Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 03, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC BEHAVIOR OF PULLBACK ATTRACTORS FOR NON-AUTONOMOUS MICROPOLAR FLUID FLOWS IN 2D UNBOUNDED DOMAINS

WENLONG SUN, YEPING LI

Communicated by Jesus Ildefonso Diaz

Abstract. In this article, we investigate the pullback asymptotic behavior of solutions for a non-autonomous micropolar fluid flows in 2D unbounded channel-like domains. First, applying the technique of truncation functions, decomposition of spatial domain, and the energy method, we show the existence of the pullback attractor in the space $\widehat{H}(\Omega)$ (has L^2-regularity). In fact, we can deduce the existence of pullback attractor in space $\widehat{V}(\Omega)$ (has H^1-regularity). Also the tempered behavior of the pullback attractor is verified. Moreover, when the spatial domain varies from $\Omega_m(\{\Omega_m\}_{m=1}^{\infty}$ be an expanding sequence of simply connected, bounded and smooth subdomains of Ω such that $\cup_{m=1}^{\infty}\Omega_m = \Omega)$ to Ω, the upper semicontinuity of the pullback attractor is discussed.

1. Introduction

The micropolar fluid model is firstly derived by Eringen [9] in 1966, which is used to describe fluids consisting of randomly oriented particles suspended in a viscous medium. The model has the form:

$$
\begin{gathered}
\frac{\partial u}{\partial t} - (\nu + \nu_\theta)\Delta u - 2\nu_\theta \operatorname{rot}\omega + (u\cdot\nabla)u + \nabla p = f,\\
\nabla\cdot u = 0,\\
\begin{aligned}
&\frac{\partial \omega}{\partial t} - (c_a + c_d)\Delta\omega + 4\nu_\theta\omega + (u\cdot\nabla)\omega\\
&- (c_0 + c_d - c_a)\nabla\operatorname{div}\omega - 2\nu_\theta\operatorname{rot}u = \tilde{f},
\end{aligned}
\end{gathered}
\tag{1.1}
$$

where $u = (u_1, u_2, u_3)$ is velocity, $\omega = (\omega_1, \omega_2, \omega_3)$ represents the angular velocity field of rotation of particles, $f = (f_1, f_2, f_3)$ and $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)$ stand for the external force and moments, respectively. p is pressure. The positive parameters $\nu, \nu_\theta, c_0, c_a$ and c_d are viscosity coefficients. Indeed, ν is the usual Newtonian viscosity and ν_θ is called microrotation viscosity. Note that when the gyration is neglected, the micropolar fluid equations are reduce to the classical Navier-Stokes equations.

2010 *Mathematics Subject Classification.* 35B40, 35B41, 35Q30.
Key words and phrases. Micropolar fluid flow; pullback attractor; truncation function; tempered behavior; upper semicontinuity.

Submitted March 6, 2017. Published January 4, 2018.

Micropolar fluid model plays important role in the fields of applied and computational mathematics, there are lots of literatures on the mathematical theory of micropolar fluid model (1.1). The existence and uniqueness of solutions for the micropolar fluids has been investigated in [8, 12, 13]. At the same time, lots of works are devoted to the long time behavior of solutions for the micropolar fluids. More precisely, Chen, Chen and Dong proved the existence of H^2-compact global attractors in a bounded domain in [4] and verified the existence of uniform attractors in non-smooth domains in [5]. Chen[6] showed the existence of L^2-pullback attractor for the micropolar fluid flows in a Lipschitz bounded domain with non-homogeneous boundary conditions. Łukaszewicz[13] verified the estimates of Hausdorff and fractal dimension of the L^2-global attractor. Later, Łukaszewicz and Tarasińska[18] proved the existence of H^1-pullback attractor for non-autonomous micropolar fluid equation in a bounded domain. As for the long time behavior of solutions for the micropolar fluid flows on unbounded domain, Dong and Chen [7] investigated the existence and regularity of the global attractors in 2D unbounded domains. Later, Zhao, Zhou and Lian [25] established the existence of H^1-uniform attractor and further proved the L^2-uniform attractor belongs to the H^1-uniform attractor in 2D unbounded domains. Nowakowski [21] investigated the existence of H^1-uniform attractor and long time behavior of solutions in 3D cylindrical domains. So far, to our knowledge, there is no results about pullback attractors of the micropolar fluid model in 2D unbounded domains. Here, we will give a positive answer for this problem.

Since we investigate the pullback asymptotic behavior of solutions for the micropolar fluid model in 2D unbounded domains, we assume that the velocity component u_3 in the x_3 direction is zero and the axes of rotation of particles are parallel to the x_3 axis. Then, the form of $u, \omega, f, \tilde{f}$ are that $u = (u_1, u_2, 0)$, $\omega = (0, 0, \omega_3)$, $f = (f_1, f_2, 0)$, $\tilde{f} = (0, 0, \tilde{f}_3)$. Further, the equations (1.1) can be reduced to the following 2D non-autonomous dynamical system:

$$
\begin{gathered}
\frac{\partial u}{\partial t} - (\nu + \nu_\theta)\Delta u - 2\nu_\theta \nabla \times \omega + (u \cdot \nabla)u + \nabla p = f(t, x), \\
\frac{\partial \omega}{\partial t} - \alpha \Delta \omega + 4\nu_\theta \omega - 2\nu_\theta \nabla \times u + (u \cdot \nabla)\omega = \tilde{f}(t, x), \\
\nabla \cdot u = 0, \quad \text{in } (\tau, T) \times \Omega,
\end{gathered}
\tag{1.2}
$$

where $\Omega := \mathbb{R} \times (-L, L)$ for some $L > 0$, $\alpha := c_a + c_d$, and $x := (x_1, x_2) \in \Omega \subseteq \mathbb{R}^2$, $u := (u_1, u_2)$, $f := (f_1, f_2)$. ω and $\tilde{f}$ are scalar functions,

$$
\nabla \times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \quad \text{and} \quad \nabla \times \omega := (\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}).
$$

To complete the formulation of the initial boundary value problem to the system (1.2), We give the following initial data and boundary conditions:

$$
w(\tau, x) = (u(\tau, x), \omega(\tau, x)) = (u_0(x), \omega_0(x)), \quad x \in \Omega,\ \tau \in \mathbb{R}, \tag{1.3}
$$

$$
u = 0, \quad \omega = 0, \quad \text{on } (\tau, T) \times \partial\Omega. \tag{1.4}
$$

In fact, there have been some papers in the literature on the pullback asymptotic behaviors of solutions for the Navier-Stokes equations and the non-Newtonian fluid in some unbounded domains. For example, Caraballo, Łukaszewicz and Real [2] established the existence of a pullback attractor for a non-autonomous Navier-Stokes. Langa, Łukaszewicz and Real [16] proved the existence of pullback attractor for the non-autonomous Navier-Stokes equations in 2D unbounded domains. In particular, they gave sufficient conditions for their pullback attractor to have finite fractal dimension. Wang and Li [23] studied the existence of a pullback attractor for a non-autonomous 2D Navier-Stokes equation with linear dampness. Zhao [27] showed the existence of pullback attractor and their upper semicontinuity for the non-autonomous non-Newtonian fluid in 2D unbounded domain. Here, borrowing the ideas and argument in [27], we discuss the existence of pullback attractor and their tempered behavior and upper semicontinuity in 2D unbounded channel-like domains.

For the sake of convenience, we introduce the following useful operators:

$$
\begin{gathered}
\langle Aw,\phi\rangle := (\nu+\nu_\theta)(\nabla u,\nabla\Phi)+\alpha(\nabla\omega,\nabla\phi_3),\\
\forall w=(u,\omega),\ \varphi=(\Phi,\phi_3)\in\widehat{V},\\
\langle B(u,w),\phi\rangle := ((u\cdot\nabla)w,\phi),\quad \forall u\in V,\ w=(u,\omega)\in\widehat{V},\ \forall\phi\in\widehat{V},\\
N(w) := (-2\nu_\theta\nabla\times\omega,-2\nu_\theta\nabla\times u+4\nu_\theta\omega),\quad \forall w=(u,\omega)\in\widehat{V}.
\end{gathered}
\tag{1.5}
$$

Then, equations (1.2)-(1.4) can be represented into the abstract form

$$
\begin{gathered}
\frac{\partial w}{\partial t}+Aw+B(u,w)+N(w)=F(t,x),\quad \text{in } (\tau,+\infty)\times\Omega,\\
\nabla\cdot u=0,\quad \text{in } (\tau,+\infty)\times\Omega,\\
w=(u,\omega)=0,\quad \text{on } (\tau,+\infty)\times\partial\Omega,\\
w(\tau,x)=(u(\tau,x),\omega(\tau,x))=w_\tau(x),\quad x\in\Omega,\ \tau\in\mathbb{R},
\end{gathered}
\tag{1.6}
$$

where $F(t,x)=(f(t,x),\tilde{f}(t,x))$.

Before stating our results, we first give some notation. We denote by $L^p(\Omega)$ and $W^{m,p}(\Omega)$ the usual Lebesgue space and Sobolev space (see [1]) endowed with norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, respectively. For example, $\|\varphi\|_{L^p}=(\int_\Omega|\varphi|^p\mathrm{d}x)^{1/p}$ and $\|\varphi\|_{m,p}:=(\sum_{|\beta|\leqslant m}\int_\Omega|D^\beta\varphi|^p\mathrm{d}x)^{1/p}$. Especially, we denote $H^m(\Omega):=W^{m,2}(\Omega)$ and $H_0^1(\Omega)$ the closure of $\{\varphi\in\mathcal{C}_0^\infty(\Omega)\}$ with respect to $H^1(\Omega)$ norm. Then, we introduce the following function spaces:

- $\mathcal{V}:=\{\varphi\in\mathcal{C}_0^\infty(\Omega)$
- $\mathcal{C}_0^\infty(\Omega)|\,\varphi=(\varphi_1,\varphi_2),\nabla\cdot\varphi=0\}$,
- H is the closure of $\mathcal{V}$ in $L^2(\Omega)\times L^2(\Omega)$ with the norm $\|\cdot\|_H$ and dual space H^*,
- V is the closure of $\mathcal{V}$ in $H^1(\Omega)\times H^1(\Omega)$ with the norm $\|\cdot\|_V$ and dual space V^*,
- $\widehat{H}:=H\times L^2(\Omega)$ with the norm $\|\cdot\|_{\widehat{H}}$ and dual space $\widehat{H}^*$,
- $\widehat{V}:=V\times H_0^1(\Omega)$ with the norm $\|\cdot\|_{\widehat{V}}$ and dual space $\widehat{V}^*$,
- $\mathcal{O}_\sigma(B):=\{w\in\widehat{V}:\inf_{v\in B}\|w-v\|_{\widehat{V}}<\sigma\}$.

Using the above notation, we further denote

- $L^p(I;X)$ is the space of strongly measurable functions on the closed interval I, with values in a Banach space X, endowed with norm
$$\|\varphi\|_{L^p(I;X)}:=(\int_I\|\varphi\|_X^p\mathrm{d}t)^{1/p},\quad \text{for } 1\leqslant p<\infty;$$
- $\mathcal{C}(I;X)$ is the space of continuous functions on the interval I, with values in the Banach space X, endowed with the usual norm;

- $L^2_{\rm loc}(I;X)$ is the space of locally square integrable functions on the interval I, with values in the Banach space X, endowed with the usual norm;
- $L^2_b(I;X)$ is the set of functions $F \in L^2_{\rm loc}(I;X)$ satisfying
$$\|F\|_{L^2_b(I;X)} = \sup_{\tau\in\mathbb{R}} \int_\tau^{\tau+1} \|F(\zeta)\|^2 d\zeta < +\infty.$$

Here
$$\|(u,v)\|_H := (\|u\|_2^2 + \|v\|_2^2)^{1/2}, \quad \|(u,v)\|_V := (\|u\|_{H^1}^2 + \|v\|_{H^1}^2)^{1/2},$$
$$\|(u,v,w)\|_{\widehat{H}} := (\|(u,v)\|_H^2 + \|w\|_2^2)^{1/2}, \quad \|(u,v,w)\|_{\widehat{V}} := (\|(u,v)\|_V^2 + \|w\|_{H^1}^2)^{1/2}.$$

Subsequently, we simplify $\|\cdot\|_2$, $\|\cdot\|_H$ and $\|\cdot\|_{\widehat{H}}$ by the same notation $\|\cdot\|$ if there is no confusion. In addition, we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, H or $\widehat{H}$, and $\langle\cdot,\cdot\rangle$ the dual pairing between V and V^* or between $\widehat{V}$ and $\widehat{V}^*$. We also denote the compact embedding between spaces by $\hookrightarrow\hookrightarrow$, and use $\rm dist_M(X,Y)$ to represent the Hausdorff semidistance between $X \subseteq M$ and $Y \subseteq M$ with $\rm dist_M(X,Y) = \sup_{x\in X}\inf_{y\in Y} dist_M(x,y)$.

Now, we state the first results of this paper in the following theorem.

Theorem 1.1. *Assume $F(t,x) \in L^2_b(\mathbb{R};\widehat{H})$, $\int_{-\infty}^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2 {\rm d}s < \infty$, for all $t \in \mathbb{R}$ and*
$$\lim_{r\to+\infty} \int_{-\infty}^t \int_{|x|\geqslant r} e^{\frac{\delta_1}{2}s}|F(s,x)|^2 {\rm d}x{\rm d}s = 0.$$
Then system (1.6) *possesses a unique pullback $\mathcal{D}$-attractor $\mathcal{A}_{\widehat{H}}(t)$ in $\widehat{H}$.*

Remark 1.2. In fact, we point out that the pullback $\mathcal{D}$-attractor $\mathcal{A}_{\widehat{V}}(t)$ in $\widehat{V}$ can be obtained by using similar proof as that in $\widehat{H}$. Specifically, based on Lemma 2.4(2), there exists a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\widehat{V}$. Applying energy method, we can prove the existence of pullback $\mathcal{D}$-absorbing set for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\widehat{V}$. Then, the technique of truncation functions and decomposition of spatial domain enable us to obtain the uniform a priori estimates for the far-field values of solutions. Further, we can show the pullback asymptotic compactness of the process in $\widehat{V}$. Finally, we can obtain the existence of pullback $\mathcal{D}$-attractor $\mathcal{A}_{\widehat{V}}(t)$ in $\widehat{V}$.

Based on Theorem 1.1, we further verify the tempered behavior and upper semicontinuity of the pullback attractor obtained in Theorem 1.1. That is the following two theorems.

Theorem 1.3. *Under the conditions of Theorem 1.1, it holds that*
$$\lim_{t\to-\infty} \big(e^{\frac{\delta_1}{2}t} \sup_{w\in\mathcal{A}_{\widehat{H}}(t)} \|w\|^2\big) = 0, \tag{1.7}$$
$$\lim_{t\to-\infty} \big(e^{\frac{\delta_1}{2}t} \sup_{w\in\mathcal{A}_{\widehat{H}}(t)} \|w\|^2_{\widehat{V}(\Omega)}\big) = 0. \tag{1.8}$$

Theorem 1.4. *Assume the conditions of Theorem 1.1 hold. Then for any $t \in \mathbb{R}$, we have*
$$\lim_{m\to\infty} {\rm dist}_{\widehat{H}(\Omega)}(\mathcal{A}_{\widehat{H}(\Omega_m)}(t), \mathcal{A}_{\widehat{H}(\Omega)}(t)) = 0, \tag{1.9}$$

where $\mathcal{A}_{\widehat{H}(\Omega)}(t)$ *and* $\mathcal{A}_{\widehat{H}(\Omega_m)}(t)$ *are the pullback attractors of system* (1.6) *and system* (4.1)*, respectively.*

Remark 1.5. In [25], the authors proved the existence of H^1-uniform attractor and pointed out that the L^2-uniform attractor belongs to the H^1-uniform attractor, under the proper conditions of $F = (f, \tilde{f})$. Here, we established the existence of pullback attractor in $\widehat{H}(\Omega)$ (has L^2-regularity), similarly, in $\widehat{V}(\Omega)$ (has H^1-regularity). Further, we investigated the tempered behavior and upper semicontinuity (in the sense of (1.9)) of the pullback attractors. About the differences and more details between pullback attractor and uniform attractor, we can refer to [17] and some references therein.

The outline of proofs for Theorem 1.1–1.4 is as follows. First, we construct the continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ by solution maps in the space $\widehat{H}(\Omega)$, then show the asymptotic compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in space $\widehat{H}(\Omega)$. The main difficulty comes from two aspects. The first one is that the usual Sobolev embedding is no longer compact in unbounded domain Ω. The other is from the angular velocity field ω of the micropolar particles in the micropolar fluid flows, which leads to a different nonlinear term $B(u, w)$ and an additional term $N(u)$ in the abstract equation (see (1.5)). To overcome the first difficulty, borrowing the arguments and ideas in [27], we show the existence of the pullback absorbing set by establishing some a priori estimates of the solutions. Further, we use the technique of truncation function and the decomposition of spatial domain, to prove the asymptotic compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in space $\widehat{H}(\Omega)$. To deal with the second difficulty, more delicate estimates and analysis for the solutions are required in our study. Next, using the arguments in [11, 15, 26, 27], we can show the tempered behavior and upper semicontinuity of the pullback attractor $\mathcal{A}_{\widehat{H}(\Omega)}$. The tempered behavior of the pullback attractor $\mathcal{A}_{\widehat{H}(\Omega)}$ is relatively easy to obtain. To prove the upper semicontinuity of the attractor, we first let $\{\Omega_m\}_{m=1}^{\infty}$ be an expanding sequence of simply connected, bounded and smooth subdomains of Ω such that $\cup_{m=1}^{\infty}\Omega_m = \Omega$. Then we consider the Cauchy problem (1.2)-(1.4) in Ω_m. We will conclude that there exists a pullback attractor $\mathcal{A}_{\widehat{H}(\Omega_m)}$ for the problem (1.2)-(1.4) in each Ω_m. Finally, we establish the upper semicontinuity by showing $\lim_{m\to\infty} \operatorname{dist}_{\widehat{H}(\Omega)}(\mathcal{A}_{\widehat{H}(\Omega_m)}(t), \mathcal{A}_{\widehat{H}(\Omega)}(t)) = 0,\ \forall\, t \in \mathbb{R}$.

The rest of this paper is organized as follows. In section 2, we make some preliminaries. That is, we introduce several important definitions and recall some known results of non-autonomous micropolar fluid flows. Section 3 is committed to the proof of Theorem 1.1, that is to prove the existence of pullback attractors in $\widehat{H}(\Omega)$. Further, the tempered behavior and upper semicontinuity of the pullback attractors, i.e. Theorem 1.3 and Theorem 1.3, will be verified in section 4.

2. Preliminaries

In this section, we first make some necessary preliminaries. That is, we first give some useful properties and estimates about those operators (1.5). Then, we give some definitions and recall some key results for the non-autonomous micropolar fluid model. To begin with, we have

Lemma 2.1. *The operator* A *is a linear continuous operator both from* V *to* V^* *and from* $D(A) := V \cap \big(H^2(\Omega)\big)^3$ *to* H*. Indeed,* $A = -\mathbb{P}\Delta$*, where* $\mathbb{P}$ *is the Leray* *projector from $\mathbb{L}^2(\Omega)$ to H. The operator $B(\cdot,\cdot)$ is continuous from $V\times V$ to V^*. Moreover, for any $u\in V, w\in V$, it holds*

$$\langle B(u,w),\varphi\rangle = -\langle B(u,\varphi),w\rangle. \tag{2.1}$$

Proof. The linearity and continuity of the operator A can be deduced directly from its definition. Similarly, the continuity of the operator $B(\cdot,\cdot)$ can be obtained easily from its definition. We only need to verify (2.1). In fact, for any $u\in V, w\in\widehat{V}$, we have

$$\begin{aligned}
\langle B(u,w),w\rangle &= ((u\cdot\nabla)w,w)\\
&= \int_\Omega (u_1\frac{\partial}{\partial x_1}+u_2\frac{\partial}{\partial x_2}(w_1,w_2,w_3)(w_1,w_2,w_3)\mathrm{d}x\\
&= \sum_{j=1}^3\sum_{i=1}^2\int_\Omega u_i\frac{\partial w_j}{\partial x_i}w_j\mathrm{d}x\\
&= \sum_{j=1}^3\sum_{i=1}^2\frac12\int_\Omega u_i\frac{\partial w_j^2}{\partial x_i}\mathrm{d}x\\
&= \frac12\sum_{j=1}^3\sum_{i=1}^2(u_iw_j^2|_{\partial\Omega}-\int_\Omega w_j^2D_iu_i\mathrm{d}x)\\
&= -\frac12\sum_{j=1}^3\sum_{i=1}^2\int_\Omega w_j^2D_iu_i\mathrm{d}x\\
&= -\frac12\sum_{j=1}^3\int_\Omega w_j^2(\nabla\cdot u)\mathrm{d}x=0.
\end{aligned}\tag{2.2}$$

Hence, (2.1) is valid as a consequence of (2.2). The proof is complete. □

About the operators $A(w)$ and $N(w)$, we have the following result.

Lemma 2.2 (see [13, 25, 29])**.** (1) *There are two positive constants c_1 and c_2 such that*

$$c_1\langle Aw,w\rangle \leqslant \|w\|_{\widehat{V}}^2 \leqslant c_2\langle Aw,w\rangle,\quad \forall w\in\widehat{V}. \tag{2.3}$$

(2) *There exists a positive constant $c(\nu_\theta)$ such that*

$$\|N(w)\|\leqslant c(\nu_\theta)\|w\|_{\widehat{V}},\ \ \forall w\in\widehat{V}, \tag{2.4}$$

$$\langle Aw,w\rangle+\langle N(w),w\rangle\geqslant \delta_1\|w\|_{\widehat{V}}^2,\ \ \forall w\in\widehat{V}, \tag{2.5}$$

where $\delta_1:=\min\{\nu,\alpha\}$.

Next, we give the definition and existence result of weak solutions for (1.6).

Definition 2.3. For each $T>\tau,\tau\in\mathbb{R}$, function w is called a weak solution of (1.6) if, $w=(u,\omega)\in L^2(\tau,T;\widehat{V})\cap L^\infty(\tau,T;\widehat{H})$ such that for $t\in(\tau,T)$ and any $\varphi\in\widehat{V}$,

$$\begin{gathered}
\frac{\mathrm{d}}{\mathrm{d}t}(w(t),\varphi)+\langle Aw(t),\varphi\rangle+\langle B(u(t),w(t)),\varphi\rangle+\langle N(w(t)),\varphi\rangle=\langle F(t,x),\varphi\rangle,\\
w\big|_{t=\tau}=w_\tau=(u_\tau,\omega_\tau)=w_0
\end{gathered}\tag{2.6}$$

holds in the sense of $\mathcal{D}'(\tau,T)$.

Using decomposition of spatial domain and the Galerkin method, we immediately have the following well-posedness of solutions to (1.6) on unbounded domain Ω.

Lemma 2.4. *Assume $F(t,x)=(f(t,x),\tilde f(t,x))\in L^2_b(\mathbb{R};\widehat H(\Omega))$.*

(1) *If $w_\tau\in\widehat H$, then system (1.6) has a unique solution $w=(u,\omega)$ satisfying*
$$w\in L^\infty(\tau,+\infty;\widehat H)\cap\mathcal{C}([\tau,+\infty);\widehat H)\cap L^2_{\rm loc}(\tau,+\infty;\widehat V),\ w'\in L^2_{\rm loc}(\tau,+\infty;\widehat V^*).$$

(2) *If $w_\tau\in\widehat V$, then problem (1.6) has a unique solution $w=(u,\omega)$ satisfying*
$$w\in L^\infty(\tau,+\infty;\widehat V)\cap\mathcal{C}([\tau,+\infty);\widehat V)\cap L^2_{\rm loc}(\tau,+\infty;D(A)),\ w'\in L^2_{\rm loc}(\tau,+\infty;\widehat H).$$

Since the proof is standard, we omit it. The interesting readers are referred to [14].

On the basis of Lemma 2.4, the biparametric map defined by
$$U(t,\tau):w_\tau\mapsto U(t,\tau)w_\tau=w(t),\ t\geqslant\tau,\ w_\tau\in\widehat H(\Omega)\text{ or }w_\tau\in\widehat V(\Omega),\tag{2.7}$$
generates a continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\widehat H(\Omega)$ or $\widehat V(\Omega)$, which satisfies the following properties:

(i) $U(\tau,\tau)w_\tau=w_\tau$,
(ii) $U(t,s)U(s,\tau)w_\tau=U(t,\tau)w_\tau=w(t)$.

Finally, we introduce some definitions related to the pullback attractor (see [10, 22, 19, 28]). For convenience, we denote by X the space $\widehat H$ or $\widehat V$ and by $\mathcal{P}(X)$ the family of all nonempty subsets of X. A universe $\mathcal{D}(X)$ in $\mathcal{P}(X)$ represents the class of families parameterized in time $\widehat B(X)=\{B(t)\big|t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$.

Definition 2.5. A family of sets $\widehat B_0=\{B_0(t):t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ is called pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ in X if for any $t\in\mathbb{R}$ and any $\widehat B=\{B(t):t\in\mathbb{R}\}\in\mathcal{D}$, there exists a $\tau_0(t,\widehat B)\leqslant t$ such that $U(t,\tau)B(\tau)\subseteq B_0(t)$ for all $\tau\leqslant\tau_0(t,\widehat B)$.

Definition 2.6. The process $\{U(t,\tau)\}_{t\geqslant\tau}$ is said to be pullback $\widehat B_0$-asymptotically compact in X if for any $t\in\mathbb{R}$, any sequences $\{\tau_n\}\subseteq(-\infty,t]$ and $\{x_n\}\subseteq X$ satisfying $\tau_n\to-\infty$ as $n\to\infty$ and $x_n\in B_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n;x_n)\}$ is relatively compact in X. $\{U(t,\tau)\}_{t\geqslant\tau}$ is called pullback $\mathcal{D}$-asymptotically compact in X if it is pullback $\widehat B$-asymptotically compact for any $\widehat B\in\mathcal{D}$.

Definition 2.7. A family of sets $\mathcal{A}_X=\{\mathcal{A}_X(t):t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ is called a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geqslant\tau}$ on X if it has the following properties:

- Compactness: for any $t\in\mathbb{R},\mathcal{A}_X(t)$ is a nonempty compact subset of X;
- Invariance: $U(t,\tau)\mathcal{A}_X(\tau)=\mathcal{A}_X(t)$, $\forall\,t\geqslant\tau$;
- Pullback attracting: $\mathcal{A}_X$ is pullback $\mathcal{D}$-attracting in the following sense:
$$\lim_{\tau\to-\infty}{\rm dist}_X\left(U(t,\tau)B(\tau),\mathcal{A}_X(t)\right)=0,\quad\forall\widehat B=\{B(s)|\ s\in\mathbb{R}\}\in\mathcal{D},t\in\mathbb{R};$$
- Minimality: the family of sets $\mathcal{A}_X$ is the minimal in the sense that if $\widehat O=\{O(t):t\in\mathbb{R}\}\subseteq\mathcal{P}(X)$ is another family of closed sets such that
$$\lim_{\tau\to-\infty}{\rm dist}_X(U(t,\tau)B(\tau),O(t))=0,\ \text{ for any }\widehat B=\{B(t)|\,t\in\mathbb{R}\}\in\mathcal{D},$$
then $\mathcal{A}_X(t)\subseteq O(t)$ for $t\in\mathbb{R}$.

3. Existence of the pullback $\mathcal{D}$-attractor in $\widehat{H}$

In this section, we are devoted to proving the existence of the pullback attractor in $\widehat{H}$, i.e. Theorem 1.1. Throughout this section, we simply denote $w(t;\tau,w_\tau)$ by $w(t)$ if there is no confusion. First, we have

Lemma 3.1. *Assume $F(t,x)\in L_b^2(\mathbb{R};\widehat{H})$, then for any $w_\tau=(u_\tau,\omega_\tau)\in\widehat{H}$, it holds that*

$$\|w(t;\tau,w_\tau)\|^2\leqslant e^{\frac{\delta_1}{2}(\tau-t)}\|w_\tau\|^2+\frac{e^{-\frac{\delta_1}{2}t}}{\delta_1}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s, \tag{3.1}$$

$$\int_\tau^t e^{\frac{\delta_1}{2}s}\|w(s)\|_{\widehat{V}}^2\mathrm{d}s\leqslant\frac{2}{\delta_1}e^{\frac{\delta_1}{2}\tau}\|w_\tau\|^2+\frac{2}{\delta_1^2}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s. \tag{3.2}$$

Proof. Multiplying $(1.6)_1$ by $w(t)$, we obtain from (2.2) and (2.5) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\delta_1\|w(t)\|_{\widehat{V}}^2\leqslant(F(t,x),w(t))\leqslant\frac{1}{2\delta_1}\|F(t)\|^2+\frac{\delta_1}{2}\|w(t)\|_{\widehat{V}}^2,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\delta_1\|w(t)\|^2\leqslant\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|^2+\delta_1\|w(t)\|_{\widehat{V}}^2\leqslant\frac{1}{\delta_1}\|F(t)\|^2. \tag{3.3}$$

Further, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\frac{\delta_1 t}{2}}\|w(t)\|^2)\leqslant\frac{e^{\frac{\delta_1 t}{2}}}{\delta_1}\|F(t,x)\|^2.$$

Changing the time variable t by s and integrating it over $[\tau,t]$, we obtain

$$e^{\frac{\delta_1 t}{2}}\|w(t)\|^2\leqslant e^{\frac{\delta_1\tau}{2}}\|w_\tau\|^2+\frac{1}{\delta_1}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s,$$

hence

$$\|w(t)\|^2\leqslant e^{\frac{\delta_1(\tau-t)}{2}}\|w_\tau\|^2+\frac{e^{-\frac{\delta_1 t}{2}}}{\delta_1}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s. \tag{3.4}$$

From (3.3), we can also deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\frac{\delta_1}{2}t}\|w(t)\|^2)+\frac{\delta_1 e^{\frac{\delta_1}{2}t}}{2}\|w(t)\|_{\widehat{V}}^2\leqslant\frac{e^{\frac{\delta_1}{2}t}}{\delta_1}\|F(t)\|^2.$$

Further, one have

$$e^{\frac{\delta_1}{2}t}\|w(t)\|^2+\frac{\delta_1}{2}\int_\tau^t e^{\frac{\delta_1}{2}s}\|w(s)\|_{\widehat{V}}^2\mathrm{d}s\leqslant\frac{1}{\delta_1}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s+e^{\frac{\delta_1}{2}\tau}\|w_\tau\|^2,$$

which implies

$$\int_\tau^t e^{\frac{\delta_1}{2}s}\|w(s)\|_{\widehat{V}}^2\mathrm{d}s\leqslant\frac{2}{\delta_1}e^{\frac{\delta_1}{2}\tau}\|w_\tau\|^2+\frac{2}{\delta_1^2}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s.$$

This completes the proof. □

Set

$$\mathcal{R}_{\frac{\delta_1}{2}}:=\{\rho(s):\mathbb{R}\to\mathbb{R}_+:\lim_{s\to-\infty}e^{\frac{\delta_1}{2}s}\rho^2(s)=0\}. \tag{3.5}$$

$\mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H})$ denotes the class of families $\widehat{B}=\{B(s)\big|s\in\mathbb{R}\}\subseteq\mathcal{P}(\widehat{H})$ such that

$$B(s)\subseteq\bar{\mathcal{B}}(0,\rho_{\widehat{B}}(s))\text{ for some }\rho_{\widehat{B}}(s)\in\mathcal{R}_{\frac{\delta_1}{2}},$$

where $\bar{\mathcal{B}}(0,\rho_{\widehat{B}}(s))$ is the closed ball in $\widehat{H}$ with center 0 and radius $\rho_{\widehat{B}}(s)$. Then, based on Lemma 3.1, there exists a pullback $\mathcal{D}$-absorbing set in $\widehat{H}$.

Lemma 3.2. *Under the condition of Lemma 3.1. For any $t\in\mathbb{R}$, $\widehat{B}=\{B(s)\big|s\in\mathbb{R}\}\in\mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H})$ and $w_\tau\in B(\tau)$, the family $\widehat{\mathcal{B}}=\{\mathcal{B}(t)\big|t\in\mathbb{R}\}$ defined by*

$$\mathcal{B}(t)=\{w\in\widehat{H}:\|w\|\leqslant\rho(t)\} \tag{3.6}$$

is pullback $\mathcal{D}$-absorbing in $\widehat{H}$, where

$$\rho^2(t):=\frac{2e^{-\frac{\delta_1}{2}t}}{\delta_1}\int_{-\infty}^{t}e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s. \tag{3.7}$$

In the following, we focus on proving the pullback $\mathcal{D}$-asymptotical compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$. First, we have

Lemma 3.3. *Under the conditions of Theorem 1.1. For any $\epsilon>0$, $t\in\mathbb{R}$ and $\widehat{B}=\{B(s)\big|s\in\mathbb{R}\}\in\mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H})$, there exist $r_0:=r_0(\epsilon,t,\widehat{B})>0$ and $\tau_0:=\tau_0(\epsilon,t,\widehat{B})<t$ such that for any $r\geqslant r_0,\tau\leqslant\tau_0$ and $w_\tau\in B(\tau)$, it holds*

$$\|w(t;\tau,w_\tau)\|^2_{\mathbb{L}^2(\Omega\backslash\Omega_r)}\leqslant\epsilon, \tag{3.8}$$

where $\Omega_r=\{x\in\Omega\big||x|<r\}$.

Proof. First, we take a function $\chi(\cdot)\in C^2(\mathbb{R}^2)$, $\chi(x)\in[0,1]$ for all $x\in\mathbb{R}^2$ such that

$$\chi(x)=\begin{cases}0, & |x|\leqslant 1,\\ 1, & |x|\geqslant 2,\end{cases}$$

and

$$\|\nabla\chi(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)}\leqslant c_0,\quad \|D^2\chi(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)}\leqslant c_0,$$

where c_0 is a constant. In particular, set $\chi_r(x)=\chi(\frac{x}{r})$ with $r\geqslant 1$, we have

$$\|\nabla\chi_r(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)}\leqslant\frac{c_0}{r},\quad \|D^2\chi_r(x)\|_{\mathbb{L}^\infty(\mathbb{R}^2)}\leqslant\frac{c_0}{r^2}. \tag{3.9}$$

Then, taking the inner product of $(1.6)_1$ with $\chi_r^2w=(\chi_r^2u,\chi_r^2\omega)$ and considering the term ∇p yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_rw\|^2+(\nu+\nu_\theta)(\nabla u,\nabla(\chi_r^2u))+\alpha(\nabla\omega,\nabla(\chi_r^2\omega))+((u\cdot\nabla)w,\chi_r^2w)\\
&+((-2\nu_\theta\nabla\times\omega,-2\nu_\theta\nabla\times u+4\nu_\theta\omega),(\chi_r^2u,\chi_r^2\omega))+(\nabla p,\chi_r^2u)\\
&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_rw\|^2+\langle A(\chi_rw),\chi_rw\rangle-(\nu+\nu_\theta)\int_\Omega|u\nabla\chi_r|^2\mathrm{d}x-\alpha\int_\Omega|\omega\nabla\chi_r|^2\mathrm{d}x\\
&\quad+((u\cdot\nabla)w,\chi_r^2w)+\langle N(\chi_rw),\chi_rw\rangle+(\nabla p,\chi_r^2u)\\
&=(F(t,x),\chi_r^2w).
\end{aligned} \tag{3.10}$$

Let us estimate the terms in the above equality one by one. From (3.9) and Hölder inequality, we have

$$\begin{aligned}
(\nu+\nu_\theta)\int_\Omega|u\nabla\chi_r|^2\mathrm{d}x&\leqslant(\nu+\nu_\theta)\|\nabla\chi_r\|^2_{\mathbb{L}^\infty(\Omega)}\|u\|^2\\
&\leqslant c_0^2(\nu+\nu_\theta)r^{-2}\|u\|^2.
\end{aligned} \tag{3.11}$$

Similarly, we have

$$\alpha \int_{\Omega} |\omega \nabla \chi_r|^2 \mathrm{d}x \leqslant \alpha \|\nabla \chi_r\|^2_{\mathbb{L}^\infty(\Omega)} \|\omega\|^2 \leqslant c_0^2 \alpha r^{-2} \|\omega\|^2. \tag{3.12}$$

Integrating by parts and using that $\nabla \cdot u = 0$, we obtain

$$\begin{aligned}
&((u\cdot\nabla)w, \chi_r^2 w) = \sum_{i,j=1}^{2} \int_{\Omega} u_i \frac{\partial u_j}{\partial x_i} \chi_r^2 u_j \mathrm{d}x + \sum_{i=1}^{2} \int_{\Omega} u_i \frac{\partial \omega}{\partial x_i} \chi_r^2 \omega \mathrm{d}x \\
&= -\sum_{i,j=1}^{2} (\int_{\Omega} u_i u_j \chi_r^2 \frac{\partial u_j}{\partial x_i} \mathrm{d}x + 2\int_{\Omega} u_i u_j^2 \chi_r \frac{\partial \chi_r}{\partial x_i} \mathrm{d}x) \\
&\quad - \sum_{i=1}^{2} (\int_{\Omega} u_i \omega \chi_r^2 \frac{\partial \omega}{\partial x_i} \mathrm{d}x + 2\int_{\Omega} u_i \omega^2 \chi_r \frac{\partial \chi_r}{\partial x_i} \mathrm{d}x) \\
&= -((u\cdot\nabla)w, \chi_r^2 w) - 2\sum_{i,j=1}^{2} \int_{\Omega} u_i u_j^2 \chi_r \frac{\partial \chi_r}{\partial x_i} \mathrm{d}x - 2\sum_{i=1}^{2} \int_{\Omega} u_i \omega^2 \chi_r \frac{\partial \chi_r}{\partial x_i} \mathrm{d}x,
\end{aligned}$$

which combine with (3.9), Hölder inequality, Gagliardo-Nirenberg inequality and Young's inequality yields

$$\begin{aligned}
&|((u\cdot\nabla)w, \chi_r^2 w)| = |\sum_{i,j=1}^{2} \int_{\Omega} u_i u_j^2 \chi_r \frac{\partial \chi_r}{\partial x_i} \mathrm{d}x + \sum_{i=1}^{2} \int_{\Omega} u_i \omega^2 \chi_r \frac{\partial \chi_r}{\partial x_i} \mathrm{d}x| \\
&\leqslant \|\nabla \chi_r\|_{\mathbb{L}^\infty(\Omega)} \|u\| \|w\|^2_{\mathbb{L}^4(\Omega)} \leqslant c_0 r^{-1} \|u\| \|w\| \|w\|_{\widehat{V}} \leqslant \frac{c_0}{r} (\|w\|^4 + \|w\|^2_{\widehat{V}}).
\end{aligned} \tag{3.13}$$

Since $F(t,x) \in L^2_b(\mathbb{R}; \widehat{H})$, it follows from Lemma 3.2 that for any $t \in \mathbb{R}, \tau \leqslant \tau_0$,

$$\begin{aligned}
&\|w(t;\tau,w_\tau)\|^2 \leqslant \rho^2(t) \\
&\leqslant \frac{2e^{-\frac{\delta_1}{2}t}}{\delta_1} \int_{-\infty}^{t} e^{\frac{\delta_1}{2}s} \|F(s)\|^2 \mathrm{d}s \\
&= \frac{2}{\delta_1} \Big(\int_{t-1}^{t} e^{-\frac{\delta_1}{2}(t-s)} \|F(s)\|^2 \mathrm{d}s + \int_{t-2}^{t-1} e^{-\frac{\delta_1}{2}(t-s)} \|F(s)\|^2 \mathrm{d}s + \dots \Big) \\
&\leqslant \frac{2}{\delta_1} \Big(\int_{t-1}^{t} \|F(s)\|^2 \mathrm{d}s + e^{-\frac{\delta_1}{2}} \int_{t-2}^{t-1} \|F(s)\|^2 \mathrm{d}s \\
&\quad + e^{-\frac{2\delta_1}{2}} \int_{t-3}^{t-2} \|F(s)\|^2 \mathrm{d}s + \dots \Big) \\
&\leqslant \frac{2}{\delta_1} (1 + e^{-\frac{\delta_1}{2}} + e^{-\frac{2\delta_1}{2}} + \dots) \|F\|^2_{L^2_b(\mathbb{R};\widehat{H})} = \frac{2}{\delta_1 (1 - e^{-\frac{\delta_1}{2}})} \|F\|^2_{L^2_b(\mathbb{R};\widehat{H})} \\
&\leqslant \frac{2}{\delta_1} (1 + \frac{2}{\delta_1}) \|F\|^2_{L^2_b(\mathbb{R};\widehat{H})}.
\end{aligned} \tag{3.14}$$

By $(1.6)_3$, (3.9) and the fact $\nabla \cdot u = 0$, we also have

$$\begin{aligned}
|(\nabla p, \chi_r^2 u)| &= |\sum_{i=1}^{2} \int_{\Omega} \frac{\partial p}{\partial x_i} \chi_r^2 u_i \mathrm{d}x| = |\sum_{i=1}^{2} \int_{\Omega} 2p\chi_r \frac{\partial \chi_r}{\partial x_i} u_i \mathrm{d}x| \\
&\leqslant 2\|p\| \|\nabla \chi_r\|_{L^\infty(\Omega)} \|\chi_r u\| \leqslant 2c_0 r^{-1} \|p\| \|\chi_r u\|.
\end{aligned} \tag{3.15}$$

Taking (2.5) and (3.10)-(3.15) into account, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_r w\|^2+\delta_1\|\chi_r w\|^2_{\widehat{V}}\\
&\leqslant \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_r w\|^2+\langle A(\chi_r w),\chi_r w\rangle+\langle N(\chi_r w),\chi_r w\rangle\\
&=(\nu+\nu_\theta)\int_\Omega |u\nabla\chi_r|^2\mathrm{d}x+\alpha\int_\Omega|\omega\nabla\chi_r|^2\mathrm{d}x-((u\cdot\nabla)w,\chi_r^2 w)\\
&\quad+(F(t,x),\chi_r^2 w)-(\nabla p,\chi_r^2 u)\\
&\leqslant \frac{c_0^2(\nu+\nu_\theta)}{r^2}\|u\|^2+\frac{c_0^2\alpha}{r^2}\|\omega\|^2+\frac{c_0}{r}(\|w\|^4+\|w\|^2_{\widehat{V}})+\|\chi_r F\|\|\chi_r w\|\\
&\quad+\frac{2c_0}{r}\|p\|\|\chi_r u\|\\
&\leqslant \frac{c_0^2\cdot\max\{\nu+\nu_\theta,\alpha\}}{r^2}\|w\|^2+\frac{4c_0}{r\delta_1^2}(1+\frac{2}{\delta_1})^2\|F\|^4_{L_b^2(\mathbb{R};\widehat{H})}+\frac{c_0}{r}\|w\|^2_{\widehat{V}}\\
&\quad+\frac{1}{\delta_1}\|\chi_r F\|^2+\frac{4c_0^2}{\delta_1 r^2}\|p\|^2+\frac{\delta_1}{2}\|\chi_r w\|^2.
\end{aligned}
\tag{3.16}
$$

Further, by Lemma 3.2 and that $\|u\|\leqslant\|w\|$ and $\|w\|\leqslant\|w\|_{\widehat{V}}$, one can deduce that there exist constants c_3,c_4,c_5 such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\chi_r w\|^2+\frac{\delta_1}{2}\|\chi_r w\|^2+\frac{\delta_1}{2}\|\chi_r w\|^2_{\widehat{V}}\\
&\leqslant \frac{c_3}{r^2}\rho^2(t)+\frac{c_4}{r}\|F\|^4_{L_b^2(\mathbb{R};\widehat{H})}+\frac{c_0}{r}\|w\|^2_{\widehat{V}}+\frac{1}{\delta_1}\|\chi_r F\|^2+\frac{c_5}{r^2}\|p\|^2,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(e^{\frac{\delta_1}{2}t}\|\chi_r w\|^2)\leqslant{}& \frac{c_3}{r^2}e^{\frac{\delta_1}{2}t}\rho^2(t)+\frac{c_4}{r}e^{\frac{\delta_1}{2}t}\|F\|^4_{L_b^2(\mathbb{R};\widehat{H})}+\frac{c_0}{r}e^{\frac{\delta_1}{2}t}\|w\|^2_{\widehat{V}}\\
&+\frac{c_5}{r^2}e^{\frac{\delta_1}{2}t}\|p\|^2+\frac{1}{\delta_1}e^{\frac{\delta_1}{2}t}\|\chi_r F\|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\chi_r w(t)\|^2\\
&\leqslant e^{-\frac{\delta_1}{2}(t-\tau)}\|\chi_r w_\tau\|^2+\frac{c_3}{r^2}e^{-\frac{\delta_1}{2}t}\int_\tau^t e^{\frac{\delta_1}{2}s}\rho^2(s)\mathrm{d}s\\
&\quad+\frac{c_4}{r}e^{-\frac{\delta_1}{2}t}\int_\tau^t e^{\frac{\delta_1}{2}s}\|F\|^4_{L_b^2(\mathbb{R};\widehat{H})}\mathrm{d}s+\frac{c_0}{r}e^{-\frac{\delta_1}{2}t}\int_\tau^t e^{\frac{\delta_1}{2}s}\|w\|^2_{\widehat{V}}\mathrm{d}s\\
&\quad+\frac{c_5}{r^2}e^{-\frac{\delta_1}{2}t}\int_\tau^t e^{\frac{\delta_1}{2}s}\|p\|^2\mathrm{d}s+\frac{1}{\delta_1}e^{-\frac{\delta_1}{2}t}\int_\tau^t e^{\frac{\delta_1}{2}s}\|\chi_r F\|^2\mathrm{d}s.
\end{aligned}
\tag{3.17}
$$

In the following, we make a more detailed detailed estimate for each term in (3.17). First, for any $\epsilon>0$, there exists a $\tau_1=\tau_1(\epsilon,t,\widehat{D})$ such that

$$
e^{-\frac{\delta_1}{2}(t-\tau)}\|\chi_r w_\tau\|^2\leqslant e^{-\frac{\delta_1}{2}(t-\tau)}\|w_\tau\|^2<\frac{\epsilon}{6}\quad\text{for all }\tau\leqslant\tau_1.
\tag{3.18}
$$

Then, from the condition

$$
\int_{-\infty}^t e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s<\infty,\quad\forall t\in\mathbb{R},
$$

it is not difficult to check that there exists a $r_1 = r_1(\epsilon, t, \widehat{D})$ such that for any $r \geqslant r_1$,

$$\frac{c_3}{r^2} e^{-\frac{\delta_1}{2}t} \int_\tau^t e^{\frac{\delta_1}{2}s} \rho^2(s) \mathrm{d}s < \frac{\epsilon}{6}, \tag{3.19}$$

$$\frac{c_4}{r} e^{-\frac{\delta_1}{2}t} \int_\tau^t e^{\frac{\delta_1}{2}s} \|F\|^4_{L^2_b(\mathbb{R};\widehat{H})} \mathrm{d}s < \frac{\epsilon}{6}, \tag{3.20}$$

$$\frac{1}{\delta_1} e^{-\frac{\delta_1}{2}t} \int_\tau^t e^{\frac{\delta_1}{2}s} \|\chi_r F\|^2 \mathrm{d}s \leqslant \frac{1}{\delta_1} e^{-\frac{\delta_1}{2}t} \int_{-\infty}^t \int_{|x|\geqslant r} e^{\frac{\delta_1}{2}s} |F(s,x)|^2 \mathrm{d}x\mathrm{d}s < \frac{\epsilon}{6}. \tag{3.21}$$

Moreover, it follows from (1.2) that $\nabla p \in L^2_{\rm loc}(\tau, +\infty; H^{-1}(\Omega))$, which implies $p \in L^2_{\rm loc}(\tau, +\infty; L^2(\Omega))$. In addition, noting that

$$\int_\tau^t e^{\frac{\delta_1}{2}s} \|p(s)\|^2 \mathrm{d}s \leqslant c \int_\tau^t e^{\frac{\delta_1}{2}s} \|w(s)\|^2_{\widehat{V}} \mathrm{d}s,$$

and using (3.2) and the condition

$$\lim_{r\to+\infty} \int_{-\infty}^t \int_{|x|\geqslant r} e^{\frac{\delta_1}{2}s} |F(s,x)|^2 \mathrm{d}x\mathrm{d}s = 0,$$

we obtain that there exists $r_2 = r_2(\epsilon, t, \widehat{D})$ such that for any $r \geqslant r_2$, it holds that

$$\frac{c_5}{r^2} e^{-\frac{\delta_1}{2}t} \int_\tau^t e^{\frac{\delta_1}{2}s} \|p\|^2 \mathrm{d}s \leqslant \frac{cc_5}{r^2} e^{-\frac{\delta_1}{2}t} \int_\tau^t e^{\frac{\delta_1}{2}s} \|w(s)\|^2_{\widehat{V}} \mathrm{d}s < \frac{\epsilon}{6}, \tag{3.22}$$

and

$$\frac{c_0}{r} e^{-\frac{\delta_1}{2}t} \int_\tau^t e^{\frac{\delta_1}{2}s} \|w(s)\|^2_{\widehat{V}} \mathrm{d}s < \frac{\epsilon}{6}. \tag{3.23}$$

Substituting (3.18)-(3.23) into (3.17), we immediately have (3.8). This completes the proof. □

Lemma 3.4. *Assume the conditions of Theorem 1.1 hold, then for any $t \in \mathbb{R}$ and $\widehat{B} = \{B(s)|s \in \mathbb{R}\} \in \mathcal{D}(\widehat{H})$, there exists a $\tau^*(\widehat{B}, t)$ such that the weak solution $w(t) := w(t; \tau, w_\tau)$ of (1.6) with initial value $w_\tau \in B(\tau)$ is bounded in $\widehat{V}$.*

Proof. To complete the proof, we need a higher regularity of the solutions. Hence, we consider the Galerkin approximate solutions. For each integer $n \geqslant 1$, we denote by

$$w_n(t) = w_n(t; \tau, w_\tau) := \sum_{i=1}^n \xi_{ni}(t) e_i, \tag{3.24}$$

the Galerkin approximation of the solution $w(t)$ of system (1.6), where $\xi_{ni}(t)$ is the solution of the following Cauchy problem of ODEs:

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t}(w_n(t), e_i) + \langle A w_n(t) + B(u_n(t), w_n(t)) + N(w_n(t)), e_i \rangle = (F(t), e_i), \\ (w_n(\tau), e_i) = (w_\tau, e_i), \ \ i = 1, 2, \dots, n, \end{gathered} \tag{3.25}$$

here $\{e_i : i \geqslant 1\} \subseteq D(A)$, which forms a Hilbert basis of $\widehat{V}$ and is orthonormal in $\widehat{H}$. Multiplying equation (3.25) by $A\xi_{ni}(t)$ and summing them for $i = 1$ to n, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n(t), w_n(t)\rangle + \|Aw_n(t)\|^2 + \langle B(u_n(t), w_n(t)), Aw_n(t)\rangle \\
&+ \langle N(w_n(t)), Aw_n(t)\rangle \\
&= (F(t), Aw_n(t)).
\end{aligned} \tag{3.26}$$

Now, we give a further estimate for the above equation. According to the definition of $B(\cdot,\cdot)$ and the facts

$$\|u_n\|^2 \leqslant \|w_n\|^2, \quad \|\nabla u_n\|^2 \leqslant \|w_n\|_{\widehat{V}}^2,$$

and using the Hölder inequality, Gagliardo-Nirenberg inequality and Young inequality, we conclude that there exists a constant c_6 such that

$$\begin{aligned}
-\langle B(u_n, w_n), Aw_n\rangle &\leqslant |\langle B(u_n, w_n), Aw_n\rangle| \\
&\leqslant c_6\|u_n\|^{1/2}\|\nabla u_n\|^{1/2}\|\nabla w_n\|^{1/2}\|Aw_n\|^{3/2} \\
&\leqslant \frac{1}{4}\|Aw_n\|^2 + c_6^4\|w_n\|^2\|w_n\|_{\widehat{V}}^4,
\end{aligned}$$

which together with (2.4) and (3.26) implies that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n, w_n\rangle \\
&= -\|Aw_n\|^2 + \langle F(t), Aw_n\rangle - \langle B(u_n, w_n), Aw_n\rangle - \langle N(w_n), Aw_n\rangle \\
&\leqslant -\|Aw_n\|^2 + \frac{1}{4}\|Aw_n\|^2 + \|F(t)\|^2 + \frac{1}{4}\|Aw_n\|^2 + c_6^4\|w_n\|^2\|w_n\|_{\widehat{V}}^4 \\
&\quad + \frac{c^2(\nu_\theta)}{2}\|w_n\|_{\widehat{V}}^2 + \frac{1}{2}\|Aw_n\|^2 \\
&= \|F(t)\|^2 + \|w_n\|_{\widehat{V}}^2\Big(c_6^4\|w_n\|^2\|w_n\|_{\widehat{V}}^2 + \frac{c^2(\nu_\theta)}{2}\Big).
\end{aligned}$$

Further, from (2.3) and the above inequality, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle Aw_n(t), w_n(t)\rangle \\
&\leqslant 2\|F(t)\|^2 + \langle Aw_n(t), w_n(t)\rangle\Big(2c_2c_6^4\|w_n(t)\|^2\|w_n(t)\|_{\widehat{V}}^2 + c_2c^2(\nu_\theta)\Big).
\end{aligned} \tag{3.27}$$

Let us set

$$\begin{gathered}
H_n(\theta) := \langle Aw_n(\theta), w_n(\theta)\rangle,\ I(\theta) := 2\|F(\theta)\|^2, \\
K_n(\theta) := 2c_2c_6^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^2 + c_2c^2(\nu_\theta).
\end{gathered}$$

Replacing the variable t with θ in (3.27), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\theta}H_n(\theta) \leqslant K_n(\theta)H_n(\theta) + I(\theta). \tag{3.28}$$

Using Gronwall inequality to (3.28), for all $\tau \leqslant t-1 \leqslant s \leqslant t$, we have

$$H_n(t) \leqslant \big(H_n(s) + \int_{t-1}^t I(\theta)\mathrm{d}\theta\big)\exp\Big\{\int_{t-1}^t K_n(\theta)\mathrm{d}\theta\Big\}. \tag{3.29}$$

Integrating (3.29) from $s = t-1$ to $s = t$, we obtain

$$H_n(t) \leqslant \Big(\int_{t-1}^t H_n(s)\mathrm{d}s + \int_{t-1}^t I(\theta)\mathrm{d}\theta\Big)\exp\Big\{\int_{t-1}^t K_n(\theta)\mathrm{d}\theta\Big\}. \tag{3.30}$$

In addition, it follows from (2.3) and (3.2) that

$$
\begin{aligned}
\int_{t-1}^{t} H_n(s)\mathrm{d}s + \int_{t-1}^{t} I(\theta)\mathrm{d}\theta &= \int_{t-1}^{t} \langle Aw_n(s), w_n(s)\rangle \mathrm{d}s + \int_{t-1}^{t} 2\|F(\theta)\|^2\mathrm{d}\theta \\
&\leqslant {c_1}^{-1}\int_{t-1}^{t} \|w_n(s)\|_{\widehat{V}}^2\mathrm{d}s + 2\int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta \\
&\leqslant c_7\big(\|w_n(t-1)\|^2 + \int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big),
\end{aligned}
$$

where $c_7 := \max\{2c_1^{-1}\delta_1^{-1}, 2+2c_1^{-1}\delta_1^{-2}e^{\frac{\delta_1}{2}}\}$. From (3.1), it holds

$$
\begin{aligned}
&\int_{t-1}^{t} K_n(\theta)\mathrm{d}\theta \\
&= \int_{t-1}^{t} \big(2c_2c_6^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\widehat{V}}^2 + c_2c^2(\nu_\theta)\big)\mathrm{d}\theta \\
&\leqslant 2c_2c_6^4\big(e^{-\frac{\delta_1}{2}}\|w_n(t-1)\|^2 + \frac{1}{\delta_1}\int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big)\int_{t-1}^{t} \|w_n(\theta)\|_{\widehat{V}}^2\mathrm{d}\theta + c_2c^2(\nu_\theta) \\
&\leqslant 2c_2c_6^4\big(e^{-\frac{\delta_1}{2}}\|w_n(t-1)\|^2 + \frac{1}{\delta_1}\int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big) \\
&\quad\times \big(\frac{2\|w_n(t-1)\|^2}{\delta_1} + 2{\delta_1}^{-2}e^{\frac{\delta_1}{2}}\int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big) + c_2c^2(\nu_\theta) \\
&\leqslant c_8\Big[\big(\|w_n(t-1)\|^2 + \int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big)^2 + 1\Big],
\end{aligned}
$$

where

$$
c_8 := \max\big\{2c_2c_6^4\max\big\{e^{-\frac{\delta_1}{2}}, \delta_1^{-1}\big\}\max\big\{2\delta_1^{-1}, 2{\delta_1}^{-2}e^{\frac{\delta_1}{2}}\big\},\ c_2c^2(\nu_\theta)\big\}.
$$

With the aid of (2.3), substituting the above two inequalities into (3.30), yields

$$
\begin{aligned}
\|w_n(t)\|_{\widehat{V}}^2 &\leqslant c_2H_n(t) \\
&\leqslant c_2c_7\big(\|w_n(t-1)\|^2 + \int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big) \\
&\quad\times \exp\big\{c_8\big[\big(\|w_n(t-1)\|^2 + \int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big)^2 + 1\big]\big\}.
\end{aligned} \tag{3.31}
$$

Observe that $w_n(t;\tau,w_\tau) \in L^\infty(\tau,t;\widehat{V}) \cap \mathcal{C}([\tau,t];\widehat{V})$ and $w(t;\tau,w_\tau) \in \mathcal{C}([\tau,t];\widehat{H})$, by the lower semicontinuity of the norm, we can pass to the limit in (3.31) and obtain that

$$
\begin{aligned}
\|w(t)\|_{\widehat{V}}^2 &\leqslant c_2c_7\big(\|w(t-1)\|^2 + \int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big) \\
&\quad\times \exp\big\{c_8\big[\big(\|w(t-1)\|^2 + \int_{t-1}^{t} \|F(\theta)\|^2\mathrm{d}\theta\big)^2 + 1\big]\big\},
\end{aligned} \tag{3.32}
$$

which together with Lemma 3.2 implies the result of Lemma 3.4. This completes the proof. $\square$

On the basis of the above results, we can prove the pullback asymptotical compactness of the process $\{U(t,\tau)\}_{t\geqslant\tau}$, that is the following Lemma.

Lemma 3.5. *Under the conditions of Theorem 1.1, the process $\{U(t,\tau)\}_{t\geqslant\tau}$ generated by* (2.7) *is pullback $\mathcal{D}$-asymptotically compact in $\widehat{H}$.*

Proof. For any fixed $t\in\mathbb{R}$, any family $\widehat{B}=\{B(s)|s\in\mathbb{R}\}\in\mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H})$, any sequences $\{\tau_n\}\subseteq(-\infty,t]$ satisfying $\tau_n\to-\infty$ as $n\to+\infty$ and $\{w_{\tau_n}\}\in\mathcal{D}(\tau_n)$, it suffice to show the sequence $\{w^n(t)\}_{n\geqslant1}$ defined by

$$w^n(\cdot):=w^n(\cdot;\tau_n,w_{\tau_n})=U(\cdot,\tau_n;w_{\tau_n})$$

is relatively compact in $\widehat{H}$.

In fact, by Lemma 3.2, there exists a time $\tau_0(\widehat{B},t)<t$ such that the sequence $\{w^n(t)|\tau_n\leqslant\tau_0(\widehat{B},t)\}$ is uniformly bounded in $\widehat{H}$. Since $\widehat{H}$ is a reflect Banach space, it follows from the diagonal procedure that there exists a function $w(t)$ such that (by extracting a subsequence if necessary)

$$w^n(t)\rightharpoonup w(t)\quad\text{weakly in }\widehat{H}\text{ as }n\to\infty.$$

Moreover, from Lemma 3.3, for any $\epsilon>0$, there exist $\tau_2:=\tau_2(\epsilon,t,\widehat{B})$, $r_3:=r_3(\epsilon,t,\widehat{B})>0$ such that

$$\|w^n(t;\tau_n,w_{\tau_n})\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)}\leqslant\frac{\epsilon}{3},\quad\forall\tau_n\leqslant\tau_2,r\geqslant r_3. \tag{3.33}$$

Observe that, for any fixed $t\in\mathbb{R},w(t)\in\widehat{H}$ is fixed. Hence for the above $\epsilon>0$, there exists a $r_4>0$ such that

$$\|w(t)\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)}\leqslant\frac{\epsilon}{3},\quad\forall r\geqslant r_4. \tag{3.34}$$

Now, we define respectively the restrictions of w^n and w in Ω_r by

$$w^n(t)|_{\Omega_r}=w^n(t;\tau_n,w_{\tau_n})|_{\Omega_r}:=\begin{cases}w^n(t), & x\in\Omega_r,\\ 0, & x\in\Omega\setminus\Omega_r,\end{cases}$$

$$w(t)|_{\Omega_r}:=\begin{cases}w(t), & x\in\Omega_r,\\ 0, & x\in\Omega\setminus\Omega_r.\end{cases}$$

From Lemma 3.4, it follows that, for any $r>0$, the sequence $\{w^n(t)|_{\Omega_r}\}_{n\geqslant1}$ is bounded in $\widehat{V}(\Omega_r)$. Since the embedding $\widehat{V}(\Omega_r)\hookrightarrow\widehat{H}(\Omega_r)$ is compact, there exists a subsequence (denoting by the same symbol) $\{w^n(t)|_{\Omega_r}\}_{n\geqslant1}$ satisfying

$$\|w^n(t)-w(t)\|_{\widehat{H}(\Omega_r)}\to0\quad\text{as }n\to\infty, \tag{3.35}$$

which together with (3.33)-(3.34) implies that there exists a $N_0\in\mathbb{N}$ such that for any $n\geqslant N_0$,

$$\begin{aligned}\|w^n(t)-w(t)\|_{\widehat{H}}&=\|w^n(t)-w(t)\|_{\widehat{H}(\Omega_r)}+\|w^n(t)-w(t)\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)}\\&\leqslant\|w^n(t)-w(t)\|_{\widehat{H}(\Omega_r)}+\|w^n(t)\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)}\\&\quad+\|w(t)\|_{\mathbb{L}^2(\Omega\setminus\Omega_r)}\leqslant\epsilon.\end{aligned} \tag{3.36}$$

Therefore, the sequence $\{w^n(t)\}_{n\geqslant1}$ is relatively compact in $\widehat{H}$. This completes the proof. □

Proof of Theorem 1.1. According to definitions 2.5-2.7, the family $\widehat{\mathcal{B}}=\{\mathcal{B}(t)|t\in\mathbb{R}\}$ is pullback $\mathcal{D}$-absorbing which can be obtained directly from Lemma 3.2. Further, it follows from Lemma 3.5 that the continuous process $\{U(t,\tau)\}_{t\geqslant\tau}$ is pullback

$\mathcal{D}$-asymptotically compact in $\widehat{H}$. Then, using [3, Theorem 7], we can show the existence and uniqueness of the pullback $\mathcal{D}$-attractor $\mathcal{A}_{\widehat{H}}(t)$ for $\{U(t,\tau)\}_{t\geqslant\tau}$ in $\widehat{H}$. □

4. Tempered behavior and upper semicontinuity of the pullback attractor

In this section, we will show the tempered behavior and upper semicontinuity of the pullback attractor $\mathcal{A}_{\widehat{H}}(t)$, which is obtained in section 3.

Proof of Theorem 1.3. According to Theorem 1.1, we know that $\mathcal{A}_{\widehat{H}}(t)\in\mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H})$. Therefore, (1.7) holds. Since $F(t,x)\in L^2_b(\mathbb{R};\widehat{H})$, (1.8) is a consequence of (1.7) and (3.32). This completes the proof. □

The rest of this section is devoted to verifying the upper semicontinuity of the pullback attractors with respect to the spatial domain, that is, we give the proof of Theorem 1.4. Followed the arguments in [27], let $\{\Omega_m\}_{m=1}^{\infty}$ be an expanding sequence of simply connected, bounded and smooth subdomains of Ω such that $\cup_{m=1}^{\infty}\Omega_m=\Omega$. We will prove the upper semicontinuity of the pullback attractor $\mathcal{A}_{\widehat{H}}$ in Ω from the pullback attractor $\mathcal{A}_{\widehat{H}(\Omega_m)}$ in Ω_m.

First, we consider equations (1.2)-(1.4) in each Ω_m and define the operators $A, B(\cdot,\cdot)$ and $N(\cdot)$ as before with the spatial domain Ω replaced by Ω_m. Then we can rewrite (1.2)-(1.4) in the abstract form

$$
\begin{gathered}
\frac{\partial w_m}{\partial t}+Aw_m+B(u_m,w_m)+N(w_m)=F(t,x),\quad \text{in } (\tau,+\infty)\times\Omega_m,\\
\nabla\cdot u_m=0,\quad \text{in } (\tau,+\infty)\times\Omega_m,\\
w_m=(u_m,\omega_m)=0,\quad \text{on } (\tau,+\infty)\times\partial\Omega_m,\\
w_m(\tau,x)=(u_m(\tau,x),\omega_m(\tau,x))=w_{m\tau}(x),\quad x\in\Omega_m,\ \tau\in\mathbb{R}.
\end{gathered}
\tag{4.1}
$$

For each bounded domain Ω_m, the global existence and uniqueness of the weak solutions of system (4.1) hold. That is,

Lemma 4.1. *[13] Assume the conditions of Theorem 1.4 hold and $w_\tau\in\widehat{H}(\Omega_m)$, then system (4.1) has a unique solution w_m satisfying*

$$
\begin{gathered}
w_m\in L^\infty(\tau,+\infty;\widehat{H}(\Omega_m))\cap\mathcal{C}([\tau,+\infty);\widehat{H}(\Omega_m))\cap L^2_{\rm loc}(\tau,+\infty;\widehat{V}(\Omega_m)),\\
w_m'\in L^2_{\rm loc}(\tau,+\infty;\widehat{V}^*(\Omega_m)).
\end{gathered}
$$

Moreover, the solution w_m depends continuously on the initial value w_τ with respect to the $\widehat{H}(\Omega_m)$ norm.

According to Lemma 4.1, the maps of solution operators defined by

$$
U_m(t,\tau): w_{m\tau}\mapsto U_m(t,\tau;w_{m\tau})=w_m(t),\quad t\geqslant\tau \tag{4.2}
$$

generates a continuous process $\{U_m(t,\tau)\}_{t\geqslant\tau}$ in $\widehat{H}(\Omega_m)$. Moreover, on any smooth bounded domain Ω_m, we have the following result.

Lemma 4.2. *Under the conditions of Theorem 1.4.*

(1) *For any $t\in\mathbb{R}$, $\widehat{B}^{\widehat{H}(\Omega_m)}=\{B^{\widehat{H}(\Omega_m)}(s)|s\in\mathbb{R}\}\in\mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H}(\Omega_m))$ and $w_{m\tau}\in B^{\widehat{H}(\Omega_m)}(\tau)$, the family $\widehat{\mathcal{B}}^{\widehat{H}(\Omega_m)}=\{\mathcal{B}^{\widehat{H}(\Omega_m)}(t)\big|t\in\mathbb{R}\}$ given by*

$$
\mathcal{B}^{\widehat{H}(\Omega_m)}(t)=\{w_m\in\widehat{H}(\Omega_m)\big|\|w_m\|_{\widehat{H}(\Omega_m)}\leqslant\rho(t)\}
$$

is pullback $\mathcal{D}^{\widehat{H}(\Omega_m)}$-absorbing in $\widehat{H}(\Omega_m)$, where $\rho(t)$ is defined by (3.8).

(2) *For any $\epsilon > 0, t \in \mathbb{R}, \widehat{B}^{\widehat{H}(\Omega_m)} = \{B^{\widehat{H}(\Omega_m)}(s) | s \in \mathbb{R}\} \in \mathcal{D}_{\frac{\delta_1}{2}}(\widehat{H}(\Omega_m))$ and $w_{m\tau} \in B^{\widehat{H}(\Omega_m)}(\tau)$, there exists $r_{0m} = r_{0m}(\epsilon, t, \widehat{B}^{\widehat{H}(\Omega_m)}) > 0$ and a time $\tau_{0m} = \tau_{0m}(\epsilon, t, \widehat{B}^{\widehat{H}(\Omega_m)}) < t$ such that for any $r \in [r_{0m}, m]$ and $\tau \leqslant \tau_{0m}$, it holds*

$$\|w_m(t;\tau, w_{m\tau})\|_{\mathbb{L}^2(\Omega_m \setminus \Omega_r)} \leqslant \epsilon.$$

(3) *The process $\{U_m(t,\tau)\}_{t \geqslant \tau}$ is pullback $\mathcal{D}^{\widehat{H}(\Omega_m)}$-asymptotically compact in $\widehat{H}(\Omega_m)$.*

Since the proof is similar to those of Lemma 3.2, Lemma 3.3 and Lemma 3.5, we can omit it here. As a consequence of Lemma 4.2, we have the following result.

Proposition 4.3. *Assume the conditions of Theorem 1.4 hold, then system* (4.1) *has a unique pullback $\mathcal{D}^{\widehat{H}(\Omega_m)}$-attractor $\mathcal{A}_{\widehat{H}(\Omega_m)} = \{\mathcal{A}_{\widehat{H}(\Omega_m)}(t)\}_{t \in \mathbb{R}}$ in $\widehat{H}(\Omega_m)$.*

Next, let us consider the convergence of solutions for (4.1) with m. That is, we show that the sequence $\{w_m\}_{m \geqslant 1}$ of solutions to system (4.1) converges to the solution of system (1.6) as $m \to \infty$.

For $w_m \in \widehat{H}(\Omega_m)$, we extend its domain from Ω_m to Ω by setting

$$\tilde{w}_m = \begin{cases} w_m, & x \in \Omega_m, \\ 0, & x \in \Omega \setminus \Omega_m, \end{cases}$$

then

$$\|w_m\|_{\widehat{H}(\Omega)} = \|\tilde{w}_m\|_{\widehat{H}(\Omega)} = \|\tilde{w}_m\|_{\widehat{H}(\Omega_m)} = \|w_m\|_{\widehat{H}(\Omega_m)}.$$

By Lemma 3.2 and Lemma 4.2 (1), there exists a $\tau(t, \widehat{B}^{\widehat{H}(\Omega)})$ (independent of m) such that

$$\begin{gathered} U(t,\tau)B(\tau) \subseteq \mathcal{B}(t), \quad \forall \tau \leqslant \tau(t, \widehat{B}^{\widehat{H}(\Omega)}), \\ U_m(t,\tau)B^{\widehat{H}(\Omega_m)}(\tau) \subseteq \mathcal{B}^{\widehat{H}(\Omega_m)}(t), \quad \forall \tau \leqslant \tau(t, \widehat{B}^{\widehat{H}(\Omega)}). \end{gathered} \tag{4.3}$$

The following result can be obtained by using the same proof as that of [14, Lemma 8.1].

Lemma 4.4. *Under the conditions of Theorem 1.4, let $\{w_{m\tau}\}_{m \geqslant 1}$ be a sequence in $\widehat{H}(\Omega_m)$ satisfying*

$$w_{m\tau} \rightharpoonup w_\tau \quad \text{weakly in } \widehat{H}(\Omega) \text{ as } m \to \infty,$$

then

$$w_m(t;\tau, w_{m\tau}) \rightharpoonup w(t;\tau, w_\tau) \quad \text{weakly in } \widehat{H}(\Omega),\ \forall t \geqslant \tau,$$

and

$$w_m(\cdot;\tau, w_{m\tau}) \rightharpoonup w(\cdot;\tau, w_\tau) \quad \text{weakly in} L^2(\tau, T; \widehat{V}(\Omega)),\ \forall T > \tau. \tag{4.4}$$

Lemma 4.5. *Assume the conditions of Lemma 4.4 hold, then for any $t \in \mathbb{R}$ and any sequence $\{w_m\}_{m \geqslant 1}$ with $w_m(\tau) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(\tau)$, $m = 1, 2, \dots$, there exists $w(t) \in \mathcal{A}_{\widehat{H}(\Omega)}(t)$ such that*

$$w_m(\cdot) \to w(\cdot) \quad \text{strongly in } \widehat{H}(\Omega) \text{ as } m \to \infty. \tag{4.5}$$

Proof. First, it follows from (4.3) and the invariant of the pullback attractor that the sequence $\{w_m(\tau)\}_{m\geqslant 1}$ is bounded in $\widehat{H}(\Omega)$. Hence, one can deduce that there exists a $w_\tau \in \mathcal{A}_{\widehat{H}(\Omega)}(\tau)$ and a subsequence $\{w_m(\tau)\}_{m\geqslant 1}$ (denoted by the same) such that

$$w_m(\tau) \rightharpoonup w_\tau \quad \text{weakly in } \widehat{H}(\Omega) \text{ as } m \to \infty. \tag{4.6}$$

By Lemma 4.4 and the invariant of the pullback attractor, we see that for any $t \in \mathbb{R}$ and $w_m(t;\tau,w_m(\tau)) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(t)$ with $w_m(\tau) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(\tau)$, there exists a $w(t;\tau,w_\tau) \in \mathcal{A}_{\widehat{H}(\Omega)}(t)$ with $w_\tau \in \mathcal{A}_{\widehat{H}(\Omega)}(\tau)$ such that

$$w_m(t) \rightharpoonup w(t) \quad \text{weakly in } \widehat{H}(\Omega) \text{ as } m \to \infty,$$

which together with the lower semicontinuity of the norm implies

$$\|w(t)\| \leqslant \liminf_{m\to\infty} \|w_m(t)\|. \tag{4.7}$$

Next, we shall prove

$$\|w(t)\|^2 \geqslant \limsup_{m\to\infty} \|w_m(t)\|^2. \tag{4.8}$$

In fact, multiplying $(4.1)_1$ by $w_m(t)$ and integrating the resultant equality over Ω, we obtain with the aid of (2.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(t)\|^2 + 2\langle Aw_m(t), w_m(t)\rangle + 2\langle N(w_m(t)), w_m(t)\rangle = 2(F(t), w_m(t)). \tag{4.9}$$

Then, from (2.3), (2.4) and (2.5), we see that

$$\begin{aligned}
(c_1^{-1} + c(\nu_\theta))\|w\|^2_{\widehat{V}(\Omega)} &\geqslant \ll w, w \gg \\
&= \langle Aw, w\rangle + \langle N(w), w\rangle - \frac{\delta_1}{4}\|w\|^2 \\
&\geqslant \delta_1\|w\|^2_{\widehat{V}(\Omega)} - \frac{\delta_1\|w\|^2}{4} \geqslant \frac{3\delta_1}{4}\|w\|^2_{\widehat{V}(\Omega)},
\end{aligned} \tag{4.10}$$

where the bilinear mapping $\ll \cdot,\cdot \gg$ is defined by

$$\ll \varphi, \phi \gg = \langle A\varphi, \phi\rangle + \langle N(\varphi), \phi\rangle - \frac{\delta_1(\varphi,\phi)}{4}, \quad \forall \varphi, \phi \in \widehat{V}(\Omega).$$

By (4.9) and (4.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_m(t)\|^2 + \frac{\delta_1}{2}\|w_m(t)\|^2 = 2(F(t), w_m(t)) - 2 \ll w_m(t), w_m(t) \gg,$$

which yields

$$\begin{aligned}
\|w_m(t)\|^2 &= e^{-\frac{\delta_1}{2}(t-\tau)}\|w_m(\tau)\|^2 \\
&\quad + 2\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}\big[(F(s), w_m(s)) - \ll w_m(s), w_m(s) \gg \big]\mathrm{d}s,
\end{aligned} \tag{4.11}$$

for all $t \geqslant \tau$.

Next, we estimate the terms on the right-half side of (4.11) one by one. First, since $w_m(\tau) \in \mathcal{A}_{\widehat{H}(\Omega_m)}(\tau)$, it follows from (3.7) and (4.3) that

$$\begin{aligned}
e^{-\frac{\delta_1}{2}(t-\tau)}\|w_m(\tau)\|^2 &\leqslant e^{-\frac{\delta_1}{2}(t-\tau)}\rho^2(\tau) \\
&= \frac{2}{\delta_1}e^{-\frac{\delta_1}{2}t}\int_{-\infty}^{\tau} e^{\frac{\delta_1}{2}s}\|F(s)\|^2\mathrm{d}s \to 0 \quad \text{as } \tau \to -\infty.
\end{aligned} \tag{4.12}$$

Next, from (4.4) and (4.6), we have

$$\lim_{m\to\infty}\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}(F(s),w_m(s))\mathrm{d}s=\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}(F(s),w(s))\mathrm{d}s. \tag{4.13}$$

Finally, we see from (4.10) that $\ll w,w\gg$ is equivalent to $\|w\|^2_{\widehat{V}(\Omega)}$, which combines with (4.4) and (4.6) implies

$$\ll w_m(s),w_m(s)\gg\rightharpoonup\ll w(s),w(s)\gg \quad \text{weakly in } L^2(\tau,t;\widehat{V}(\Omega)),\ \forall t>\tau.$$

It follows from the lower semicontinuity of the norm that

$$\int_\tau^t \ll w(s),w(s)\gg \mathrm{d}s\leqslant\liminf_{m\to\infty}\int_\tau^t\ll w_m(s),w_m(s)\gg\mathrm{d}s. \tag{4.14}$$

Inserting (4.13)-(4.14) into (4.11) leads to

$$\begin{aligned}\limsup_{m\to\infty}\|w_m(t)\|^2 &\leqslant e^{-\frac{\delta_1}{2}(t-\tau)}\|w_m(\tau)\|^2+2\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}(F(s),w(s))\mathrm{d}s\\ &\quad-2\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}\ll w(s),w(s)\gg\mathrm{d}s.\end{aligned}\tag{4.15}$$

Similar to (4.11), the following energy equality for $w(t)$ hold:

$$\begin{aligned}\|w(t)\|^2 &= e^{-\frac{\delta_1}{2}(t-\tau)}\|w(t)\|^2+2\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}(F(s),w(s))\mathrm{d}s\\ &\quad-2\int_\tau^t e^{-\frac{\delta_1}{2}(t-s)}\ll w(s),w(s)\gg\mathrm{d}s.\end{aligned}\tag{4.16}$$

From the above two inequalities, we obtain

$$\limsup_{m\to\infty}\|w_m(t)\|^2\leqslant e^{-\frac{\delta_1}{2}(t-\tau)}\|w_m(\tau)\|^2+\|w(t)\|^2,$$

which together with (4.12) implies (4.8) when τ is small enough. This completes the proof. □

Proof of Theorem 1.4. Suppose that (1.9) is false, then there exist $t_0\in\mathbb{R},\epsilon_0>0$ and $w_m\in\mathcal{A}_{\widehat{H}(\Omega_m)}(t_0)$, $m=1,2,\dots$, such that

$$\operatorname{dist}_{\widehat{H}(\Omega)}(w_m,\mathcal{A}_{\widehat{H}(\Omega)}(t_0))\geqslant\epsilon_0>0,\quad m=1,2,\dots. \tag{4.17}$$

However, by Lemma 4.5, we see that there exists a subsequence $\{w_{m_k}\}_{k\geqslant1}\subseteq\{w_m\}_{m\geqslant1}$ such that

$$\lim_{k\to\infty}\operatorname{dist}_{\widehat{H}(\Omega)}(w_{m_k},\mathcal{A}_{\widehat{H}(\Omega)}(t_0))=0,$$

which leads to a contradiction with (4.17). Therefore, (1.9) follows. This completes the proof. □

Acknowledgements. We are grateful to two anonymous referees for valuable comments which greatly improved our original manuscript. The research is supported in partial by the National Science Foundation of China (Grant No. 11671134).

References

[1] R. A. Adams; *Sobolev Spaces*, Academic Press, New York, 1975.

[2] T. Caraballo, G. Łukaszewicz, J. Real; Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains, *Comptes Rendus Mathematique*, **342** (2006), 263-268.

[3] T. Caraballo, G. Łukaszewicz, J. Real; Pullback attractors for asymptotically compact non-autonomous dynamical systems, *Nonlinear Anal.*, **64** (2006), 484-498.

[4] J. Chen, Z. Chen, B. Dong; Existence of H^2-global attractors of two-dimensional micropolar fluid flows, *J. Math. Anal. Appl.*, **322** (2006), 512-522.

[5] J. Chen, B. Dong, Z. Chen; Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains, *Nonlinearity*, **20** (2007), 1619-1635.

[6] G. Chen; Pullback attractor for non-homogeneous micropolar fluid flows in non-smooth domains, *Nonlinear Anal.*, **10** (2009), 3018-3027.

[7] B. Dong, Z. Chen; Global attractors of two-dimensional micropolar fluid flows in some unbounded domains, *Appl. Math. Comp.*, **182** (2006), 610-620.

[8] B. Dong, Z. Zhang; Global regularity of the 2D micropolar fluid flows with zero angular viscosity, *J. Differential Equations*, **249** (2010), 200-213.

[9] A. C. Eringen; Theory of micropolar fluids, *J. Math. Mech.*, **16** (1966), 1-18.

[10] J. García-Luengo, P. Marín-Rubio, J. Real; Pullback attractors in V for non-autonomous 2D-Navier-Stokes equations and their tempered behavior, *J. Differential Equations*, **252** (2012), 4333-4356.

[11] J. K. Hale, L. Xin, G. Raugel; Upper semicontinuity of attractors for approximations of semigroups and partial differential equations, *Math. Comput.*, **5** (1988), 89-123.

[12] G. Łukaszewicz; *Micropolar Fluids: Theory and Applications*, Birkhäuser, Boston, 1999.

[13] G. Łukaszewicz; Long time behavior of 2D micropolar fluid flows, *Math. Comput. Modelling*, **34** (2001), 487-509.

[14] G. Łukaszewicz, W. Sadowski; Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains, *Z. Angew. Math. Phys.*, **55** (2004), 247-257.

[15] K. Lu, B. Wang; Upper semicontinuity of attractors for the Klein-Gordon-Schrödinger equation, *Inter. J. Bifur. Chaos*, **15** (2005), 157-168.

[16] J. A. Langa, G. Łukaszewicz, J. Real; Finite fractal dimension of pullback attractors for non-autonomous 2D Navier-Stokes equations in some unbounded domains, *Nonlinear Anal.*, **66** (2007), 735-749.

[17] J. A. Langa, J. C. Robinson, A. Rodríguez-Bernal, A. Suárez, A. Vidal-López; Existence and nonexistence of unbounded forward attractor for a class of non-autonomous reaction diffusion equations, *Discrete Contin. Dyn. Sys.*, **18** (2007), 483-497.

[18] G. Łukaszewicz, A. Tarasińska; On H^1-pullback attractors for nonautonomous micropolar fluid equations in a bounded domain, *Nonlinear Anal.*, **71** (2009), 782-788.

[19] P. Marín-Rubio, J. Real; On the relation between two different concepts of pullback attractors for non-autonomous dynamical systems, *Nonlinear Anal.*, **71** (2009), 3956-3963.

[20] A. M. Márquez; Existence and uniqueness of solutions, and pullback attractor for a system of globally modified 3D-Navier-Stokes equations with finite delay, *Bol. Soc. Esp. Mat. Apl.*, **51** (2010), 117-124.

[21] B. Nowakowski; Long-time behavior of micropolar fluid equations in cylindrical domains, *Nonlinear Anal.: RWA*, **14** (2013), 2166-2179.

[22] H. Song, H. Wu; Pullback attractors of nonautonomous reaction-diffusion equations, *J. Math. Anal. Appl.*, **325** (2007), 1200-1215.

[23] X. H. Wang, S. Y. Li; Pullback attractors for non-autonomous 2D Navier-Stokes equations with linear damping in some unbounded domains, *Acta Mathematica Sinica-English Series*, **4** (2009), 873-881.

[24] P. J. Yeoul, P. S. Hye; Pullback attractors for the non-autonomous Benjamin-Bona-Mahony equation in unbounded domains, *Sci. China Math.*, **54** (2011), 741-752.

[25] C. Zhao, S. Zhou, X. Lian; H^1-uniform attractor and asymptotic smoothing effect of solutions for a nonautonomous micropolar fluid flow in 2D unbounded domains, *Nonlinear Anal.: RWA*, **9** (2008), 608-627.

[26] C. Zhao, J. Duan; Upper semicontinuity of global attractors for 2D Navier-Stokes equations, *Inter. J. Bifur. Chaos*, **22** (2012), 1-7.

[27] C. Zhao; Pullback asymptotic behavior of solutions for a non-autonomous non-Newtonian fluid on two-dimensional unbounded domains, *J. Math. Phys.*, **53** (2012), 122702-1-23.

[28] C. Zhao, G. Liu, W. Wang; Smooth pullback attractors for a non-autonomous 2D non-Newtonian fluid and their tempered behavior, *J. Math. Fluid Mech.*, **16** (2013), 243-262.

[29] C. Zhao, W. Sun, C. Hsu; Pullback dynamical behaviors of the non-autonomous micropolar fluid flows, *Dynamics of PDE*, **12** (2015), 265-288.

Wenlong Sun
Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China
E-mail address: wenlongsun1988@163.com

Yeping Li (corresponding author)
Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China
E-mail address: yplee@ecust.edu.cn