h_2(t,s)$ is nondecreasing with respect to t on $[0,s]$, one has

$$\begin{aligned}\min_{\xi\le t\le 1}H(t,s)&=\begin{cases}\min_{\xi\le t\le 1}\{h_1(t,s),h_2(t,s)\}, & 0\le s<\xi,\\ \min_{\xi\le t\le 1}\{h_1(t,s),h_2(t,s)\}, & \xi\le s<1,\end{cases}\\&=\begin{cases}h_1(t,s), & 0\le s<\xi,\\ \lambda_2(s), & \xi\le s<1,\end{cases}\end{aligned}$$

where $\lambda_2(s) = \min\{h_1(1,s), h_2(\xi,s)\}$, $\lambda_2(s) \in C(\xi,1)$ and $\lambda_2(s) > 0$. Let

$$\rho(s) = \begin{cases} \frac{h_1(t,s)}{H(s,s)}, & 0 \le s < \xi, \\ \frac{\lambda_2(s)}{H(s,s)}, & \xi \le s < 1, \end{cases}$$

where $H(s,s) = \frac{1}{\Gamma(\beta)}[s(1-s)]^{\beta-1}$. Therefore, we obtain that

$$\min_{s\in[\xi,1]} H(t,s) \ge \rho(s) \max_{t\in[0,1]} H(t,s) = \rho(s)H(s,s), \quad 0 < s < 1.$$

The proof is complete. $\square$

5. Existence of a unique positive solution

In this section, we discuss the existence of a unique positive solution to fractional boundary-value problem (1.3) by using the mixed monotone fixed point theorem. We need the following assumptions:

- (H1) Let $f(t,x_1,x_2,\dots,x_{n-1}) = g(t,x_1,x_2,\dots,x_{n-1}) + h(t,x_1,x_2,\dots,x_{n-1})$, where $g:(0,1)\times[0,+\infty)\times\mathbb{R}^{n-1}\to[0,+\infty)$ and $h:(0,1)\times[0,+\infty)\times(\mathbb{R}/0)^{n-2}\to[0,+\infty)$ are continuous;
- (H2) $g(t,x_1,x_2,\dots,x_{n-1})$ is nondecreasing in t and x_i and $h(t,x_1,x_2,\dots,x_{n-1})$ is nonincreasing in t and x_i, where $t, x_i \in (0,1)\times[0,+\infty)\times(\mathbb{R}/0)^{n-2}, i = 1,2,\dots,n-1$;
- (H3) There exists a constant $b\in(0,1)$ such that
$$g(t,kx_1,kx_2,\dots,kx_{n-1}) \ge k^b g(t,x_1,x_2,\dots,x_{n-1}), \quad k\in(0,1),$$
$$h(t,k^{-1}x_1,k^{-1}x_2,\dots,k^{-1}x_{n-1}) \ge k^b h(t,x_1,x_2,\dots,x_{n-1}), k\in(0,1).$$
where $x_i > 0$ and $i = 1,2,\dots,n-1$;
- (H4) $k^r:[0,1]\to[0,+\infty)$ is continuous and $\int_0^1 \phi_q(s^{-b(\alpha-1)})ds < +\infty$, where $0 \le r < 1$.

Let us denote $E_1 = C(0,1)$ equipped with the norm $||v|| = \sup_{t\in[0,1]}|v(t)|$, then E_1 is a Banach space. Let P be a normal cone of E_1 defined by

$$P = \big\{v\in E_1 : v(t)\ge 0, t\in[0,1]\big\}.$$

Define

$$Q_e = \Big\{v\in P : \frac{1}{M}e(t) \le v(t) \le Me(t), t\in[0,1]\Big\},$$

where $e(t) = t^{\alpha-n+1}$, M is a positive constant defined by

$$\begin{aligned} M > \min\Big\{1, \Big[&\frac{(g(1,1,\dots,1))^{q-1}\int_0^1\phi_q(s^{\beta-1})ds}{\Gamma(\alpha-n+1)(\Gamma(\beta))^{q-1}} \\ &+ \frac{(\zeta^{-b}h(0,1,1,\dots,1))^{q-1}\int_0^1\phi_q(s^{\beta-1}\int_0^1\tau^{-b(\alpha-1)}d\tau)ds}{\Gamma(\alpha-n+1)(\Gamma(\beta))^{q-1}}\Big]^{\frac{1}{b(1-q)}}, \\ &\Big[(h(1,1,1,\dots,1))^{q-1}\int_\xi^1\gamma(s)G(s,s)\phi_q\Big(\int_\xi^1\rho(\tau)H(\tau,\tau)\tau^{b(\alpha-1)}\Big)d\tau)ds \\ &+ (h(1,1,1,\dots,1))^{q-1}\int_\xi^1\gamma(s)G(s,s)\phi_q\Big(\int_\xi^1\rho(\tau)H(\tau,\tau)d\tau\Big)ds\Big]^{\frac{1}{b(1-q)}}\Big\}, \end{aligned}$$

where $0<\zeta<\min\big\{1,\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)},\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha-1)},\dots,\frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)-n+3}\big\}$. It is easy to obtain that $e\in P$ and $\|e\|=1$, $e\neq\theta$. The operator T is defined by

$$\begin{aligned} T(v,w)(t)=\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)(g(\tau,I_{0^+}^{n-2}v(\tau),,\dots,I_{0^+}^1v(\tau),v(\tau))\\ +h(\tau,I_{0^+}^{n-2}w(\tau),,\dots,I_{0^+}^1w(\tau),w(\tau))d\tau\Big)ds,\quad t\in(0,1). \end{aligned}$$

Theorem 5.1. *Assume that* (H1)–(H4) *are satisfied. Then the fractional boundary-value problem* (1.3) *has a unique positive solution.*

Proof. By the definition of the operator T and its properties, it suffices to show that all conditions of Lemma 2.8 are satisfied with respect to t.

Firstly. we show that $T:Q_e\times Q_e\to Q_e$. Let $x_i=1$, Assumption (H3) implies that

$$g(t,k,k,\dots,k)\geq k^b g(t,1,1,\dots,1),\quad k\in(0,1).$$

Set $\bar{x}:=x_1=x_2=\dots=x_{n-1}$, and $k=\frac{1}{\bar{x}}$, $\bar{x}>1$, one has

$$g(t,\bar{x},\dots,\bar{x})\leq\bar{x}^b g(t,1,\dots,1),\ \bar{x}>1.$$

Similarly, from (H3), for $x_i>0$, if we let $k^{-1}x_i=y_i$, $i=1,2,\dots,n-1$, then

$$h(t,y_1,\dots,y_{n-1})\geq k^b h(t,ky_1,\dots,ky_{n-1}),\quad k\in(0,1),\ y_i>0,\ i=1,2,\dots,n-1.$$

Now, let $y_i=1, i=1,2,\dots,n-1$, we obtain

$$h(t,1,\dots,1)\geq k^b h(t,k,\dots,k),\quad k\in(0,1).$$

From the above discussion, we have

$$\begin{gathered} h(t,k^{-1},\dots,k^{-1})\geq k^b h(t,1,\dots,1),\\ h(t,ky_1,\dots,ky_{n-1})\leq k^{-b}h(t,y_1,\dots,y_{n-1}),\\ h(t,k,\dots,k)\leq k^{-b}h(t,1,\dots,1), \end{gathered}$$

where $k\in(0,1)$, $y_i>0$, $i=1,2,\dots,n-1$.

Since $v\in Q_e$ and the monotonicity of Riemann-Liouville fractional integral $I_{0^+}^{\delta}$, we obtain that

$$\begin{aligned} &I_{0^+}^{n-2}v(t)>0,\ I_{0^+}^{n-3}v(t)>0,\ \dots,\ I_{0^+}^1v(t)>0,v(t)>0,\\ &g\Big(t,I_{0^+}^{n-2}v(t),I_{0^+}^{n-3}v(t),\dots,I_{0^+}^1v(t),v(t)\Big)\\ &\leq g\Big(t,I_{0^+}^{n-2}Me(t),I_{0^+}^{n-3}Me(t),\dots,I_{0^+}^1Me(t),Me(t)\Big)\\ &\leq g\Big(t,I_{0^+}^{n-2}M,I_{0^+}^{n-3}M,\dots,I_{0^+}^1M,M\Big)\\ &=g\Big(t,\frac{M}{(n-2)!}t^{n-2},\frac{M}{(n-3)!}t^{n-3},\dots,Mt,M\Big)\\ &\leq g(t,M,M,\dots,M,M)\\ &\leq M^b g(t,1,1,\dots,1,1)\\ &\leq M^b g(1,1,1,\dots,1,1), \end{aligned}$$

and

$$h\Big(t,I_{0^+}^{n-2}w(t),I_{0^+}^{n-3}w(t),\dots,I_{0^+}^1w(t),w(t)\Big)$$

$$
\begin{aligned}
&\le h\Big(t, I_{0^+}^{n-2}\frac{1}{M}e(t), I_{0^+}^{n-3}\frac{1}{M}e(t), \dots, I_{0^+}^{1}\frac{1}{M}e(t), \frac{1}{M}e(t)\Big)\\
&= h\Big(t, \frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha)}t^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha-1)}t^{\alpha-2}, \dots,\\
&\quad \frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha-n+3)}t^{\alpha-n+2}, \frac{1}{M}t^{\alpha-n+1}\Big)\\
&\le h\Big(t, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}, \dots, \frac{\zeta}{M}t^{\alpha-n+3}, \frac{\zeta}{M}t^{\alpha-n+2}\Big)\\
&\le h\Big(t, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}, \dots, \frac{\zeta}{M}t^{\alpha-n+4}, \frac{\zeta}{M}t^{\alpha-n+3}\Big)\\
&\le \dots\\
&\le h\Big(t, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}, \dots, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}\Big)\\
&\le (\frac{\zeta}{M})^{-b}t^{-b(\alpha-1)}h(t,1,1,\dots,1,1)\\
&\le (\frac{\zeta}{M})^{-b}t^{-b(\alpha-1)}h(0,1,1,\dots,1,1),
\end{aligned}
$$

where

$$
0<\zeta<\min\Big\{1, \frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)}, \frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha-1)}, \dots, \frac{\Gamma(\alpha-n+2)}{\Gamma(\alpha)-n+3}\Big\},
$$

for $0<\frac{\zeta}{M}t^{\alpha-1}<1$. We also obtain

$$
\begin{aligned}
&g\Big(t, I_{0^+}^{n-2}v(t), I_{0^+}^{n-3}v(t), \dots, I_{0^+}^{1}v(t), v(t)\Big)\\
&\ge g\Big(t, I_{0^+}^{n-2}\frac{1}{M}e(t), I_{0^+}^{n-3}\frac{1}{M}e(t), \dots, I_{0^+}^{1}\frac{1}{M}e(t), \frac{1}{M}e(t)\Big)\\
&= g\Big(t, \frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha)}t^{\alpha-1}, \frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha-1)}t^{\alpha-2}, \dots,\\
&\quad \frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha-n+3)}t^{\alpha-n+2}, \frac{1}{M}t^{\alpha-n+1}\Big)\\
&\ge g\Big(t, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}, \dots, \frac{\zeta}{M}t^{\alpha-n+3}, \frac{\zeta}{M}t^{\alpha-n+2}\Big)\\
&\ge g\Big(t, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}, \dots, \frac{\zeta}{M}t^{\alpha-n+4}, \frac{\zeta}{M}t^{\alpha-n+3}\Big)\\
&\ge \dots\\
&\ge g\Big(t, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}, \dots, \frac{\zeta}{M}t^{\alpha-1}, \frac{\zeta}{M}t^{\alpha-1}\Big)\\
&\ge (\frac{\zeta}{M})^{-b}t^{b(\alpha-1)}g(t,1,1,\dots,1,1)\\
&\ge (\frac{\zeta}{M})^{-b}t^{b(\alpha-1)}g(0,1,1,\dots,1,1),
\end{aligned}
$$

where $0<\frac{\zeta}{M}t^{\alpha-1}<1$, and

$$
\begin{aligned}
&h\Big(t, I_{0^+}^{n-2}w(t), I_{0^+}^{n-3}w(t), \dots, I_{0^+}^{1}w(t), w(t)\Big)\\
&\ge h\Big(t, I_{0^+}^{n-2}Me(t), I_{0^+}^{n-3}Me(t), \dots, I_{0^+}^{1}Me(t), Me(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\geq h\Big(t, I_{0^+}^{n-2}M, I_{0^+}^{n-3}M, \dots, I_{0^+}^{1}M, M\Big)\\
&= h\Big(t, \frac{M}{(n-2)!}t^{n-2}, \frac{M}{(n-3)!}t^{n-3}, \dots, Mt, M\Big)\\
&\geq h(t, M, M, \dots, M, M)\\
&\geq M^{-b}h(t, 1, 1, \dots, 1, 1)\\
&\geq M^{-b}h(1, 1, 1, \dots, 1, 1).
\end{aligned}
$$

From the above and Lemma 4.4 it follows that $T(v,w) \in C([0,1],[0,+\infty))$. Then

$$
\begin{aligned}
&\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^{1}v(\tau), v(\tau))d\tau\Big)ds\\
&\leq \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\int_0^1 \phi_q\Big(\frac{s^{\beta-1}}{\Gamma(\beta)}\int_0^1 g(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^{1}v(\tau), v(\tau))d\tau\Big)ds\\
&\leq \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\int_0^1 \phi_q\Big(\frac{s^{\beta-1}M^b g(1,1,\dots,1)}{\Gamma(\beta)}\Big)ds\\
&\leq \frac{t^{\alpha-n+1}(M^b g(1,1,\dots,1))^{q-1}}{\Gamma(\alpha-n+1)(\Gamma(\beta))^{q-1}}\int_0^1 \phi_q(s^{\beta-1})ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau, I_{0^+}^{n-2}w(\tau), \dots, I_{0^+}^{1}w(\tau), w(\tau))d\tau\Big)ds\\
&\leq \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\int_0^1 \phi_q\Big(\frac{s^{\beta-1}}{\Gamma(\beta)}\int_0^1 h(\tau, I_{0^+}^{n-2}w(\tau), \dots, I_{0^+}^{1}w(\tau), w(\tau))d\tau\Big)ds\\
&\leq \frac{t^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\int_0^1 \phi_q\Big(\frac{s^{\beta-1}}{\Gamma(\beta)}\int_0^1 (\frac{\zeta}{M})^{-b}\tau^{-b(\alpha-1)}h(0,1,1,\dots,1)d\tau\Big)ds\\
&\leq \frac{t^{\alpha-n+1}(\zeta^{-b}M^b h(0,1,1,\dots,1))^{q-1}}{\Gamma(\alpha-n+1)(\Gamma(\beta))^{q-1}}\int_0^1 \phi_q\Big(s^{\beta-1}\int_0^1 \tau^{-b(\alpha-1)}d\tau\Big)ds.
\end{aligned}
$$

Then

$$
T(v,w)(t) \leq Mt^{\alpha-n+1} = Me(t), \quad t \in (0,1).
$$

From the inequalities

$$
\begin{aligned}
&\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^{1}v(\tau), v(\tau))d\tau\Big)ds\\
&\geq \int_\xi^1 G(t,s)\phi_q\Big(\int_\xi^1 H(s,\tau)g(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^{1}v(\tau), v(\tau))d\tau\Big)ds\\
&\geq \int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)g(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^{1}v(\tau), v(\tau))d\tau\Big)ds\\
&\geq \int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)(\frac{\zeta}{M})^b\tau^{b(\alpha-1)}g(0,1,1,\dots,1)d\tau\Big)ds\\
&= \big(\zeta^b M^{-b}g(0,1,1,\dots,1)\big)^{q-1}\int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)\tau^{b(\alpha-1)}d\tau\Big)ds\\
&\geq t^{\alpha-n+1}\big(\zeta^b M^{-b}g(0,1,1,\dots,1)\big)^{q-1}
\end{aligned}
$$

$$\times \int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)\tau^{b(\alpha-1)}d\tau\Big)ds.$$

and

$$\begin{aligned}
&\int_0^1 G(t,s)\phi_q(\int_0^1 H(s,\tau)h\Big(\tau, I_{0^+}^{n-2}w(\tau),\dots,I_{0^+}^1 w(\tau),w(\tau)\Big)d\tau)ds\\
&\geq \int_\xi^1 G(t,s)\phi_q\Big(\int_\xi^1 H(s,\tau)h(\tau, I_{0^+}^{n-2}w(\tau),\dots,I_{0^+}^1 w(\tau),w(\tau))d\tau\Big)ds\\
&\geq \int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)h(\tau, I_{0^+}^{n-2}w(\tau),\dots,I_{0^+}^1 w(\tau),w(\tau))d\tau\Big)ds\\
&\geq \int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)M^{-b}h(1,1,1,\dots,1)d\tau\Big)ds\\
&\geq \big(M^{-b}h(1,1,1,\dots,1)\big)^{q-1}\int_\xi^1 \gamma(s)G(s,s)\phi_q(\int_\xi^1 \rho(\tau)H(\tau,\tau)d\tau)ds\\
&\geq t^{\alpha-n+1}\big(M^{-b}h(1,1,1,\dots,1)\big)^{q-1}\int_\xi^1 \gamma(s)G(s,s)\phi_q\Big(\int_\xi^1 \rho(\tau)H(\tau,\tau)d\tau\Big)ds,
\end{aligned}$$

we deduce that

$$T(v,w)(t) \geq \frac{1}{M}t^{\alpha-n+1} = \frac{1}{M}e(t), \quad t\in(0,1).$$

Therefore, we concluded that $T: Q_e\times Q_e \to Q_e$.

Secondly, we prove that $T: Q_e\times Q_e\to Q_e$ is a mixed monotone operator. Let $v_1,v_2\in Q_e$ and $v_1\leq v_2$, we obtain

$$\begin{aligned}
&\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau, I_{0^+}^{n-2}v_1(\tau),\dots,I_{0^+}^1 v_1(\tau),v_1(\tau))d\tau\Big)ds\\
&\leq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g(\tau, I_{0^+}^{n-2}v_2(\tau),\dots,I_{0^+}^1 v_2(\tau),v_2(\tau))d\tau\Big)ds,
\end{aligned}$$

i.e.,

$$T(v_1,w)(t)\leq T(v_2,w)(t), \quad w\in Q_e. \tag{5.1}$$

Thus $T(v,w)(t)$ is nondecreasing in v for any $w\in Q_e$.

Let $w_1,w_2\in Q_e$ and $w_1\geq w_2$. Then

$$\begin{aligned}
&\int_0^1 G(t,s)\phi_q(\int_0^1 H(s,\tau)h(\tau, I_{0^+}^{n-2}w_1(\tau),\dots,I_{0^+}^1 w_1(\tau),w_1(\tau))d\tau)ds\\
&\leq \int_0^1 G(t,s)\phi_q(\int_0^1 H(s,\tau)h(\tau, I_{0^+}^{n-2}w_2(\tau),\dots,I_{0^+}^1 w_2(\tau),w_2(\tau))d\tau)ds,
\end{aligned}$$

i.e.,

$$T(v,w_1)(t)\leq T(v,w_2)(t), \quad w\in Q_e. \tag{5.2}$$

Therefore $T(v,w)(t)$ is nonincreasing in w for any $v\in Q_e$. Consequently, according to (5.1) and (5.2), we conclude that the operator $T: Q_e\times Q_e\to Q_e$ is a mixed monotone operator.

Finally, we show that the operator T has a fixed point. If $v,w\in Q_e$, it follows from (H3) that

$$\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g\Big(\tau, I_{0^+}^{n-2}tv(\tau),\dots,I_{0^+}^1 tv(\tau),tv(\tau)\Big)d\tau\Big)ds$$

$$
\begin{aligned}
&= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g\Big(\tau, tI_{0^+}^{n-2}v(\tau),\dots,tI_{0^+}^1 v(\tau),tv(\tau)\Big)d\tau\Big)ds\\
&\geq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)t^b g\Big(\tau, I_{0^+}^{n-2}v(\tau),\dots,I_{0^+}^1 v(\tau),v(\tau)\Big)d\tau\Big)ds\\
&\geq t^b\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g\Big(\tau, I_{0^+}^{n-2}v(\tau),\dots,I_{0^+}^1 v(\tau),v(\tau)\Big)d\tau\Big)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau, I_{0^+}^{n-2}t^{-1}w(\tau),\dots,I_{0^+}^1 t^{-1}w(\tau),t^{-1}w(\tau))d\tau\Big)ds\\
&= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau, t^{-1}I_{0^+}^{n-2}w(\tau),\dots,t^{-1}I_{0^+}^1 w(\tau),t^{-1}w(\tau))d\tau\Big)ds\\
&\geq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)t^b h(\tau, I_{0^+}^{n-2}w(\tau),\dots,I_{0^+}^1 w(\tau),w(\tau))d\tau\Big)ds\\
&\geq t^b\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)h(\tau, I_{0^+}^{n-2}w(\tau),\dots,I_{0^+}^1 w(\tau),w(\tau))d\tau\Big)ds,
\end{aligned}
$$

we obtain

$$
T\Big(tx,\frac{1}{t}y\Big)\geq t^b T(x,y),\quad x,y\in Q_e,\ t\in(0,1),\ b\in(0,1).
$$

Therefore, from Lemma 2.8 it follows that the operator T has a fixed point. That is to say, the fractional differential equation (3.1) has a unique positive solution $v(t)$, $v\in Q_e$. By Lemma 3.1, we know that the fractional boundary-value problem (1.3) has a unique positive solution $u(t)$, such that

$$
\begin{aligned}
\frac{\Gamma(\alpha-n+2)}{M\Gamma(\alpha)}t^{\alpha-1} &= \frac{1}{M}I_{0^+}^{n-2}e(t)\leq u(t)\\
&\leq MI_{0^+}^{n-2}e(t)=\frac{M\Gamma(\alpha-n+2)}{\Gamma(\alpha)}t^{\alpha-1},\quad t\in(0,1).
\end{aligned}
$$

The proof is complete. □

Now we introduce the following assumptions:

(H5) $f(t,x_1,x_2,\dots,x_{n-1})=g(t,x_1,x_2,\dots,x_{n-1})\times h(t,x_1,x_2,\dots,x_{n-1})$, where $g:(0,1)\times[0,+\infty)\times\mathbb{R}^{n-1}\to[0,+\infty)$ and $h:(0,1)\times[0,+\infty)\times(\mathbb{R}/0)^{n-2}\to[0,+\infty)$ are continuous;

(H6) For $x_i>0$, $i=1,2,\dots,n-1$, there exist constants $b_1,b_2>0$, $0<b_1+b_2<1$, such that

$$
g(t,kx_1,kx_2,\dots,kx_{n-1})\geq k^{b_1}g(t,x_1,x_2,\dots,x_{n-1}),k\in(0,1),
$$

$$
h(t,k^{-1}x_1,k^{-1}x_2,\dots,k^{-1}x_{n-1})\geq k^{b_2}h(t,x_1,x_2,\dots,x_{n-1}),k\in(0,1);
$$

Corollary 5.2. *Assume that* (H2), (H4), (H5) *and* (H6) *are satisfied, then the fractional boundary-value problem* (1.3) *has a unique positive solution.*

The proof is done in the same way as the proof of Theorem 5.1; we omit it.

6. Existence of at least one positive solution

In this section, we show the existence of at least one positive solution to the fractional boundary-value problem (1.3) by using the upper and lower solution method.

Let $E_2 = \{v : v(t) \in C^2[0,1]$ and $\phi_p(D_{0^+}^{\alpha-n+2}v(t)) \in C^2[0,1]\}$ denote the Banach space endowed with the norm $\| v \| = \max_{t\in[0,1]}\{\sup|v(t)|, \sup|\phi_p(D_{0^+}^{\alpha-n+2}v(t)|\}$.

The operator F is defined by

$$(Fv)(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^1 v(\tau), v(\tau))d\tau\Big)ds,$$

for $t \in (0,1)$. Now we have two definitions on the lower and upper solutions of the fractional differential equation (3.1).

Definition 6.1. A function $m(t)$ is called a lower solution of the fractional differential equation (3.1), if $m(t) \in E_2$, and $m(t)$ satisfies

$$\begin{gathered} D_{0^+}^{\beta}(\phi_p D_{0^+}^{\alpha-n+2}m(t)) \geq f(t, I_{0^+}^{n-2}m(t), I_{0^+}^{n-3}m(t), \dots, I_{0^+}^1 m(t), m(t)), \quad t \in (0,1), \\ m(0) \leq 0, m(1) \leq am(\xi), D_{0^+}^{\alpha-n+2}m(0) \geq D_{0^+}^{\alpha-n+2}m(1). \end{gathered} \tag{6.1}$$

Definition 6.2. A function $n(t)$ is called an upper solution of the fractional differential equation (3.1), if $n(t) \in E_2$, and $n(t)$ satisfies

$$\begin{gathered} D_{0^+}^{\beta}(\phi_p D_{0^+}^{\alpha-n+2}n(t)) \leq f(t, I_{0^+}^{n-2}n(t), I_{0^+}^{n-3}n(t), \dots, I_{0^+}^1 n(t), n(t)), \quad t \in (0,1), \\ n(0) \geq 0, n(1) \geq an(\xi), D_{0^+}^{\alpha-n+2}n(0) \leq D_{0^+}^{\alpha-n+2}n(1). \end{gathered} \tag{6.2}$$

We introduce the following assumptions:

(H9) $f(t, x_1, x_2, \dots, x_{n-1}) \in C((0,1)\times(0,+\infty)^{n-1}, [0,+\infty))$ is nonincressing relative to x_i, , and there exists a constant $L_1 > 0$, such that $| f(t, x_1, x_2, \dots, x_{n-1}) | \leq L_1$, where $x_i > 0$, $i = 1, 2, \dots, n-1$;

(H10) For any constant $\mu > 0$, we have

$$0 < \int_0^1 H(t,t)f(t, I_{0^+}^{n-2}\mu t^{\alpha-n+1}, \dots, I_{0^+}^1 \mu t^{\alpha-n+1}, \mu t^{\alpha-n+1})dt < +\infty;$$

(H11) There exists a continuous function $p(t), t \in [0,1]$, such that

$$\begin{gathered} \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau, I_{0^+}^{n-2}p(\tau), \dots, I_{0^+}^1 p(\tau), p(\tau))d\tau\Big)ds = q(t) \geq p(t), \\ \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau, I_{0^+}^{n-2}q(\tau), \dots, I_{0^+}^1 q(\tau), q(\tau))d\tau\Big)ds \geq p(t). \end{gathered}$$

Theorem 6.3. *If* (H9)–(H11) *are satisfied, then problem* (1.3) *has at least one positive solution.*

Proof. We divide our proof into four steps.

Step 1. Let $M_1 := \max_{[0,1]\times[0,1]} G(t,s), M_2 := \max_{[0,1]\times[0,1]} H(t,s)$. Set $\Omega_1 = \{v \in E_2 : ||v|| \leq M_1\phi_q(M_2L_1)\}$, we prove that $F(\Omega_1) \subset \Omega_1$.

For any $v \in \Omega_1$, we obtain that

$$\begin{aligned}(Fv)(t) &= \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)f(\tau, I_{0^+}^{n-2}v(\tau),\dots,I_{0^+}^1 v(\tau), v(\tau))d\tau\Big)ds\\ &\le M_1\phi_q(M_2L_1).\end{aligned}$$

Consequently $F(\Omega_1) \subset \Omega_1$.

By computations, we have

$$\begin{gathered}\int_0^1 G(t,s)\phi_p\Big(\int_0^1 H(s,\tau)(Fv)(t)d\tau\Big)ds = f\Big(t, I_{0^+}^{n-2}v(t),\dots,I_{0^+}^1 v(t), v(t)\Big),\\ \text{for } t \in (0,1),\\ (Fv)(0) = 0, \quad (Fv)(1) = a(Fv)(\xi),\\ D_{0^+}^{\alpha-n+2}(Fv)(0) = D_{0^+}^{\alpha-n+2}(Fv)(1) = 0.\end{gathered} \tag{6.3}$$

Step 2. Set $m(t) = Fq(t), n(t) = Fp(t)$, in this step, we prove that $m(t), n(t)$ are lower and upper solutions of the fractional differential equation (3.1), respectively. From the assumptions (H9) and (H11), we obtain

$$p(t) \le q(t) = Fp(t), \quad Fq(t) \le q(t) = Fp(t), \quad t \in [0,1], \tag{6.4}$$

this means that $m(t) \le n(t)$. Since $F(\Omega_1) \subset \Omega_1$, there is $m(t), n(t) \in \Omega_1$. According to (6.1), (6.2), we have

$$\begin{gathered}D_{0^+}^{\beta}\phi_p\Big(D_{0^+}^{\alpha-n+2}m(t)\Big) - f\Big(t, I_{0^+}^{n-2}m(t), I_{0^+}^{n-3}m(t),\dots,I_{0^+}^1 m(t), m(t)\Big)\\ \le D_{0^+}^{\beta}\phi_p\Big(D_{0^+}^{\alpha-n+2}(Fq)(t)\Big) - f\Big(t, I_{0^+}^{n-2}q(t), I_{0^+}^{n-3}q(t),\dots,I_{0^+}^1 q(t), q(t)\Big) = 0,\\ m(0) = 0, \quad m(1) = am(\xi), \quad D_{0^+}^{\alpha-n+2}m(0) = D_{0^+}^{\alpha-n+2}m(1) = 0.\end{gathered} \tag{6.5}$$

and

$$\begin{gathered}D_{0^+}^{\beta}\phi_p\Big(D_{0^+}^{\alpha-n+2}n(t)\Big) - f\Big(t, I_{0^+}^{n-2}n(t), I_{0^+}^{n-3}n(t),\dots,I_{0^+}^1 n(t), n(t)\Big)\\ \ge D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha-n+2}(Fp)(t)\big) - f\Big(t, I_{0^+}^{n-2}p(t), I_{0^+}^{n-3}p(t),\dots,I_{0^+}^1 p(t), p(t)\Big) = 0,\\ n(0) = 0, \quad n(1) = an(\xi), \quad D_{0^+}^{\alpha-n+2}n(0) = D_{0^+}^{\alpha-n+2}n(1) = 0.\end{gathered} \tag{6.6}$$

Hence, $m(t)$ and $n(t)$ are lower and upper solutions of the fractional differential equation (3.1), respectively.

Step 3. Let

$$\begin{aligned}&g_1\Big(t, I_{0^+}^{n-2}v(t), I_{0^+}^{n-3}v(t),\dots,I_{0^+}^1 v(t), v(t)\Big)\\ &= \begin{cases} f\big(t, I_{0^+}^{n-2}m(t), I_{0^+}^{n-3}m(t),\dots,I_{0^+}^1 m(t), m(t)\big), & v(t) < m(t),\\ f\big(t, I_{0^+}^{n-2}v(t), I_{0^+}^{n-3}v(t),\dots,I_{0^+}^1 v(t), v(t)\big), & m(t) \le v(t) \le n(t),\\ f\big(t, I_{0^+}^{n-2}n(t), I_{0^+}^{n-3}n(t),\dots,I_{0^+}^1 n(t), n(t)\big), & v(t) > n(t).\end{cases}\end{aligned} \tag{6.7}$$

Consider the fractional differential equation

$$\begin{gathered}D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha-n+2}n(t)\big) + g_1\big(t, I_{0^+}^{n-2}v(t), I_{0^+}^{n-3}v(t),\dots,I_{0^+}^1 v(t), v(t)\big) = 0,\\ 0 < t < 1,\\ v(0) = 0, \quad v(1) = av(\xi), \quad D_{0^+}^{\alpha-n+2}v(0) = D_{0^+}^{\alpha-n+2}v(1) = 0.\end{gathered} \tag{6.8}$$

Set $\Omega_2 = \{v \in E_2 : ||v|| \leq M_1\phi_q(M_2L_2)\}$, then Ω_2 is a closed, bounder and convex set, where

$$L_2 := \sup_{t\in[0,1],v\in\Omega_2} |g_1(t, I_{0^+}^{n-2}v(t), \dots, I_{0^+}^1 v(t), v(t))| + 1.$$

The operator $A : \Omega_2 \to E_2$ is defined by

$$Av(t) = \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g_1\big(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^1 v(\tau), v(\tau)\big)d\tau\Big)ds,$$

where $G(t,s)$ and $H(s,\tau)$ are defined in Lemma 4.1 and Lemma 4.3. From Lemma 4.2, if $v \in \Omega_2$, we have $Av(t) \geq 0$, and the fixed point of the operator A is the solution of the fractional differential equation (6.8).

Now, we show that A is a completely continuous operator. Let $v \in \Omega_2$, it follow from the nonnegative and continuity of $G(t,s), H(t,s)$ and Lemma 4.4 that $A : \Omega_2 \to \Omega_2$ is continuous.

For any $v \in \Omega_2$,

$$\begin{aligned}
&|(Av)(t)| \\
&= \Big|\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)g_1(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^1 v(\tau), v(\tau))d\tau\Big)ds\Big| \\
&\leq \int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)|g_1(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^1 v(\tau), v(\tau))|d\tau\Big)ds \\
&\leq L_2^{q-1}\int_0^1 G(t,s)\phi_q\Big(\int_0^1 H(s,\tau)d\tau\Big)ds \\
&\leq L_2^{q-1}\int_0^1 G(s,s)\phi_q\Big(\int_0^1 H(\tau,\tau)d\tau\Big)ds \\
&< +\infty,
\end{aligned}$$

which means that A is uniformly bounded.

The function $G(t,s)$ is a continuous function for $t,s \in [0,1]\times[0,1]$; then it is uniformly continuous for $t,s \in [0,1]\times[0,1]$. Hence for fixed $s \in [0,1]$ and any $\varepsilon > 0$, there exists a constant $\delta > 0$, such that

$$G(t_1,s) - G(t_2,s) < \frac{\varepsilon}{L_2^{q-1}\phi_q(\int_0^1 H(\tau,\tau)d\tau)},$$

where $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$. Therefore

$$\begin{aligned}
&|Av(t_2) - Av(t_1)| \\
&\leq \int_0^1 |G(t_2,s) - G(t_1,s)|\phi_q\Big(\int_0^1 H(s,\tau)g_1(\tau, I_{0^+}^{n-2}v(\tau), \dots, I_{0^+}^1 v(\tau), v(\tau))d\tau\Big)ds \\
&\leq L_2^{q-1}\int_0^1 |G(t_2,s) - G(t_1,s)|\phi_q(\int_0^1 H(\tau,\tau)d\tau)ds \\
&\leq L_2^{q-1}\phi_q(\int_0^1 H(\tau,\tau)d\tau)\int_0^1 |G(t_2,s) - G(t_1,s)|ds \\
&< \varepsilon \quad \text{for any } v \in \Omega_2,
\end{aligned}$$

which implies that A is equicontinuous. It follows from the Arzela-Ascoli theorem that the operator $A : \Omega_2 \to \Omega_2$ is completely continuous.

Similar to Step 1, we have $A(\Omega_2) \subset \Omega_2$. According to the Schauder's fixed point theorem, the operator A has a fixed point, that is to say, the fractional differential equation (6.8) has a positive solution.

Step 4. We prove that the fractional differential equation (1.3) has at least one positive solution. Suppose that $d(t)$ is a solution of (6.8), then

$$d(0) = 0, \quad d(1) = ad(\xi), \quad D_{0^+}^{\alpha-n+2}d(0) = D_{0^+}^{\alpha-n+2}d(1) = 0.$$

From (H9), we know that

$$\begin{aligned}
&f\Big(t, I_{0^+}^{n-2}n(t), I_{0^+}^{n-3}n(t), \dots, I_{0^+}^{1}n(t), n(t)\Big) \\
&\le g_1\Big(t, I_{0^+}^{n-2}d(t), I_{0^+}^{n-3}d(t), \dots, I_{0^+}^{1}d(t), d(t)\Big) \\
&\le f(t, I_{0^+}^{n-2}m(t), I_{0^+}^{n-3}m(t), \dots, I_{0^+}^{1}m(t), m(t)), \quad t \in [0,1].
\end{aligned}$$

According to (H11) and (6.4),

$$\begin{aligned}
&f\Big(t, I_{0^+}^{n-2}q(t), I_{0^+}^{n-3}q(t), \dots, I_{0^+}^{1}q(t), q(t)\Big) \\
&\le g_1\Big(t, I_{0^+}^{n-2}d(t), I_{0^+}^{n-3}d(t), \dots, I_{0^+}^{1}d(t), d(t)\Big) \\
&\le f\Big(t, I_{0^+}^{n-2}p(t), I_{0^+}^{n-3}p(t), \dots, I_{0^+}^{1}p(t), p(t)\Big), \quad t \in [0,1].
\end{aligned}$$

It follows from $p(t) \in \Omega_2$ and (6.3) that

$$\begin{aligned}
D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha-n+2}n(t)\big) &= D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha-n+2}(Fp)(t)\big) \\
&= f\big(t, I_{0^+}^{n-2}p(t), I_{0^+}^{n-3}p(t), \dots, I_{0^+}^{1}p(t), p(t)\big), \quad t \in [0,1].
\end{aligned}$$

From the above discussions, we obtain

$$\begin{aligned}
&D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha-n+2}n(t)\big) - D_{0^+}^{\beta}\phi_p\big(D_{0^+}^{\alpha-n+2}d(t)\big) \\
&= f\Big(t, I_{0^+}^{n-2}p(t), I_{0^+}^{n-3}p(t), \dots, I_{0^+}^{1}p(t), p(t)\Big) \\
&\quad - g_1\Big(t, I_{0^+}^{n-2}d(t), I_{0^+}^{n-3}d(t), \dots, I_{0^+}^{1}d(t), d(t)\Big) \\
&\ge 0, \quad t \in [0,1].
\end{aligned}$$

and

$$\begin{gathered}
(n-d)(0) = 0, \quad (n-d)(1) = a(n-d)(\xi), \\
D_{0^+}^{\alpha-n+2}(n-d)(0) = D_{0^+}^{\alpha-n+2}(n-d)(1) = 0.
\end{gathered}$$

If we let $z(t) = \phi_p(D_{0^+}^{\alpha-n+2}n(t)) - \phi_p(D_{0^+}^{\alpha-n+2}d(t))$, then $z(0) = z(1) = 0$. By Lemma 4.2, we have $z(t) \le 0$. Hence,

$$\phi_p(D_{0^+}^{\alpha-n+2}n(t)) \le \phi_p(D_{0^+}^{\alpha-n+2}d(t)), \quad t \in [0,1].$$

Since ϕ_p is monotone increasing,

$$D_{0^+}^{\alpha-n+2}n(t) \le D_{0^+}^{\alpha-n+2}d(t),$$

that is

$$D_{0^+}^{\alpha-n+2}(n-d)(t) \le 0.$$

By Lemma 4.2, we have $(n-d)(t) \ge 0$. Thus we conclude that $n(t) \ge d(t), t \in [0,1]$.

In the same way, it is easy to prove that $m(t) \le d(t), t \in [0,1]$. Hence $d(t)$ is a positive solution of the boundary-value problem (3.1), that is, Lemma 3.1 imolies that $u(t) = I_{0^+}^{n-2}d(t)$ is a positive solution of boundary-value problem (1.3). The proof is complete. $\square$

7. Examples

In this section, we give two simple examples to illustrate our main results.

Example 7.1. We consider the boundary-value problem

$$\begin{gathered} D_{0^+}^{4/3}\phi_p(D_{0^+}^{7/2}u(t)) + t(u^{1/4}(t) + (u')^{1/2}(t) + (u'')^{3/4}(t)) \\ + t^{-1/2}(u^{-1/4}(t) + (u')^{-1/2}(t) + (u'')^{-3/4}(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, u''(1) = au''(\xi), D_{0^+}^{7/2}u(0) = D_{0^+}^{7/2}u(1) = 0. \end{gathered} \tag{7.1}$$

Proof. Set

$$\begin{aligned} g(t,u(t),u'(t),u''(t)) &= t(u^{1/4}(t) + (u')^{1/2}(t) + (u'')^{3/4}(t)), \\ h(t,u(t),u'(t),u''(t)) &= t^{-1/2}(u^{-1/4}(t) + (u')^{-1/2}(t) + (u'')^{-3/4}(t)). \end{aligned}$$

where $0 < t < 1$ and $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$. Obviously, $g(t,u(t),u'(t),u''(t))$ is nondecreasing relative to t and $u(t),u'(t),u''(t)$, and $h(t,u(t),u'(t),u''(t))$ is non-increasing relative to t and $u(t),u'(t),u''(t)$.

By $k \in (0,1)$ and $u(t) > 0, u'(t) > 0, u''(t) > 0$, we have

$$\begin{aligned} g(t,ku(t),ku'(t),ku''(t)) &= t(k^{1/4}u^{1/4}(t) + k^{1/2}(u')^{1/2}(t) + k^{3/4}(u'')^{3/4}(t)) \\ &\ge tk^{3/4}(u^{1/4}(t) + (u')^{1/2}(t) + (u'')^{3/4}(t)) \\ &= k^{3/4}g(t,u(t),u'(t),u''(t)), \end{aligned}$$

and

$$\begin{aligned} &h(t,k^{-1}u(t),k^{-1}u'(t),k^{-1}u''(t)) \\ &= t^{-1/2}(k^{1/4}u^{-1/4}(t) + k^{1/2}(u')^{-1/2}(t) + k^{3/4}(u'')^{-3/4}(t)) \\ &\le t^{-1/2}k^{3/4}(u^{-1/4}(t) + (u')^{-1/2}(t) + (u'')^{-3/4}(t)) \\ &= k^{3/4}h(t,u(t),u'(t),u''(t)). \end{aligned}$$

From above discussion, the assumptions (H1)–(H4) are satisfied. It follows from Theorem 5.1 that the boundary-value problem (7.1) has a unique positive solution. The proof is complete. $\square$

Next we give an example which is difficult to obtain the existence of positive solution to the fractional boundary-value problem (7.2) by using Theorem 5.1.

Example 7.2. We consider the boundary-value problem

$$\begin{gathered} D_{0^+}^{7/3}\phi_p\big(D_{0^+}^{7/2}u(t)\big) = t^2 + \frac{u(t)}{2} + \frac{u'(t)}{3} + \frac{u''(t)}{4}, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \frac{1}{2}u''(\frac{1}{3}), \quad D_{0^+}^{7/2}u(0) = D_{0^+}^{7/2}u(1) = 0. \end{gathered} \tag{7.2}$$

Proof. The function

$$D_{0^+}^{7/3}\phi_p\big(D_{0^+}^{7/2}u(t)\big) = t^2 + \frac{u(t)}{2} + \frac{u'(t)}{3} + \frac{u''(t)}{4}$$

is changed into the form of

$$D_{0^+}^{7/3}\phi_p\big(D_{0^+}^{7/2}u(t)\big) - t^2 - \frac{u(t)}{2} - \frac{u'(t)}{3} - \frac{u''(t)}{4} = 0.$$

Let $u(t) = I_{0^+}^2 v(t)$, we have

$$\begin{gathered}
D_{0^+}^{7/3}\phi_p\big(D_{0^+}^{3/2}v(t)\big) - t^2 - \frac{1}{2}I_{0^+}^2 v(t) - \frac{1}{3}I_{0^+}^1 v(t) - \frac{1}{4}v(t) = 0, \quad 0 < t < 1, \\
v(0) = 0, v(1) = \frac{1}{2}v(\frac{1}{3}), D_{0^+}^{3/2}v(0) = D_{0^+}^{3/2}v(1) = 0,
\end{gathered} \tag{7.3}$$

then the hypotheses (H9) are satisfied. For any $\mu > 0$, we obtain

$$\begin{aligned}
&\int_0^1 H(\tau,\tau) f\big(\tau, I_{0^+}^2\mu\tau^{1/2}, I_{0^+}^1\mu\tau^{1/2}, \mu\tau^{1/2}\big)d\tau \\
&= \int_0^1 H(\tau,\tau)\big(t^2 + \frac{1}{2}I_{0^+}^2\mu\tau^{1/2} + \frac{1}{3}I_{0^+}^1\mu\tau^{1/2} + \frac{1}{4}\mu\tau^{1/2}\big)d\tau < +\infty,
\end{aligned}$$

which implies that (H10) holds.

$$\begin{aligned}
f\Big(t, \frac{1}{2}I_{0^+}^2 v(t), \frac{1}{3}I_{0^+}^1 v(t), \frac{1}{4}v(t)\Big) &= t^2 + \frac{1}{2}I_{0^+}^2 v(t) + \frac{1}{3}I_{0^+}^1 v(t) + \frac{1}{4}v(t) \\
&\le r^{-1/2}\Big(t^2 + \frac{1}{2}I_{0^+}^2 v(t) + \frac{1}{3}I_{0^+}^1 v(t) + \frac{1}{4}v(t)\Big) \\
&= r^{-1/2} f\Big(t, \frac{1}{2}I_{0^+}^2 v(t), \frac{1}{3}I_{0^+}^1 v(t), \frac{1}{4}v(t)\Big).
\end{aligned}$$

Let $a(t) = t^{1/2}$, we obtain

$$b(t) := Ta(t) = \int_0^1 G(t,s)\phi_p(\int_0^1 H(\tau,\tau) f(\tau, I_{0^+}^2\mu\tau^{1/2}, I_{0^+}^1\mu\tau^{1/2}, \mu\tau^{1/2})d\tau)ds \in \Omega_1,$$

and $Tb(t) = T^2a(t) \in \Omega_1$. Hence, there exist two positive numbers μ_1, μ_2, such that

$$Ta(t) \ge \mu_1 a(t), T^2a(t) \ge \mu_2 a(t).$$

Let $0 < \mu_0 \le \min\{1, \mu_1, \mu_2^{7/3}\}$, by the monotonicity of T, we obtain

$$T(\mu_0 a(t)) \ge Ta(t) \ge \mu_1 a(t) \ge \mu_0 a(t),$$

and

$$T^2(\mu_0 a(t)) \ge \mu_0^{1/4} T^2 a(t) \ge \mu_0^{1/4}\mu_2 a(t) \ge \mu_0 a(t).$$

If we take $p(t) = \mu_0 t^{1/2}$, then

$$\begin{aligned}
q(t) &= Tp(t) \\
&= \int_0^1 G(t,s)\phi_p\Big(\int_0^1 H(\tau,\tau) f(\tau, I_{0^+}^2\mu\tau^{1/2}, I_{0^+}^1\mu\tau^{1/2}, \mu\tau^{1/2})d\tau\Big)ds \\
&\ge \mu_0 t^{1/2} = p(t),
\end{aligned}$$

and

$$\begin{aligned}
&Tq(t) = T^2p(t) \\
&= \int_0^1 G(t,s)\phi_p\Big(\int_0^1 H(\tau,\tau) f(\tau, T(I_{0^+}^2\mu\tau^{1/2}), T(I_{0^+}^1\mu\tau^{1/2}), T(\mu\tau^{1/2}))d\tau\Big)ds \\
&\ge \mu_0 t^{1/2} = p(t).
\end{aligned}$$

Therefore, condition (H11) holds. By Theorem 6.3, the fractional boundary-value problem (7.3) has at least one positive solution, that is to say, it follows from Lemma 3.1 that the fractional boundary-value problem (7.2) has at least one positive solution. The proof is complete. □

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You-Hui Su (corresponding author)
School of Mathematics and Physics, Xuzhou University of Technology, Xuzhou, Jiangsu 221018, China
E-mail address: suyh02@163.com, suyouhui@xzit.edu.cn, fax 86-0516-85608333

Yongzhen Yun
College of Science, Hohai University, Nanjing, Jiangsu 211100, China.
School of Mathematics and Physics, Xuzhou University of Technology, Xuzhou, Jiangsu 221018, China
E-mail address: yongzhen0614@163.com

Dongdong Wang
School of Mathematics and Physics, Xuzhou University of Technology, Xuzhou, Jiangsu 221018, China
E-mail address: wdd@xzit.edu.cn

Weimin Hu
School of Mathematics and Statistic, Yili Normal University, Yining, Xinjiang 835000, China
E-mail address: hwm680702@163.com