Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 94, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

HOMOCLINIC SOLUTIONS FOR SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS WITH JACOBI OPERATORS

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Communicated by Paul H. Rabinowitz

Abstract. We obtain sufficient conditions for the existence of a nontrivial homoclinic solution to a second-order nonlinear difference equation with Jacobi operator. To do this, we use variational methods and critical point theory. An example is provided to illustrate our main result.

1. Introduction

Difference equations, the discrete analogs of differential equations [8, 9, 15, 28], occur in numerous settings and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamical systems, economics, biology and other fields. For the general background of difference equations, we refer to the monographs [1, 2, 5].

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of all natural numbers, integers and real numbers respectively. For a, $b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a,b) = \{a, a+1, \dots, b\}$ when $a \le b$. Moreover, I denotes the identity operator.

In this article, we consider the second-order nonlinear difference equation

$$Lu(t) - \omega u(t) = f(t, u(t+\Gamma), \dots, u(t), \dots, u(t-\Gamma)), \quad t \in \mathbb{Z} \tag{1.1}$$

containing both advances and retardations. Here the operator L is the Jacobi operator

$$Lu(t) = a(t)u(t+1) + a(t-1)u(t-1) + b(t)u(t),$$

where $a(t)$ and $b(t)$ are real valued for each $t \in \mathbb{Z}$, $\omega \in \mathbb{R}$, $f \in C(\mathbb{R}^{2\Gamma+2}, \mathbb{R})$, Γ is a given nonnegative integer, $a(t)$, $b(t)$ and $f(t, y_\Gamma, \dots, y_0, \dots, y_{-\Gamma})$ are all M-periodic in t for a given positive integer M.

Jacobi operators appear in a variety of applications [27]. They can be viewed as the discrete analogue of Sturm-Liouville operators and their investigation has many similarities with Sturm-Liouville theory. Whereas numerous books about Sturm-Liouville operators have been written, only few on Jacobi operators exist. In particular, there are currently fewer researches available which cover some basic topics (like stability, attractivity, positive solutions, periodic operators, homoclinic

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2010 *Mathematics Subject Classification.* 34C37, 37J45, 39A12, 47J22.
Key words and phrases. Homoclinic solutions; nonlinear difference equations;
Jacobi operators; critical point theory.

Submitted August 11, 2016. Published March 30, 2017.

solutions, boundary value problems, etc.) typically found in textbooks on Sturm-Liouville operators [17].

We may regard (1.1) as being a discrete analog of the second-order differential equation

$$Su(s) - \omega u(s) = f(s, u(s+\Gamma), \dots, u(s), \dots, u(s-\Gamma)), \quad s \in \mathbb{R}, \tag{1.2}$$

where S is the Sturm-Liouville differential expression and $\omega \in \mathbb{R}$, Γ is a given nonnegative integer, $f \in C(\mathbb{R}^{2\Gamma+2}, \mathbb{R})$.

Equation (1.2) includes the equation

$$c^2 u''(s) = V'(u(s+1) - u(s)) - V'(u(s) - u(s-1)), \quad s \in \mathbb{R}. \tag{1.3}$$

Equations similar in structure to (1.3) arise in the study of the existence of solitary waves of lattice differential equations and the existence of homoclinic solutions for functional differential equations, see [10, 11, 26] and the references cited therein.

Assuming that $f(t, 0, \dots, 0, \dots, 0) = 0$ for $t \in \mathbb{Z}$, then $\{u(t)\}_{t\in\mathbb{Z}} = \{0\}$ is a solution of (1.1), which is called the trivial solution. As usual, we say that a solution $\{u(t)\}_{t\in\mathbb{Z}}$ of (1.1) is homoclinic (to 0) if (1.1) holds. In addition, if $\{u(t)\}_{t\in\mathbb{Z}} \neq \{0\}$, then u is called a nontrivial solution.

It is well known that homoclinic solutions (homoclinic orbits) play a very important role in the study of chaos in dynamical systems. It has been proved that the system must be chaotic provided it has the transversely intersected homoclinic solutions. Homoclinic solutions have been extensively studied since the time of Poincaré, see [3, 14, 19, 21, 22, 23, 24, 25, 28, 30] and the references therein. Therefore, it possesses important theoretical significance and practical value to investigate the existence of homoclinic solutions of (1.1) emanating from zero.

By using the Symmetric Mountain Pass Theorem, Chen and Tang [4] established some existence criteria to guarantee the fourth-order difference system containing both advance and retardation

$$\Delta^4 u(t-2) + q(t)u(t) = f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbb{Z} \tag{1.4}$$

has infinitely many homoclinic solutions.

Deng, Liu, Shi and Zhou [7] in 2011 proved the existence of nontrivial homoclinic solutions for a second-order nonlinear p-Laplacian difference equation

$$\Delta(\varphi_p(\Delta u(t-1))) - \varphi_p(u(t)) = \lambda(t) f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbb{Z}, \tag{1.5}$$

without any assumptions on periodicity using the critical point theory.

When $\Gamma = 1$, (1.1) reduces to the special equation

$$Lu(t) - \omega u(t) = f(t, u(t+1), u(t), u(t-1)), \quad t \in \mathbb{Z}, \tag{1.6}$$

containing both advance and retardation. Liu, Zhang and Shi [13] considered the existence of a nontrivial homoclinic solution for (1.6) by using the Mountain Pass Lemma in combination with periodic approximations.

In 2016, Shi, Liu and Zhang [22] obtained the existence of a nontrivial homoclinic solution for a second-order p-Laplacian difference equation containing both advance and retardation

$$\Delta(\varphi_p(\Delta u(t-1))) - q(t)\varphi_p(u(t)) + f(t, u(t+M), u(t), u(t-M)) = 0, \tag{1.7}$$

for $t \in \mathbb{Z}$, by using critical point theory.

Deng, Chen and Shi in [6] studied the existence of homoclinic solutions for second-order discrete Hamiltonian systems by using the critical point theory. However, to the best of our knowledge, the results on homoclinic solutions of second-order nonlinear difference equation (1.1) which contains both many advances and retardations are very scarce in the literature (see [22]), because there are only few known methods to establish the existence of homoclinic solutions of discrete systems.

Motivated by the articles [13, 22], our main purpose is to establish new criteria for the existence of nontrivial homoclinic orbits to a class of second-order nonlinear difference equations which contains both several advances and retardations with Jacobi operators. Our results do not suppose that the system satisfies the well-known global Ambrosetti-Rabinowitz superquadratic assumption. Some existing results are generalized and improved; see Remarks 1.2 and 1.3 for details.

Throughout this article, for a function F, we let $F_i'(y_1,\dots,y_i\dots,y_n)$ denote the partial derivative of F on the i variable. For basic knowledge of variational methods, the reader is referred to [18, 20].

Our main results are obtained using the following hypotheses:

- (H1) $a(t)\neq 0$, $b(t)-|a(t-1)|-|a(t)|>\omega$, for all $t\in\mathbb{Z}$;
- (H2) there exists a function $F(t,y_\Gamma,\dots,y_0)$ which is continuously differentiable in the variable from y_Γ to y_0 for every $t\in\mathbb{Z}$ and satisfies
$$F(t+M,y_\Gamma,\dots,y_0)=F(t,y_\Gamma,\dots,y_0),$$
$$\sum_{i=-\Gamma}^{0}F'_{2+\Gamma+i}(t+i,y_{\Gamma+i},\dots,y_i)=f(t,y_\Gamma,\dots,y_0,\dots,y_{-\Gamma});$$
- (H3) $\lim_{\varrho\to 0}\frac{f(t,y_\Gamma,\dots,y_0,\dots,y_{-\Gamma})}{y_0}=0$ for $t\in\mathbb{Z}$, $\varrho=(\sum_{i=-\Gamma}^{\Gamma}y_i^2)^{1/2}$;
- (H4) $\lim_{\delta\to 0}\frac{F(t,y_\Gamma,\dots,y_0)}{\delta^2}=0$ for $t\in\mathbb{Z}$, $\delta=(\sum_{i=0}^{\Gamma}y_i^2)^{1/2}$;
- (H5) $\lim_{\delta\to\infty}\frac{F(t,y_\Gamma,\dots,y_0)}{\delta^2}=\infty$ for all $t\in\mathbb{Z}$, $\delta=(\sum_{i=0}^{\Gamma}y_i^2)^{1/2}$;
- (H6) for any $t\in\mathbb{Z}$, $F(t,0,\dots,0)=0$, $F(t,y_\Gamma,\dots,y_0)\geq F(t,y_0)\geq 0$;
- (H7) for any $r>0$, there exist $p=p(r)>0$, $q=q(r)>0$ and $\nu<2$ such that
$$\Big(2+\frac{1}{p+q\big(\sum_{i=0}^{\Gamma}y_i^2\big)^{\frac{\nu}{2}}}\Big)F(t,y_\Gamma,\dots,y_0)\leq\sum_{i=-\Gamma}^{0}F'_{2+\Gamma+i}(t,y_\Gamma,\dots,y_0)y_{-i},$$
for all $t\in\mathbb{Z}$, $\big(\sum_{i=0}^{\Gamma}y_i^2\big)^{1/2}>r$.

Theorem 1.1. *Assume that* (H1)–(H7) *are satisfied. Then* (1.1) *has a nontrivial homoclinic solution.*

Remark 1.2. Theorem 1.1 extends Theorem 1.1 in [13] which is the special case of our Theorem 1.1 by letting $\Gamma=1$.

Remark 1.3. In the superquadratic case, almost all the existing results (see e.g. [6, 7, 12, 16, 21]) need the following well-known global Ambrosetti-Rabinowitz superquadratic condition:

- (AR) there exists a constant $\beta>2$ such that
$0<\beta F(t,u)\leq uf(t,u)$ for all $t\in\mathbb{Z}$ and $u\in\mathbb{R}\setminus\{0\}$.

Note that (H5)–(H7) are much weaker than the Ambrosetti-Rabinowitz condition. Therefore, our result improves that the existing ones.

Theorem 1.4. *Assume that* (H1)–(H4) *and the following assumption are satisfied:*

(H8) $F(t, y_\Gamma, \dots, y_0) \geq 0$ *and there exists a constant* $\beta > 2$ *such that*

$$0 < \beta F(t, y_\Gamma, \dots, y_0) \leq \sum_{i=0}^{\Gamma} F'_{2+i}(t, y_\Gamma, \dots, y_0) y_{\Gamma-i},$$

for all $t \in \mathbb{Z}$, $(y_\Gamma, \dots, y_0) \in \mathbb{R}^{\Gamma+1} \setminus \{(0, \dots, 0)\}$.

Then (1.1) *has a nontrivial homoclinic solution.*

The rest of this article is organized as follows. First, in Section 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of homoclinic orbits of (1.1) into that of the existence of critical points of the corresponding functional. Then, in Section 3, some related lemmas will be stated. Next, in Section 4, we shall complete the proof of the results by using variational methods and the critical point method. Finally, in Section 5, we shall give an example to illustrate the applicability of the main result.

2. Variational structure

To apply the critical point theory, the corresponding variational framework for equation (1.1) is established. We start by some basic notation for the reader's convenience.

Let S be the vector space of all real sequences of the form

$$u = \{u(t)\}_{t\in\mathbb{Z}} = (\dots, u(-t), \dots, u(-1), u(0), u(1), \dots, u(t), \dots),$$

namely $S = \{\{u(t)\} : u(t) \in \mathbb{R},\ t \in \mathbb{Z}\}$. Define

$$E = \big\{u \in S : \sum_{t=-\infty}^{+\infty} [(L - \omega I)u(t) \cdot u(t)] < +\infty\big\}.$$

The space is a Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{t=-\infty}^{+\infty} [(L - \omega I)u(t)v(t)], \quad \forall u, v \in E, \tag{2.1}$$

and the corresponding norm

$$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{\sum_{t=-\infty}^{+\infty} [(L - \omega I)u(t)u(t))]}, \quad \forall u \in E. \tag{2.2}$$

Next, we define

$$l^2 = \big\{u \in S : \sum_{t=-\infty}^{+\infty} u^2(t) < +\infty\big\}, \quad l^\infty = \big\{u \in S : \sup_{t\in\mathbb{Z}} |u(t)| < +\infty\big\},$$

and their norms are

$$\|u\|_2 = \Big(\sum_{t=-\infty}^{+\infty} u^2(t)\Big)^{1/2}, \quad \forall u \in l^2,$$
$$\|u\|_\infty = \sup_{t\in\mathbb{Z}} |u(t)|, \quad \forall u \in l^\infty,$$

respectively.

For $u \in E$, we define the functional J on E as follows:

$$\begin{aligned} J(u) &:= \sum_{t=-\infty}^{+\infty} \big[\frac{1}{2}(L-\omega I)u(t)\cdot u(t) - F(t,u(t+\Gamma),\dots,u(t))\big] \\ &= \frac{1}{2}\|u\|^2 - \sum_{t=-\infty}^{+\infty} F(t,u(t+\Gamma),\dots,u(t)). \end{aligned} \tag{2.3}$$

The functional J is a well-defined C^1 functional on E and (1.1) is easily recognized as the corresponding Euler-Lagrange equation for J. Therefore, we are looking for nonzero critical points of J.

3. Main lemmas

To apply variational methods and critical point theory for the existence of a nontrivial homoclinic solution of (1.1), we shall state some lemmas which will be used in the proofs of our main results.

Lemma 3.1 ([18]). *Let E be a real Banach space with its dual space E^* and suppose that $J \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{J(0),J(e)\} \le \eta_0 < \eta \le \inf_{\|u\|=\rho} J(u),$$

for some $\eta_0 < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by

$$c = \inf_{\gamma\in\Upsilon} \max_{0\le s\le 1} J(\gamma(s)),$$

where $\Upsilon = \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ \gamma(1) = e\}$ is the set of continuous paths joining 0 to e; then there exists $\{u_k\}_{k\in\mathbb{N}} \subset E$ such that $J(u_k) \to c$ and $(1+\|u_k\|)\|J'(u_k)\|_{E^*} \to 0$ as $k \to \infty$.

Lemma 3.2 ([13]). *Assume that* (H1) *holds. Then there exists a constant $\underline{\lambda}$ such that the following inequalities hold:*

$$\underline{\lambda}\|u\|_2^2 \le \|u\|^2, \tag{3.1}$$

$$\underline{\lambda}\|u\|_\infty^2 \le \|u\|^2, \tag{3.2}$$

where $\underline{\lambda} = \inf_{t\in\mathbb{Z}}(b(t) - \omega - |a(t-1)| - |a(t)|) > 0$.

Lemma 3.3. *Assume that* (H1)–(H7) *are satisfied. Then there exists a constant $c > 0$ and a sequence $\{u_k\}_{k\in\mathbb{N}}$ satisfying*

$$J(u_k) \to c, \quad \|J'(u_k)\|(1+\|u_k\|) \to 0, \quad k \to \infty. \tag{3.3}$$

Proof. By (H4), there exists a constant $\rho > 0$ such that for any $\sqrt{y_\Gamma^2 + \dots + y_0^2} \le \rho$,

$$F(t,y_\Gamma,\dots,y_0) \le \frac{\underline{\lambda}}{4(\Gamma+1)}(y_\Gamma^2 + \dots + y_0^2), \quad \forall t \in \mathbb{Z}. \tag{3.4}$$

If $\|u\| = \sqrt{\underline{\lambda}}\rho := \eta$, then by (3.2), $|u(t)| \le \rho$ for all $t \in \mathbb{Z}$. For any $u \in E$, $\|u\| = \rho$, it follows from (2.3) and (3.4) that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|^2 - \sum_{t=-\infty}^{+\infty} F(t,u(t+\Gamma),\dots,u(t)) \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\underline{\lambda}}{4(\Gamma+1)} \sum_{t=-\infty}^{+\infty} [u^2(t+\Gamma) + \dots + u^2(t)] \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{4}\|u\|_2^2,\\
&\geq \frac{1}{4}\|u\|^2 = \frac{1}{4}\eta^2.
\end{aligned}$$

Let $u_0(0) = 1$, $u_0(t) = 0$ for $t \neq 0$. By (H2), (H3), (H5) and (2.3), we have

$$\begin{aligned}
J(su_0) &= \frac{s^2}{2}\|u_0\|^2 - \sum_{t=-\infty}^{+\infty} F(t, su_0(t+\Gamma), \dots, su_0(t))\\
&\leq \frac{s^2}{2}\|u_0\|^2 - F(0, su_0(\Gamma), \dots, su_0(0))\\
&\leq s^2\Big[\frac{1}{2}\|u_0\|^2 - \frac{F(0, su_0(\Gamma), \dots, su_0(0))}{|su_0(0)|^2}\Big] \leq 0
\end{aligned}$$

for large enough $s > 0$.

Choose $s_1 > 1$ such that $s_1\|u_0\| > \eta$ and $J(s_1u_0) \leq 0$. Let $e = s_1u_0$, then $e \in E$, $\|e\| > \eta$ and $J(e) \leq 0$. By Lemma 3.1, there exists a constant $c \geq \frac{1}{4}\eta^2$ and a sequence $\{u_k\}_{k\in\mathbb{N}} \subset E$ such that (3.3) holds. $\square$

Lemma 3.4. *Assume that* (H1)–(H7) *are satisfied. Then any* $\{u_k\}_{k\in\mathbb{N}}$ *satisfying*

$$J(u_k) \to c > 0, \quad \langle J'(u_k), u_k\rangle \to 0, \quad k \to \infty \tag{3.5}$$

is bounded in E.

Proof. It follows from (H4) that there exists a constant $0 < \rho < 1$ such that for any $\sqrt{u_k^2(t+\Gamma) + \dots + u_k^2(t)} \leq \rho$,

$$|F(t, u_k(t+\Gamma), \dots, u_k(t))| \leq \frac{\lambda}{4(\Gamma+1)}\sum_{i=0}^{\Gamma} u_k^2(t+i), \quad \forall t \in \mathbb{Z}. \tag{3.6}$$

For $t \in \mathbb{Z}$, by (H7), we have

$$\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u_k(t+\Gamma), \dots, u_k(t))u_k(t-i) > 2F(t, u_k(t+\Gamma), \dots, u_k(t)) \geq 0, \tag{3.7}$$

and $t \in \mathbb{Z}$, $\sqrt{u_k^2(t+\Gamma) + \dots + u_k^2(t)} > \rho$, we have

$$\begin{aligned}
&F(t, u_k(t+\Gamma), \dots, u_k(t))\\
&\leq \Big[p + q\big(\sum_{i=0}^{\Gamma} u_k^2(t+i)\big)^{\frac{\nu}{2}}\Big]\Big[\sum_{t=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u_k(t+\Gamma), \dots, u_k(t))u_k(t-i)\\
&\quad - 2F(t, u_k(t+\Gamma), \dots, u_k(t))\Big].
\end{aligned} \tag{3.8}$$

By (2.1), (2.3) and (3.5), there exist constants C_1 and C_2 such that

$$\begin{aligned}
C_1 &\geq 2J(u_k) - \langle J'(u_k), u_k\rangle\\
&= \sum_{t=-\infty}^{+\infty}\Big[\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u_k(t+\Gamma), \dots, u_k(t))u_k(t-i)\\
&\quad - 2F(t, u_k(t+\Gamma), \dots, u_k(t))\Big]
\end{aligned} \tag{3.9}$$

and

$$J(u_k) \leq C_2. \tag{3.10}$$

From (2.3), (3.2), (3.6), (3.7), (3.8), (3.9) and (3.10) it follows that

$$
\begin{aligned}
&\frac{1}{2}\|u_k\|^2\\
&= J(u_k) + \sum_{t=-\infty}^{+\infty} F(t, u_k(t+\Gamma), \dots, u_k(t))\\
&= J(u_k) + \sum_{t\in\mathbb{Z}\left((\sum_{i=0}^{\Gamma} u_k^2(t+i))^{1/2}\le\rho\right)} F(t, u_k(t+\Gamma), \dots, u_k(t))\\
&\quad + \sum_{t\in\mathbb{Z}\left((\sum_{i=0}^{\Gamma} u_k^2(t+i))^{1/2}>\rho\right)} F(t, u_k(t+\Gamma), \dots, u_k(t))\\
&\le J(u_k) + \frac{\underline{\lambda}}{4(\Gamma+1)} \sum_{t\in\mathbb{Z}((\sum_{i=0}^{\Gamma} u_k^2(t+i))^{1/2}\le\rho)} \sum_{i=0}^{\Gamma} u_k^2(t+i)\\
&\quad + \sum_{t\in\mathbb{Z}\left((\sum_{i=0}^{\Gamma} u_k^2(t+i))^{1/2}>\rho\right)} \Big[p + q(\sum_{i=0}^{\Gamma} u_k^2(t+i))^{\frac{\nu}{2}}\Big]\\
&\quad \times \Big[\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u_k(t+\Gamma), \dots, u_k(t))u_k(t-i) - 2F(t, u_k(t+\Gamma), \dots, u_k(t))\Big]\\
&\le C_2 + \frac{1}{4}\|u_k\|^2 + \sum_{t\in\mathbb{Z}} \Big[p + q(\sum_{i=0}^{\Gamma} u_k^2(t+i))^{\frac{\nu}{2}}\Big]\\
&\times \Big[\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u_k(t+\Gamma), \dots, u_k(t))u_k(t-i) - 2F(t, u_k(t+\Gamma), \dots, u_k(t))\Big]\\
&\le C_2 + \frac{1}{4}\|u_k\|^2 + [p + q(\Gamma+1)\|u_k\|_\infty^\nu]\\
&\quad \times \Big[\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u_k(t+\Gamma), \dots, u_k(t))u_k(t-i) - 2F(t, u_k(t+\Gamma), \dots, u_k(t))\Big]\\
&\le C_2 + \frac{1}{4}\|u_k\|^2 + C_1[p + q(\Gamma+1)\|u_k\|_\infty^\nu]\\
&\le C_2 + \frac{1}{4}\|u_k\|^2 + C_1[p + \underline{\lambda}^{-\frac{\nu}{2}} q(\Gamma+1)\|u_k\|^\nu], \quad k\in\mathbb{N}.
\end{aligned}
$$

Since $\nu < 2$, from the above inequality it follows that $\{u_k\}_{k\in\mathbb{N}}$ is bounded. The proof is complete. □

4. Proof of main results

In this Section, we shall prove our main results by using the critical point theory.

Proof of Theorem 1.1. Lemma 3.3 implies that the existence of a sequence $\{u_k\}_{k\in\mathbb{N}} \subset E$ satisfying (3.3), and so (3.5). By Lemma 3.4, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in E. Thus, combining with (3.2), there exists a constant $C_3 > 0$ such that

$$\sqrt{\underline{\lambda}}\|u_k\|_\infty \le \|u_k\| \le C_3, \quad \forall k\in\mathbb{N}. \tag{4.1}$$

Hence, by (H2)–(H4), for $t\in\mathbb{Z}$, with $\big(\sum_{i=0}^{\Gamma}u_k^2(t+i)\big)^{1/2}\le\frac{1}{\sqrt{\underline{\lambda}}}C_3$, we have

$$
\begin{aligned}
&\big|\frac{1}{2}f(t,u_k(t+\Gamma),\dots,u_k(t),\dots,u_k(t-\Gamma))u_k(t)-F(t,u_k(t+\Gamma),\dots,u_k(t))\big|\\
&\le\frac{c\underline{\lambda}}{4C_3^2}u_k^2(t)+\frac{c\underline{\lambda}}{4(\Gamma+1)C_3^2}\sum_{i=0}^{\Gamma}u_k^2(t+i).
\end{aligned}\tag{4.2}
$$

Define $\varepsilon:=\limsup_{k\to\infty}\|u_k\|_\infty$. We state that $\varepsilon>0$. For the sake of contradiction, we assume that $\varepsilon=0$. From (H3), (2.3), (3.5) and (4.2), we have

$$
\begin{aligned}
c&=J(u_k)-\frac{1}{2}\langle J'(u_k),u_k\rangle+o(1)\\
&=\frac{1}{2}\sum_{t=-\infty}^{+\infty}f(t,u_k(t+\Gamma),\dots,u_k(t),\dots,u_k(t-\Gamma))u_k(t)\\
&\quad-\sum_{t=-\infty}^{+\infty}F(t,u_k(t+\Gamma),\dots,u_k(t))+o(1)\\
&\le\frac{c\underline{\lambda}}{4C_3^2}\sum_{t=-\infty}^{+\infty}u_k^2(t)+\frac{c\underline{\lambda}}{4(\Gamma+1)C_3^2}\sum_{t=-\infty}^{+\infty}\sum_{i=0}^{\Gamma}u_k^2(t+i)+o(1)\\
&\le\frac{c\underline{\lambda}}{4C_3^2}\|u_k\|_2^2+\frac{c\underline{\lambda}}{4C_3^2}\|u_k\|_2^2+o(1)\\
&\le\frac{c}{2}+o(1),\quad k\to\infty.
\end{aligned}
$$

This contradiction shows that $\varepsilon>0$.

First, going to a subsequence if necessary, we can assume that the existence of $t_k\in\mathbb{Z}$ depending on u_k such that

$$
|u_k(t_k)|=\|u_k\|_\infty>\frac{\varepsilon}{2}. \tag{4.3}
$$

Hence, making such shifts, we can assume that $t_k\in\mathbb{Z}(0,M-1)$ in (4.3). Moreover, passing to a subsequence of ks, we can even assume that $t_k=t_0$ is independent of k.

Next, we extract a subsequence, still denote by u_k, such that

$$
u_k(t)\to u(t),\quad k\to\infty,\quad \forall t\in\mathbb{Z}.
$$

Inequality (4.3) implies that $|u(t_0)|\ge\xi$ and, hence, $u=\{u(t)\}$ is a nonzero sequence. Moreover,

$$
\begin{aligned}
&Lu(t)-\omega u(t)-f(t,u(t+\Gamma),\dots,u(t),\dots,u(t-\Gamma))\\
&=\lim_{k\to\infty}[Lu_k(t)-\omega u_k(t)-f(t,u_k(t+\Gamma),\dots,u_k(t),\dots,u_k(t-\Gamma))]\\
&=\lim_{k\to\infty}0=0.
\end{aligned}
$$

So $u=\{u(t)\}$ is a solution of (1.1).

Finally, for any fixed $D\in\mathbb{Z}$ and k large enough, we have

$$
\sum_{t=-D}^{D}|u_k(t)|^2\le\frac{1}{\underline{\lambda}}\|u_k\|^2\le C_3^2.
$$

Since C_3^2 is a constant independent of k, passing to the limit, we have

$$\sum_{t=-D}^{D} |u(t)|^2 \le C_3^2.$$

Because of the arbitrariness of D, $u \in l^2$. Therefore, u satisfies $u(t) \to 0$ as $|t| \to \infty$. The existence of a nontrivial homoclinic solution is obtained. □

Proof of Theorem 1.4. By a proof similar to the one in Theorem 1.1 and the process in [13], we can prove Theorem 1.4. For simplicity, the proof is omitted. □

5. Example

As an application of Theorem 1.1, we give an example that illustrates our main result. For $t \in \mathbb{Z}$, assume that

$$\begin{aligned} &u(t+1) + u(t-1) - (2+\omega)u(t) \\ &= 3\sum_{j=0}^{\Gamma} \Big\{ 2u(t) \ln \Big[1 + (\sum_{i=0}^{\Gamma} u^2(t+i-j))^{1/2}\Big] \\ &\quad + \frac{\big\{ \sum_{i=0}^{\Gamma} u^2(t+i-j)\big)^{1/2} u(t)}{1 + \big(\sum_{i=0}^{\Gamma} u^2(t+i-j)\big)^{1/2}} \Big\}, \end{aligned} \tag{5.1}$$

where $\omega < -4$. We have $a(t) = a(t-1) \equiv 1$, $b(t) \equiv -2$, and

$$F(t, u(t+\Gamma), \dots, u(t)) = 3\sum_{i=0}^{\Gamma} u^2(t+i) \ln \Big[1 + (\sum_{i=0}^{\Gamma} u^2(t+i))^{1/2}\Big].$$

Then

$$\begin{aligned} &\sum_{i=-\Gamma}^{0} F'_{2+\Gamma+i}(t, u(t+\Gamma), \dots, u(t)) u(t-i) \\ &= 3\Big[2\sum_{i=0}^{\Gamma} u^2(t+i) \ln \Big[1 + (\sum_{i=0}^{\Gamma} u^2(t+i))^{1/2}\Big] + \frac{\big(\sum_{i=0}^{\Gamma} u^2(t+i)\big)^{3/2}}{1 + \big(\sum_{i=0}^{\Gamma} u^2(t+i)\big)^{1/2}}\Big] \\ &\ge \Big[2 + \frac{1}{1 + (\sum_{i=0}^{\Gamma} u^2(t+i))^{1/2}}\Big] F(t, u(t+\Gamma), \dots, u(t)) \ge 0. \end{aligned}$$

This shows that (H7) holds with $p = q = \nu = 1$. It is easy to verify that all the assumptions of Theorem 1.1 are satisfied. Consequently, (5.1) has a nontrivial homoclinic solution.

Acknowledgements. The author would like to thank the anonu=ymous referees, Prof. Julio G. Dix, and the editors for their careful reading and for making some valuable comments and suggestions on the manuscript.

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