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UNIFORM REGULARITY FOR A MATHEMATICAL MODEL IN SUPERFLUIDITY

JISHAN FAN, BESSEM SAMET, YONG ZHOU

Abstract. We prove uniform-in-μ estimates for a mathematical model in superfluidity. Consequently, the limit as $\mu \to 0$ can be established.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and ν is the unit outward normal vector to $\partial\Omega$. We consider the following mathematical model in superfluidity [6]:

$$\gamma\psi_t = \frac{1}{k^2}\Delta\psi - \frac{2i}{k}A\cdot\nabla\psi - \psi|A|^2 + i\beta\psi\operatorname{div}A - \psi(|\psi|^2 - 1 + u), \tag{1.1}$$

$$A_t = \mu\Delta A - |\psi|^2 A + \frac{i}{2k}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi) - \nabla u, \tag{1.2}$$

$$u_t - I(u)(|\psi|^2)_t = \Delta u + I(u)\nabla\cdot\big[-|\psi|^2 A + \frac{i}{2k}(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi)\big], \tag{1.3}$$

in $\Omega \times (0,\infty)$ with boundary conditions

$$A\cdot\nu = 0, \quad \operatorname{curl} A \times \nu = 0, \quad \nabla\psi\cdot\nu = 0, \tag{1.4}$$

$$\nabla u\cdot\nu = 0, \quad \text{on } \partial\Omega\times(0,\infty) \tag{1.5}$$

and initial data

$$(\psi, A, u)(\cdot, 0) = (\psi_0, A_0, u_0)(\cdot) \quad \text{in } \Omega \subseteq \mathbb{R}^3. \tag{1.6}$$

The unknowns ψ, A, and u are $\mathbb{C}$-valued, $\mathbb{R}^3$-valued, and $\mathbb{R}^+$-valued functions, respectively. $\overline{\psi}$ denotes the complex conjugate of $\psi, |\psi|^2 := \psi\overline{\psi}$ is the density of superconducting carriers, and $i := \sqrt{-1}$. γ, k, μ, and $\beta := \frac{1}{k}(k^2\gamma - 1)$ are positive constants and for simplicity we will take $k = 1, \gamma = 2$ and thus $\beta = 1$. The function $I(u)$ is defined by

$$I(u) := \begin{cases} 0, & u < 0, \\ 1, & u \geq 0. \end{cases} \tag{1.7}$$

When $u = 0$ in (1.1) and (1.2), then the system (1.1) and (1.2) is the well-known Ginzburg-Landau equations in superconductivity with the choice of the Lorentz gauge, which has received many studies [8, 9, 10, 11, 12, 13, 16, 17].

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In [6], Berti and Fabrizio proved the global-in-time existence and uniqueness of strong solutions when $\psi_0, A_0 \in H^1(\Omega)$ and $u_0 \in L^2(\Omega)$ when (1.5) is replaced by the homogeneous Dirichlet boundary condition

$$u = 0.$$

However, their proof also works here for (1.5). But their estimates depend on μ. The long-time behavior of the problem (1.1)-(1.6) has been studied in [5].

The aim of this paper is to prove global-in-time estimates for solutions of (1.1)-(1.6) uniform-in μ. We will prove the following result.

Theorem 1.1. *Let $0 < \mu < 1$. Let $\psi_0, u_0 \in H^2(\Omega), A_0 \in W^{1,q}(\Omega)$ $(3 < q \le 6)$, with $|\psi_0| \le 1$ and $u_0 \ge 0$ in Ω. Then for any $T > 0$, there exists a unique strong solution (ψ_μ, A_μ, u_μ) of (1.1)-(1.6) such that*

$$\begin{gathered}
\psi_\mu \in L^\infty(0,T;H^2) \cap L^2(0,T;H^3), \partial_t\psi_\mu \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1), \\
A_\mu \in L^\infty(0,T;W^{1,q}), \partial_t A_\mu \in L^\infty(0,T;L^2), \\
u_\mu \in L^\infty(0,T;H^2) \cap L^2(0,T;W^{2,q}), \partial_t u_\mu \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)
\end{gathered} \tag{1.8}$$

with the corresponding norms that are uniformly bounded with respect to $\mu > 0$.

Remark 1.2. As soon as the uniform-in-μ a priori estimates are established, we can easily show by standard compactness arguments that the limit as $\mu \to 0$ for (1.1)-(1.6) exists.

We now collect several vector identities and the Gauss-Green formula which will be used in the rest of the paper.

Lemma 1.3 ([3, Theorem 2.1])**.** *Let Ω be a regular bounded domain in $\mathbb{R}^3$, $A : \Omega \to \mathbb{R}^3$ be a sufficiently smooth vector field, and let $1 < p < \infty$. Then, the following identity holds.*

$$\begin{aligned}
&-\int_\Omega \Delta A \cdot A|A|^{p-2} dx \\
&= \int_\Omega |A|^{p-2}|\nabla A|^2 dx + 4\frac{p-2}{p^2}\int_\Omega \big|\nabla |A|^{p/2}\big|^2 dx \\
&\quad - \int_{\partial\Omega} |A|^{p-2}(\nu\cdot\nabla)A \cdot A\, dS.
\end{aligned} \tag{1.9}$$

Moreover, recalling the vector identity:

$$(\nu\cdot\nabla)A \cdot A = (A\cdot\nabla)A\cdot\nu + (\operatorname{curl} A \times \nu)\cdot A \tag{1.10}$$

for a sufficiently smooth vector field A, we can also deduce that

$$\begin{aligned}
-\int_\Omega \Delta A \cdot A|A|^{p-2} dx &= \int_\Omega |A|^{p-2}|\nabla A|^2 dx + 4\frac{p-2}{p^2}\int_\Omega \big|\nabla |A|^{p/2}\big|^2 dx \\
&\quad - \int_{\partial\Omega} |A|^{p-2}(A\cdot\nabla)A\cdot\nu\, dS \\
&\quad - \int_{\partial\Omega} |A|^{p-2}(\operatorname{curl} A\times\nu)\cdot A\, dS.
\end{aligned} \tag{1.11}$$

Lemma 1.4 ([4, Lemma 2.2])**.** *Assume that A is sufficiently smooth, satisfying the boundary condition (1.4) on $\partial\Omega$. Then, the following identity for $B := \operatorname{curl} A$ holds.*

$$-\frac{\partial B}{\partial\nu}\cdot B = (\epsilon_{1jk}\epsilon_{1\beta\gamma} + \epsilon_{2jk}\epsilon_{2\beta\gamma} + \epsilon_{3jk}\epsilon_{3\beta\gamma})B_j B_\beta \partial_k \nu_\gamma \tag{1.12}$$

on $\partial\Omega$, where ϵ_{ijk} denotes the totally anti-symmetric tensor such that $(a\times b)_i = \epsilon_{ijk}a_jb_k$.

Lemma 1.5 ([1, Lemma 7.44], [14, Corollary 1.7])**.** *Let a smooth and bounded open set Ω be given and let $1 < p < \infty$. Then the following inequality holds. There exists a constant $C > 0$, such that*

$$\|f\|_{L^p(\partial\Omega)} \le C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{1/p} \tag{1.13}$$

for any $f \in W^{1,p}(\Omega)$.

Lemma 1.6 ([7])**.** *There exists a constant $C > 0$, such that*

$$\|f\|_{W^{1,p}(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|\operatorname{div} f\|_{L^p(\Omega)} + \|\operatorname{curl} f\|_{L^p(\Omega)}) \tag{1.14}$$

for any $1 < p < \infty$ and all $f \in W^{1,p}(\Omega)$.

When A satisfies $A\cdot\nu = 0$ on $\partial\Omega$, we will also use the identity

$$(A\cdot\nabla)A\cdot\nu = -(A\cdot\nabla)\nu\cdot A \quad \text{on } \partial\Omega \tag{1.15}$$

for any sufficiently smooth vector field A.

Lemma 1.7 ([2])**.** *Let u be a smooth solution of the problem*

$$\begin{gathered} u_t - \Delta u = \operatorname{div} g \quad \text{in } \Omega\times(0,T), \\ \frac{\partial u}{\partial\nu} = 0 \quad \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = 0 \quad \text{in } \Omega \end{gathered}$$

for any given $T > 0$. Then there exists a constat $C > 0$, such that

$$\|\nabla u\|_{L^q(0,T;L^p(\Omega))} \le C\|g\|_{L^q(0,T;L^p(\Omega))}. \tag{1.16}$$

with $1 < p, q < \infty$.

2. Proof of main results

This section is devoted to the proof of Theorem 1.1. Since it has been proved that the problem (1.1)-(1.6) has a unique global-in-time strong solution [6], we only need to prove a priori estimates (1.8) uniformly in μ. From now on, we drop the subscript μ.

It follows from (1.3), (1.5) and (1.6) that

$$u \ge 0 \quad \text{if } u_0 \ge 0 \tag{2.1}$$

and thus $I(u) \equiv 1$ in (1.3). Then we have

$$2f_t = \Delta f - f(f^2 - 1 + u + V_s^2) \quad \text{in } \Omega\times(0,\infty), \tag{2.2}$$

$$\nabla f\cdot\nu = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{2.3}$$

$$f = f_0 \quad \text{in } \Omega \tag{2.4}$$

where

$$f := |\psi|, \psi := fe^{i\phi}, V_s := -A + \nabla\phi.$$

It follows from (2.2), (2.3), and (2.4) that

$$|\psi| \le 1 \quad \text{in } \Omega\times(0,\infty). \tag{2.5}$$

Testing (1.1) by ψ, taking the real part and using (2.1), we see that

$$\frac{d}{dt}\int|\psi|^2dx+\int|i\nabla\psi+\psi A|^2dx+\int|\psi|^4dx+\int u|\psi|^2dx=\int|\psi|^2dx,$$

which gives

$$\int_0^T\int|i\nabla\psi+\psi A|^2\,dx\,dt\le C. \tag{2.6}$$

Here and in what follows, C will denote a generic positive constant independent of $\mu>0$.

Testing (2.2) by f and using (2.1), we find that

$$\frac{d}{dt}\int f^2dx+\int|\nabla f|^2dx+\int f^2(f^2+u+V_s^2)dx=\int f^2dx,$$

which reads

$$\int_0^T\int\big|\nabla|\psi|^2\big|^2\,dx\,dt\le C. \tag{2.7}$$

We denote $w:=u-|\psi|^2$. Testing (1.3) by w, using (2.5), (2.6) and (2.7), we get

$$\begin{aligned}\frac12\frac{d}{dt}\int w^2dx+\int|\nabla w|^2dx&\le\int\big|\nabla|\psi|^2\big|\cdot|\nabla w|dx+\int|i\nabla\psi+\psi A|\cdot|\nabla w|dx\\&\le\int(\big|\nabla|\psi|^2\big|^2+|i\nabla\psi+\psi A|^2)dx+\frac12\int|\nabla w|^2dx,\end{aligned}$$

which gives

$$\begin{gathered}\|w\|_{L^\infty(0,T;L^2)}+\|w\|_{L^2(0,T;H^1)}\le C,\\\|u\|_{L^\infty(0,T;L^2)}+\|u\|_{L^2(0,T;H^1)}\le C.\end{gathered}$$

Testing (1.2) by A, using (2.5), (2.6) and (2), we deduce that

$$\begin{aligned}&\frac12\frac{d}{dt}\int A^2dx+\mu\int(|\operatorname{div}A|^2+|\operatorname{curl}A|^2)dx\\&\le\int|i\nabla\psi+\psi A||\psi||A|dx+\int|\nabla u||A|dx\\&\le\int|A|^2dx+\int|i\nabla\psi+\psi A|^2dx+\int|\nabla u|^2dx,\end{aligned}$$

which implies

$$\|A\|_{L^\infty(0,T;L^2)}+\sqrt{\mu}\|A\|_{L^2(0,T;H^1)}\le C. \tag{2.8}$$

Obviously, inequalities (2.5), (2.6) and (2.8) imply

$$\|\psi\|_{L^2(0,T;H^1)}\le C. \tag{2.9}$$

Testing (1.1) by $-\Delta\overline{\psi}$, taking the real part, and using (2.5), we have

$$
\begin{aligned}
&\frac{d}{dt}\int|\nabla\psi|^2dx+\int|\Delta\psi|^2dx\\
&\le 2\int|A||\nabla\psi||\Delta\psi|dx+\int|\psi||A|^2|\Delta\psi|dx\\
&\quad+\int|\psi||\operatorname{div}A||\Delta\psi|dx+\int|\psi|(|\psi|^2+1+|u|)|\Delta\psi|dx\\
&\le C(\|A\|_{L^4}\|\nabla\psi\|_{L^4}+\|A\|_{L^4}^2+\|\operatorname{div}A\|_{L^2}+\|u\|_{L^2}+1)\|\Delta\psi\|_{L^2}\\
&\le C(\|A\|_{L^4}\|\Delta\psi\|_{L^2}^{1/2}+\|A\|_{L^4}^2+\|\operatorname{div}A\|_{L^2}+\|u\|_{L^2}+1)\|\Delta\psi\|_{L^2}\\
&\le \frac{1}{16}\|\Delta\psi\|_{L^2}^2+C\|A\|_{L^4}^4+C\|\operatorname{div}A\|_{L^2}^2+C\|u\|_{L^2}^2+C,
\end{aligned}
\tag{2.10}
$$

where we have used the Gagliardo-Nirenberg inequality:

$$
\|\nabla\psi\|_{L^4}^2\le C\|\psi\|_{L^\infty}\|\Delta\psi\|_{L^2}. \tag{2.11}
$$

Testing (1.2) by $|A|^2A$, using (1.11), (1.15), (2.5) and (1.13), we derive

$$
\begin{aligned}
&\frac14\frac{d}{dt}\int|A|^4dx+\mu\int|A|^2|\nabla A|^2dx+\frac{\mu}{2}\int\big|\nabla|A|^2\big|^2dx\\
&=\mu\int_{\partial\Omega}|A|^2(A\cdot\nabla)\nu\cdot A\,dS-\int\nabla w\cdot|A|^2Adx-\int\nabla|\psi|^2\cdot|A|^2Adx\\
&\quad-\int\operatorname{Re}\{(i\nabla\psi+\psi A)\overline{\psi}\}|A|^2Adx\\
&\le\|\nabla\nu\|_{L^\infty}\mu\int_{\partial\Omega}|A|^4dS+(\|\nabla w\|_{L^4}+3\|\nabla\psi\|_{L^4})\|A\|_{L^4}^3\\
&\le C\mu\int|A|^4dx+\frac{1}{16}\mu\int\big|\nabla|A|^2\big|^2dx+C\|A\|_{L^4}^4\\
&\quad+\epsilon\int(|\nabla w|^4+|\nabla\psi|^4)dx
\end{aligned}
\tag{2.12}
$$

for any $0<\epsilon<1$.

It follows from (1.2), (1.4) and (1.5) that [12]:

$$
\nabla\operatorname{div}A\cdot\nu=0\quad\text{on }\partial\Omega\times(0,\infty). \tag{2.13}
$$

Taking div to (1.2), testing by $\operatorname{div}A$, using (2.5) and (2.11), we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int|\operatorname{div}A|^2dx+\mu\int|\nabla\operatorname{div}A|^2dx+\int|\psi|^2|\operatorname{div}A|^2dx\\
&\le\int|A|\big|\nabla|\psi|^2\big|\cdot|\operatorname{div}A|dx+2\int|\Delta\psi||\operatorname{div}A|dx+\int|\Delta(w+|\psi|^2)||\operatorname{div}A|dx\\
&\le C\|A\|_{L^4}\|\nabla\psi\|_{L^4}\|\operatorname{div}A\|_{L^2}+C(\|\Delta w\|_{L^2}+\|\Delta\psi\|_{L^2}+\|\nabla\psi\|_{L^4}^2)\|\operatorname{div}A\|_{L^2}\\
&\le C\|\operatorname{div}A\|_{L^2}^2+C\|A\|_{L^4}^4+\epsilon\|\Delta w\|_{L^2}^2+\epsilon\|\Delta\psi\|_{L^2}^2
\end{aligned}
\tag{2.14}
$$

for any $0<\epsilon<1$. We rewrite (1.3) as

$$
w_t-\Delta w=\Delta|\psi|^2+\nabla\cdot\big[-|\psi|^2A+\frac{i}{2}(\psi\nabla\overline{\psi}-\overline{\psi}\nabla\psi)\big]. \tag{2.15}
$$

Testing (2.15) by $-\Delta w$, using (2.5) and (2.11), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int|\nabla w|^2dx+\int|\Delta w|^2dx\\ &\le C\int(|\Delta\psi|+|\nabla\psi|^2+|\operatorname{div}A|+|\nabla\psi||A|)|\Delta w|dx\\ &\le\epsilon\int|\Delta w|^2dx+C_0\int|\Delta\psi|^2dx+C\int|\operatorname{div}A|^2dx+C\int|A|^4dx. \end{aligned} \tag{2.16}$$

By Lemma 1.7, from (2.15) and (2.5) it follows that

$$\int_0^T\int|\nabla w|^4\,dx\,dt\le C+C\int_0^T\int|\nabla\psi|^4\,dx\,dt+C\int_0^T\int|A|^4\,dx\,dt. \tag{2.17}$$

Integrating $2C_0\times(2.10)+(2.12)+(2.14)+(2.16)$ over $(0,T)$, using (2.17), (2.5) and (2.11), taking ϵ small enough, we have

$$\|\psi\|_{L^\infty(0,T;H^1)}+\|\psi\|_{L^2(0,T;H^2)}\le C, \tag{2.18}$$

$$\|A\|_{L^\infty(0,T;L^4)}+\|\operatorname{div}A\|_{L^\infty(0,T;L^2)}+\sqrt{\mu}\|\nabla\operatorname{div}A\|_{L^2(0,T;L^2)}\le C, \tag{2.19}$$

$$\|w\|_{L^\infty(0,T;H^1)}+\|w\|_{L^2(0,T;H^2)}\le C. \tag{2.20}$$

Testing (1.2) with $\operatorname{curl}^2 A$, and utilize the fact $\operatorname{curl}\nabla=0$, (2.5), (2.11), (1.14), (2.18), and (2.19), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int|\operatorname{curl}A|^2dx+\mu\int|\operatorname{curl}^2A|^2dx\\ &=-\operatorname{Re}\int\operatorname{curl}[(i\nabla\psi+\psi A)\overline{\psi}]\operatorname{curl}Adx\\ &=-\operatorname{Re}\int(i\nabla\overline{\psi}\times\nabla\psi+|\psi|^2\operatorname{curl}A+\nabla|\psi|^2\times A)\operatorname{curl}Adx\\ &\le C(\|\nabla\psi\|_{L^4}^2+\|\nabla\psi\|_{L^4}\|A\|_{L^4})\|\operatorname{curl}A\|_{L^2}\\ &\le C\|\Delta\psi\|_{L^2}^2+C\|A\|_{L^4}^4+C\|\operatorname{curl}A\|_{L^2}^2, \end{aligned}$$

which gives

$$\|A\|_{L^\infty(0,T;H^1)}+\sqrt{\mu}\|A\|_{L^2(0,T;H^2)}\le C. \tag{2.21}$$

On the other hand, from (1.1), (1.2), (1.3), (2.18), (2.19), (2.20) and (2.21) it follows that

$$\|\psi_t\|_{L^2(0,T;L^2)}+\|A_t\|_{L^2(0,T;L^2)}+\|w_t\|_{L^2(0,T;L^2)}+\|u_t\|_{L^2(0,T;L^2)}\le C. \tag{2.22}$$

Now, taking ∂_t to (1.1), testing then by $\overline{\psi}_t$, taking the real part, and employing (2.5), (2.21), and (2.20), we have

$$
\begin{aligned}
&\frac{d}{dt}\int|\psi_t|^2dx+\int|\nabla\psi_t|^2dx+\int A^2|\psi_t|^2dx\\
&\le 2\int|A_t||\nabla\psi||\psi_t|dx+2\int|A||\nabla\psi_t||\psi_t|dx+2\int|A||A_t||\psi_t|dx\\
&\quad+\Big|\int\psi\overline{\psi}_t\operatorname{div}A_tdx\Big|+C\int|\psi_t|^2dx+C\int|u||\psi_t|^2dx+C\int|u_t||\psi_t|dx\\
&\le C\|A_t\|_{L^2}\|\nabla\psi\|_{L^3}\|\psi_t\|_{L^6}+C\|A\|_{L^6}\|\nabla\psi_t\|_{L^2}\|\psi_t\|_{L^3}\\
&\quad+C\|A\|_{L^6}\|A_t\|_{L^2}\|\psi_t\|_{L^3}+\Big|\int A_t(\psi\nabla\overline{\psi}_t+\overline{\psi}_t\nabla\psi)dx\Big|\\
&\quad+C\int|\psi_t|^2dx+C\|u\|_{L^3}\|\psi_t\|_{L^3}^2+C\|u_t\|_{L^2}\|\psi_t\|_{L^2}\\
&\le C\|A_t\|_{L^2}\|\nabla\psi\|_{L^3}(\|\psi_t\|_{L^2}+\|\nabla\psi_t\|_{L^2})+C\|\nabla\psi_t\|_{L^2}\|\psi_t\|_{L^3}\\
&\quad+C\|A_t\|_{L^2}\|\psi_t\|_{L^3}+C\|A_t\|_{L^2}\|\nabla\psi_t\|_{L^2}\\
&\quad+C\int|\psi_t|^2dx+C\|\psi_t\|_{L^3}^2+C\|u_t\|_{L^2}^2\\
&\le\frac{1}{16}\|\nabla\psi_t\|_{L^2}^2+C\|\psi_t\|_{L^2}^2+C\|A_t\|_{L^2}^2+C\|u_t\|_{L^2}^2+C\|\nabla\psi\|_{L^3}^2\|A_t\|_{L^2}^2.
\end{aligned}
\tag{2.23}
$$

Taking ∂_t to (1.2), testing then by A_t, and making use of (2.5) and (2.21), we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int|A_t|^2dx+\mu\int(|\operatorname{div}A_t|^2+|\operatorname{curl}A_t|^2)dx+\int|\psi|^2|A_t|^2dx\\
&\le C\int(|\nabla\psi_t|+|\nabla\psi||\psi_t|+|\psi_t||A|)|A_t|dx+C\int|u_t||\operatorname{div}A_t|dx\\
&\le C(\|\nabla\psi_t\|_{L^2}+\|\nabla\psi\|_{L^6}\|\psi_t\|_{L^3}+\|\psi_t\|_{L^3}\|A\|_{L^6})\|A_t\|_{L^2}+C\|u_t\|_{L^2}^2\\
&\quad+C\|\operatorname{div}A_t\|_{L^2}^2\\
&\le\frac{1}{16}\|\nabla\psi_t\|_{L^2}^2+C\|\nabla\psi\|_{L^6}^2\|A_t\|_{L^2}^2+C\|\psi_t\|_{L^2}^2\\
&\quad+C\|A_t\|_{L^2}^2+C\|u_t\|_{L^2}^2+C\|\operatorname{div}A_t\|_{L^2}^2.
\end{aligned}
\tag{2.24}
$$

Taking div to (1.2), testing by $\operatorname{div}A_t$, using (2.5), (2.11), (2.21), (2.20) and (2.18), we have

$$
\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int|\nabla\operatorname{div}A|^2dx+\int|\operatorname{div}A_t|^2dx\\
&\le C\int(|\operatorname{div}A|+|A||\nabla\psi|+|\Delta u|+|\Delta\psi|+|\nabla\psi|^2)|\operatorname{div}A_t|dx\\
&\le\frac12\int|\operatorname{div}A_t|^2dx+C\int|\operatorname{div}A|^2dx+C\int|\Delta u|^2dx\\
&\quad+C\int|\Delta\psi|^2dx+C\|A\|_{L^6}^2\|\nabla\psi\|_{L^3}^2\\
&\le\frac12\int|\operatorname{div}A_t|^2dx+C+C\int(|\Delta u|^2+|\Delta\psi|^2)dx
\end{aligned}
$$

which implies

$$\int_0^T \int |\operatorname{div} A_t|^2 \, dx \, dt \le C. \tag{2.25}$$

Combining (2.23) and (2.24), using (2.18), (2.22), (2.25) and the Gronwall inequality, we arrive at

$$\|\psi_t\|_{L^\infty(0,T;L^2)} + \|\psi_t\|_{L^2(0,T;H^1)} \le C, \tag{2.26}$$

$$\|A_t\|_{L^\infty(0,T;L^2)} \le C. \tag{2.27}$$

It follows from (1.1), (2.5), (2.18), (2.21), (2.20), and (2.26) that

$$\|\psi\|_{L^\infty(0,T;H^2)} + \|\psi\|_{L^2(0,T;H^3)} \le C. \tag{2.28}$$

Taking curl to (1.2), testing by $|\operatorname{curl} A|^{q-2} \operatorname{curl} A$ $(3 < q \le 6)$, using $\operatorname{curl} \nabla = 0$, (1.9), (1.12), (2.21) and (2.28), we have

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int |\operatorname{curl} A|^q dx + \mu \int |\operatorname{curl} A|^{q-2} |\nabla \operatorname{curl} A|^2 dx + 4\frac{q-2}{q^2}\mu \int \Big|\nabla |\operatorname{curl} A|^{q/2}\Big|^2 dx \\
&\le C\mu \int_{\partial\Omega} |\nabla \nu| |\operatorname{curl} A|^q \, dS \\
&\quad - \operatorname{Re} \int (i\nabla\overline{\psi} + |\psi|^2 \operatorname{curl} A + \nabla|\psi|^2 \times A) |\operatorname{curl} A|^{q-2} \operatorname{curl} A dx \\
&\le C\mu \int_{\partial\Omega} |\operatorname{curl} A|^q \, dS + C\int (|\nabla\psi|^2 + |\nabla\psi||A|) |\operatorname{curl} A|^{q-1} dx \\
&\le C\mu \int |\operatorname{curl} A|^q dx + 2\frac{q-2}{q^2}\mu \int \Big|\nabla |\operatorname{curl} A|^{q/2}\Big|^2 dx \\
&\quad + C\|\nabla\psi\|_{L^{2q}}^2 \|\operatorname{curl} A\|_{L^q}^{q-1} + C\|\nabla\psi\|_{L^\infty} \|A\|_{L^q} \|\operatorname{curl} A\|_{L^q}^{q-1},
\end{aligned}$$

whence,

$$\frac{d}{dt}\|\operatorname{curl} A\|_{L^q} \le C\|\operatorname{curl} A\|_{L^q} + C\|\nabla\psi\|_{L^{2q}}^2 + C\|\nabla\psi\|_{L^\infty},$$

which implies

$$\|\operatorname{curl} A\|_{L^\infty(0,T;L^q)} \le C \quad (3 < q \le 6). \tag{2.29}$$

Taking ∂_t to (1.3), testing then by w_t, using (2.5), (2.26), (2.28), and (2.21), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int w_t^2 dx + \int |\nabla w_t|^2 dx \\
&\le C\int (|\nabla\psi_t| + |\psi_t||\nabla\psi| + |A_t| + |\psi_t||A|) |\nabla w_t| dx \\
&\le \frac{1}{2}\|\nabla w_t\|_{L^2}^2 + C\|\nabla\psi_t\|_{L^2}^2 + C\|\psi_t\|_{L^3}^2 \|\nabla\psi\|_{L^6}^2 + C\|A_t\|_{L^2}^2 + C\|\psi_t\|_{L^3}^2 \|A\|_{L^6}^2 \\
&\le \frac{1}{2}\|\nabla w_t\|_{L^2}^2 + C\|\nabla\psi_t\|_{L^2}^2 + C\|\psi_t\|_{L^3}^2 + C\|A_t\|_{L^2}^2,
\end{aligned}$$

which implies

$$\|w_t\|_{L^\infty(0,T;L^2)} + \|w_t\|_{L^2(0,T;H^1)} \le C. \tag{2.30}$$

Taking div to (1.2), testing by $|\operatorname{div} A|^{q-2} \operatorname{div} A$ $(3 < q \le 6)$, using (2.5), (2.21), and (2.28), we have

$$\frac{1}{q}\frac{d}{dt}\int |\operatorname{div} A|^q dx + \mu \int |\operatorname{div} A|^{q-2} |\nabla \operatorname{div} A|^2 dx + 4\frac{q-2}{q^2}\int \Big|\nabla |\operatorname{div} A|^{q/2}\Big|^2 dx$$

$$
\begin{aligned}
&+\int |\psi|^2 |\operatorname{div} A|^q dx \\
&\le C\int (|A||\nabla\psi| + |\Delta\psi| + |\Delta u|)|\operatorname{div} A|^{q-1} dx \\
&\le C(\|A\|_{L^q}\|\nabla\psi\|_{L^\infty} + \|\Delta\psi\|_{L^q} + \|\Delta u\|_{L^q})\|\operatorname{div} A\|_{L^q}^{q-1} \\
&\le C(\|\nabla\psi\|_{L^\infty} + \|\Delta\psi\|_{L^q} + \|\Delta u\|_{L^q})\|\operatorname{div} A\|_{L^q}^{q-1},
\end{aligned}
$$

whence

$$
\begin{aligned}
\frac{d}{dt}\|\operatorname{div} A\|_{L^q}^2 &\le C(\|\nabla\psi\|_{L^\infty} + \|\Delta\psi\|_{L^q} + \|\Delta u\|_{L^q})\|\operatorname{div} A\|_{L^q}, \\
&\le C(\|\nabla\psi\|_{L^\infty}^2 + \|\Delta\psi\|_{L^q}^2 + \|\Delta u\|_{L^q}^2) + C\|\operatorname{div} A\|_{L^q}^2,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|\operatorname{div} A\|_{L^q}^2 &\le C + C\int_0^t \big(\|\nabla\psi\|_{L^\infty}^2 + \|\Delta\psi\|_{L^q}^2 + \|\Delta u\|_{L^q}^2\big) ds + C\int_0^t \|\operatorname{div} A\|_{L^q}^2 ds \\
&\le C + C\int_0^t \|\Delta u\|_{L^q}^2 ds + C\int_0^t \|\operatorname{div} A\|_{L^q}^2 ds.
\end{aligned}
\tag{2.31}
$$

By the $L^2(0,T;W^{2,q})$-theory of heat equation, it follows from (1.3), (2.5), (2.21) and (2.28), we have

$$
\begin{aligned}
\|\Delta u\|_{L^2(0,t;L^q)} &\le C + C\|\Delta\psi\|_{L^2(0,t;L^q)} + C\|\operatorname{div} A\|_{L^2(0,t;L^q)} \\
&\quad + C\|A\|_{L^\infty(0,t;L^q)}\|\nabla\psi\|_{L^2(0,t;L^\infty)} \\
&\le C + C\|\operatorname{div} A\|_{L^2(0,t;L^q)}.
\end{aligned}
\tag{2.32}
$$

Inserting (2.32) into (2.31), we have

$$
\|\operatorname{div} A\|_{L^\infty(0,T;L^q)} \le C \quad (3 < q \le 6), \tag{2.33}
$$

$$
\|u\|_{L^2(0,T;W^{2,q})} \le C. \tag{2.34}
$$

It follows from (1.14), (2.29) and (2.33) that

$$
\|A\|_{L^\infty(0,T;W^{1,q})} \le C \quad (3 < q \le 6). \tag{2.35}
$$

It follows from (1.3), (2.28), (2.26), (2.30) and (2.35) that

$$
\|u\|_{L^\infty(0,T;H^2)} \le C.
$$

This completes the proof.

Remark 2.1. We do not need to assume $u_0 \ge 0$ in Ω and then we take $I(u) = 1$ in (1.3). Now we use the Lyapunov functional [6]:

$$
G(t) := \|\nabla f\|_{L^2}^2 + \frac{1}{2}\|f^2 - 1\|_{L^2}^2 + \|fV_s\|_{L^2}^2 + \mu\|\operatorname{curl} V_s\|_{L^2}^2 + \|u\|_{L^2}^2 \le G(0) < \infty,
$$

to prove that $u \in L^\infty(0,T;L^2)$. Then by the method of Stampacchia [15], it follows from (2.2), (2.3) and (2.4) that

$$
\|\psi\|_{L^\infty(0,T;L^\infty(\Omega))} \le C.
$$

Then by the same calculations above, we can complete the proof.

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Jishan Fan
Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, China
E-mail address: fanjishan@njfu.edu.cn

Bessem Samet
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail address: bessem.samet@gmail.com

Yong Zhou (corresponding author)
School of Mathematics, Sun Yat-Sen University, Zhuhai 518092, China
E-mail address: zhouyong3@mail.sysu.edu.cn