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A q-FRACTIONAL APPROACH TO THE REGULAR STURM-LIOUVILLE PROBLEMS

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Abstract. In this article, we study the regular q-fractional Sturm-Liouville problems that include the right-sided Caputo q-fractional derivative and the left-sided Riemann-Liouville q-fractional derivative of the same order, $\alpha \in (0,1)$. We prove properties of the eigenvalues and the eigenfunctions in a certain Hilbert space. We use a fixed point theorem for proving the existence and uniqueness of the eigenfunctions. We also present an example involving little q-Legendre polynomials.

1. Introduction

The q-calculus was initiated at the beginning of the 19th century. Since then, many works have been devoted to the study of q-difference equations; see e.g., [1, 2, 12]. Recently many researchers have focused their attention on certain generalizations of Sturm-Liouville problems. In particular, in [6] the authors studied a q-analogue of Sturm-Liouville eigenvalue problems and formulated a self-adjoint q-difference operator in a Hilbert space. Their results are applied and developed in different aspects; see for example [5, 8, 10, 14, 16]. Mansour [15] introduced fractional q-Sturm-Liouville problems containing the left-sided Caputo q-fractional derivative and the right-sided Riemann-Liouville q-fractional derivative which are adjoint operators in a certain Hilbert space.

In this paper, we formulate a regular q-fractional Sturm-Liouville problem that contains the right-sided Caputo q-fractional derivative and the left-sided Riemann-Liouville q-fractional derivative of the same order, $\alpha \in (0,1)$. More precisely, our problem is described as follows.

Let $0 < \alpha < 1$ and p, r, w_α be given real valued functions defined on a q-linear grid $A^*_{q,a}$ (see Section 2.) such that $p(x) \neq 0$ and $w_\alpha(x) > 0$ for all x. We consider the q-Sturm-Liouville operator

$$\mathcal{L}_{q,\alpha} y(x) := {}^cD^\alpha_{q,a^-}\big(pD^\alpha_{q,0^+}y\big)(x) + r(x)y(x),$$

and consider the fractional differential equation

$$\mathcal{L}_{q,\alpha} y(x) - \lambda w_\alpha(x) y(x) = 0, \quad x \in A^*_{q,a}, \tag{1.1}$$

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that will be called a regular fractional q-Sturm-Liouville problem (regular qFSLP). This equation is complemented with the boundary conditions

$$\beta_1(I_{q,0^+}^{1-\alpha}y)(0)+\beta_2(pD_{q,0^+}^{\alpha}y)(0)=0, \tag{1.2}$$

$$\gamma_1(I_{q,0^+}^{1-\alpha}y)(a)+\gamma_2(pD_{q,0^+}^{\alpha}y)(\frac{a}{q})=0, \tag{1.3}$$

with $\beta_1^2+\beta_2^2\neq 0$ and $\gamma_1^2+\gamma_2^2\neq 0$.

This article is organized as follows. In the next section, we state the q-definitions and present some preliminaries of fractional q-calculus which will play an important role in our main results. The properties of the associated eigenvalues and eigenfunctions of the regular qFSLP (1.1)–(1.3) are stated and proved in Section 3. In Section 4, we apply the fixed point theorem to prove the existence and uniqueness of the eigenfunctions and corresponding eigenvalues. In the last section, we give an example for a regular qFSLP involving little q-Legendre polynomials.

2. Preliminaries

Throughout this article, we assume that $0<q<1$ and we follow Gasper and Rahman [11] for the definitions of the q-shifted factorial, the q-gamma and q-beta functions, the basic hypergeometric series and Jackson q-integrals.

For $t>0$, the sets $A_{q,t}$, $A_{q,t}^*$ and $\mathcal{A}_{q,t}$ are defined by

$$A_{q,t}:=\{tq^n:n\in\mathbb{N}_0\},\quad A_{q,t}^*:=A_{q,t}\cup\{0\},\quad \mathcal{A}_{q,t}:=\{tq^k:k\in\mathbb{Z}\},$$

where $\mathbb{N}_0:=\{0,1,2,\dots\}$. Note that if $t=1$ we write A_q, A_q^*, and $\mathcal{A}_q$. A function f defined on $A_{q,t}^*$ is called q-regular at zero if it satisfies

$$\lim_{n\to\infty}f(xq^n)=f(0)\quad\text{for all } x\in A_{q,t}^*.$$

The q-derivative D_qf of an arbitrary function f is defined by

$$(D_qf)(x):=\frac{f(x)-f(qx)}{(1-q)x},\quad x\neq 0.$$

Note that

$$D_{q,x}f(\frac{x}{q})=-\frac{1}{q}D_{q^{-1},x}f(x), \tag{2.1}$$

$$D_q(fg)(x)=D_qf(x)g(x)+f(qx)D_qg(x). \tag{2.2}$$

The q-integration by parts rule on an interval $[a,b]$ (see [7]) is

$$\int_a^b f(x)D_qg(x)\,d_qx=f(x)g(x)\Big|_a^b-\int_a^b D_qf(x)g(qx)\,d_qx, \tag{2.3}$$

where f and g are q-regular at zero functions. Using (2.1) and (2.3), we obtain the q^{-1}-integration by parts rule:

$$\int_a^b f(x)D_{q^{-1}}g(x)\,d_qx=qf(x)g(\frac{x}{q})\Big|_a^b-q\int_a^b g(x)D_qf(x)\,d_qx. \tag{2.4}$$

If X is the set $A_{q,t}$ or $A_{q,t}^*$, then for $p>0$, $L_q^p(X)$ is the space of all functions defined on X and satisfying

$$\|f\|_p:=\Big(\int_0^t|f(x)|^p\,d_qx\Big)^{1/p}<\infty;$$

it is a normed space. Moreover, if $p=2$, then $L_q^2(X)$ associated with the inner product

$$\langle f,g\rangle := \int_0^t f(x)\overline{g(x)}\,d_qx \tag{2.5}$$

is a Hilbert space. The space of all functions f defined on X such that

$$\int_0^t |f(x)|^2 w(x)\,d_qx < \infty,$$

where w is a positive function defined on X is called a weighted space and denoted by $L_q^2(X,w)$. This space associated with the inner product

$$\langle f,g\rangle := \int_0^t f(x)\overline{g(x)}w(x)\,d_qx \tag{2.6}$$

is a Hilbert space.

Let $C_q(X)$ denote the space of all q-regular at zero functions defined on X with values in $\mathbb{R}$. The space of all q-absolutely continuous functions on $A_{q,t}^*$ is denoted by $\mathcal{A}C_q(A_{q,t}^*)$ and is defined as the space of all q-regular at zero functions f satisfying

$$\sum_{j=0}^{\infty} |f(xq^j)-f(xq^{j+1})| \le K \quad \text{for all } x\in A_{q,t}^*,$$

where K is a constant depending on the function f. Note that $\mathcal{A}C_q(A_{q,t}^*) \subseteq C_q(A_{q,t}^*)$.

In the following we recall some definitions, roles and properties of fractional q-calculus (for more details see [3, 4]).

Let $\alpha>0$ and $f\in L_q(A_{q,a}^*)$. The left-sided Riemann-Liouville q-fractional operator of order α is

$$I_{q,a^+}^{\alpha} f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x (qt/x;q)_{\alpha-1} f(t)\,d_qt,$$

If $f\in L_q(A_{q,b}^*)$, then the right-sided Riemann-Liouville q-fractional operator of order α is

$$I_{q,b^-}^{\alpha} f(x) := \frac{1}{\Gamma_q(\alpha)}\int_{qx}^b t^{\alpha-1}(qt/x;q)_{\alpha-1} f(t)\,d_qt.$$

The left and right side Riemann-Liouville fractional q-derivatives are defined by

$$D_{q,a^+}^{\alpha} f(x) := D_q^m I_{q,a^+}^{m-\alpha} f(x), \quad D_{q,b^-}^{\alpha} f(x) := \Big(\frac{-1}{q}\Big)^m D_{q^{-1}}^m I_{q,b^-}^{m-\alpha} f(x),$$

and the left and right sided Caputo fractional q-derivatives are defined by

$${}^cD_{q,a^+}^{\alpha} f(x) := I_{q,a^+}^{m-\alpha} D_q^m f(x), \quad {}^cD_{q,b^-}^{\alpha} f(x) := \Big(\frac{-1}{q}\Big)^m I_{q,b^-}^{m-\alpha} D_{q^{-1}}^m f(x), \tag{2.7}$$

where $m=\ulcorner\alpha\urcorner$ denotes the ceiling function. According to [7, pp. 124, 148], $D_{q,a^+}^{\alpha} f(x)$ exists if $f\in L_q(A_{q,a}^*)$ such that $I_{q,a^+}^{1-\alpha} f\in \mathcal{A}C_q(A_{q,a}^*)$, and ${}^cD_{q,a^+}^{\alpha} f$ exists if $f\in\mathcal{A}C_q(A_{q,a}^*)$.

We end this section by the following results from [15], which will be needed later.

Lemma 2.1. *Let $\alpha>0$. If f is a function defined on $A_{q,a}^*$, then*

$$I_{q,a^-}^{\alpha}\, {}^cD_{q,a^-}^{\alpha} f(x) = f(x) - f(a/q), \tag{2.8}$$

$$ {}^cD^{\alpha}_{q,a^-} I^{\alpha}_{q,a^-} f(x) = f(x) - \frac{a^{-\alpha}}{\Gamma_q(1-\alpha)} (qx/a;q)_{-\alpha} \big(I^{1-\alpha}_{q,a^-} f \big)\big(\frac{a}{q}\big). \tag{2.9} $$

Lemma 2.2. *Let $\alpha > 0$. If $f \in L^1_q(A^*_{q,a})$ and bounded, then*

$$ {}^cD^{\alpha}_{q,0^+} I^{\alpha}_{q,0^+} f(x) = f(x), \quad I^{\alpha}_{q,0^+} f \in \mathcal{AC}_q(A^*_{q,a}), \tag{2.10} $$

$$ I^{\alpha}_{q,0^+} D^{\alpha}_{q,0^+} f(x) = f(x) - \frac{f(0)}{\Gamma_q(\alpha)}\, x^{\alpha-1}, \tag{2.11} $$

$$ D^{\alpha}_{q,0^+} I^{\alpha}_{q,0^+} f(x) = f(x). \tag{2.12} $$

Lemma 2.3. *Let $\alpha \in (0,1)$. If*

- *$f \in L^1_q(X)$ and g is a bounded function on $A_{q,a}$, or*
- *$\alpha \neq 1/2$ and $f, g \in L^2_q(X)$,*

then

$$ \int_0^a g(x) I^{\alpha}_{q,0^+} f(x)\, d_q x = \int_0^a f(x) I^{\alpha}_{q,a^-} g(x)\, d_q x. \tag{2.13} $$

3. Properties of regular fractional q-Sturm-Liouville problems

Recall that a complex number λ^* is said to be an eigenvalue of problem (1.1)–(1.3) if there is a non-trivial solution $y^*(\cdot)$ which satisfies the problem for this λ^*. In this case, we say that $y^*(\cdot)$ is an eigenfunction of the regular qFSLP corresponding to the eigenvalue λ^*.

We denote by V the Hilbert subspace of $L^2_q(A^*_{q,a}) \cap C_q(A^*_{q,a})$ which consists of all q-regular at zero functions satisfying the boundary conditions (1.2) and (1.3) with inner product

$$ \langle u, v \rangle := \int_0^a u(t)\overline{v(t)}\, d_q t. $$

Note that for $f, g \in V$ and $\alpha > 0$, we have the following equation (see [15, Lemma 2.4]):

$$ \int_0^a g(x) I^{\alpha}_{q,0^+} f(x)\, d_q x = \int_0^a f(x) I^{\alpha}_{q,a^-} g(x)\, d_q x \tag{3.1} $$

Lemma 3.1. *Let $\alpha \in (0,1)$ and $f, g \in V$. Then*

$$ \langle {}^cD^{\alpha}_{q,a^-} f, g \rangle = -f\big(\frac{x}{q}\big) I^{1-\alpha}_{q,0^+} g(x) \Big|_{x=0}^{a} + \langle f, D^{\alpha}_{q,0^+} g \rangle. $$

The proof of the above lemma follows directly by using (2.4), (3.1) and the definitions of ${}^cD^{\alpha}_{q,a^-}$ and $D^{\alpha}_{q,0^+}$. We omit it. Now, we prove the following important identity known as q-Lagrange's identity.

Proposition 3.2. *Let $u, v \in V$. Then*

$$ \langle \mathcal{L}_{q,\alpha} u, v \rangle - \langle u, \mathcal{L}_{q,\alpha} v \rangle = \Big[(I^{1-\alpha}_{q,0^+} u)(x) (pD^{\alpha}_{q,0^+} v)\big(\frac{x}{q}\big) - (I^{1-\alpha}_{q,0^+} v)(x) (pD^{\alpha}_{q,0^+} u)\big(\frac{x}{q}\big) \Big]_{x=0}^{a}. $$

Proof. Using the definition of $\mathcal{L}_{q,\alpha}$ and applying Lemma 3.1, it follows that

$$ \begin{aligned} \langle \mathcal{L}_{q,\alpha} u, v \rangle &= \langle {}^cD^{\alpha}_{q,a^-} pD^{\alpha}_{q,0^+} u + ru, v \rangle \\ &= -(pD^{\alpha}_{q,0^+} u)\big(\frac{x}{q}\big) I^{1-\alpha}_{q,0^+} v(x) \Big|_{x=0}^{a} + \langle ru, v \rangle + \langle D^{\alpha}_{q,0^+} u, pD^{\alpha}_{q,0^+} v \rangle \\ &= (I^{1-\alpha}_{q,0^+} u)(x) (pD^{\alpha}_{q,0^+} v)\big(\frac{x}{q}\big) \Big|_{x=0}^{a} - (I^{1-\alpha}_{q,0^+} v)(x) (pD^{\alpha}_{q,0^+} u)\big(\frac{x}{q}\big) \Big|_{x=0}^{a} \end{aligned} $$

$$+ \langle u, {}^cD^\alpha_{q,a^-} pD^\alpha_{q,0^+} v + rv\rangle.$$

Since $\langle u, {}^cD^\alpha_{q,a^-} pD^\alpha_{q,0^+} v + rv\rangle = \langle u, \mathcal{L}_{q,\alpha} v\rangle$, we obtained the required equality. □

By using q-Lagrange's identity, we obtain the following properties of the operator $\mathcal{L}_{q,\alpha}$ on the Hilbert space V.

Proposition 3.3. *Let $\alpha \in (0,1)$. Then*

(I) *$\mathcal{L}_{q,\alpha}$ is a self-adjoint operator on V. In other words,*

$$\langle \mathcal{L}_{q,\alpha} u, v\rangle = \langle u, \mathcal{L}_{q,\alpha} v\rangle \quad u, v \in V.$$

(II) *$\mathcal{L}_{q,\alpha}$ has only real eigenvalues.*

Proof. First, we prove (I). Let $u, v \in V$. Then from the boundary condition (1.2), we have

$$\beta_1 (I^{1-\alpha}_{q,0^+} u)(0) + \beta_2 (pD^\alpha_{q,0^+} u)(0) = 0,$$
$$\beta_1 (I^{1-\alpha}_{q,0^+} v)(0) + \beta_2 (pD^\alpha_{q,0^+} v)(0) = 0.$$

That is,

$$\begin{pmatrix} I^{1-\alpha}_{q,0^+} u(0) & I^{1-\alpha}_{q,0^+} v(0) \\ (pD^\alpha_{q,0^+} u)(0) & (pD^\alpha_{q,0^+} v)(0) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But $\beta_1^2 + \beta_2^2 \neq 0$ which implies

$$I^{1-\alpha}_{q,0^+} u(0) (pD^\alpha_{q,0^+} v)(0) - I^{1-\alpha}_{q,0^+} v(0) (pD^\alpha_{q,0^+} u)(0) = 0.$$

Similarly, from the boundary condition (1.3), we obtain

$$I^{1-\alpha}_{q,0^+} u(a) (pD^\alpha_{q,0^+} v)(\frac{a}{q}) - I^{1-\alpha}_{q,0^+} v(a) (pD^\alpha_{q,0^+} u)(\frac{a}{q}) = 0.$$

Hence, using q-Lagrange's identity, we conclude that $\mathcal{L}_{q,\alpha}$ is a self-adjoint operator on V.

To prove (II), we assume that λ is an eigenvalue associated with an eigenfunction y. Then

$$\mathcal{L}_{q,\alpha} y(x) = \lambda w_\alpha(x) y(x), \tag{3.2}$$
$$\mathcal{L}_{q,\alpha} \overline{y(x)} = \bar{\lambda} w_\alpha(x) \overline{y(x)}. \tag{3.3}$$

Multiply equation (3.2) by $\bar{y}$ and (3.3) by y and then subtracting, we obtain

$$y(x) \mathcal{L}_{q,\alpha} \overline{y(x)} - \overline{y(x)} \mathcal{L}_{q,\alpha} y(x) = (\bar{\lambda} - \lambda) w_\alpha(x) y(x) \overline{y(x)}.$$

Now, the q-integration over the interval $[0, a]$, and the application of q-Lagrange's identity yield

$$0 = \int_0^a \Big(y(x) \mathcal{L}_{q,\alpha} \overline{y(x)} - \overline{y(x)} \mathcal{L}_{q,\alpha} y(x) \Big)\, d_q x = (\bar{\lambda} - \lambda) \int_0^a w_\alpha(x) |y(x)|^2 \, d_q x.$$

But y is non trivial solution and $w_\alpha > 0$, this implies $\lambda = \bar{\lambda}$. □

Proposition 3.4. *The eigenfunctions corresponding to different eigenvalues of the regular qFSLP are orthogonal on the weighted space $L^2_q(A^*_{q,a}, w_\alpha)$.*

Proof. Let u_i $(i=1,2)$ be eigenfunctions of the regular qFSLP (1.1)–(1.3) associated with different eigenvalues λ_i $(i=1,2)$. Then

$$\mathcal{L}_{q,\alpha}\{u_i\} = \lambda_i w_\alpha u_i, \quad i=1,2$$

By using Proposition 3.3, we obtain

$$(\lambda_1-\lambda_2)\int_0^a u_1(x)u_2(x)w_\alpha(x)\,d_qx = 0.$$

Since $\lambda_1 \neq \lambda_2$, then u_1 and u_2 are orthogonal on $L_q^2(A_{q,a}^*, w_\alpha)$. $\square$

4. Uniqueness of eigenfunctions of the regular qFSLP

In this section, we give a sufficient condition of λ to guarantee the existence and uniqueness of the eigenfunctions up to a multiplier constant.

Recall that the multiplicity of an eigenvalue is defined to be the number of linearly independent eigenfunctions associated with it. In particular, an eigenvalue is simple if and only if it has only one eigenfunction.

First, we study the solution of the q-difference equation

$$^cD_{q,a^-}^\alpha p(x) D_{q,0^+}^\alpha \phi_0(x) = \frac{c\,a^{-\alpha}}{\Gamma_q(1-\alpha)}(qx/a;q)_{-\alpha}, \tag{4.1}$$

where c is constant. Note that

$$I_{q,a^-}^{-\alpha}(1) = \frac{a^{-\alpha}}{\Gamma_q(1-\alpha)}(qx/a;q)_{-\alpha}.$$

So, acting on the two sides of (4.1) by the operator I_{q,a^-}^α, we obtain

$$I_{q,a^-}^\alpha\, {^cD_{q,a^-}^\alpha} p(x) D_{q,0^+}^\alpha \phi_0(x) = cI_{q,a^-}^\alpha I_{q,a^-}^{-\alpha}(1).$$

Using (2.8) and (2.11), we obtain

$$\phi_0(x) = c_1\,x^{\alpha-1} + c_2\,I_{q,0^+}^\alpha \frac{1}{p(x)},$$

where

$$c_1 = c - \Big(p(\cdot)D_{q,0^+}^\alpha \phi_0(\cdot)\Big)(a/q), quad c_2 = \frac{\phi(0)}{\Gamma_q(\alpha)}.$$

Thus, we have the following result.

Lemma 4.1. *The general solution of the q-difference equation* (4.1) *takes the form*

$$\phi_0(x) = c_1 x^{\alpha-1} + c_2\psi_\alpha(x),$$

where $\psi_\alpha(x) = I_{q,0^+}^\alpha \frac{1}{p(x)}$ and c_1, c_2 are constants.

Lemma 4.2. *Let $\alpha \in (0,1)$, $\psi_\alpha(x) = I_{q,0^+}^\alpha \frac{1}{p(x)}$ and*

$$Y_y(x) := r(x)y(x) - \lambda w_\alpha(x) y(x), \tag{4.2}$$

$$\Delta := \Gamma_q(\alpha)\Big[\beta_1\gamma_2 - \beta_2\gamma_1 + \beta_1\gamma_1(\psi_\alpha(a) - \psi_\alpha(0))\Big]. \tag{4.3}$$

If $\Delta \neq 0$, then, on the space $C(A_{q,a}^*)$, the regular qFSLP (1.1)–(1.3) *is equivalent to the q-integral equation*

$$y(x) = -\Big(I_{q,0^+}^\alpha \frac{1}{p(\cdot)} I_{q,a^-}^\alpha Y_y(\cdot)\Big)(x) + A(x)\Big(I_{q,a^-}^\alpha Y_y(\cdot)\Big)(x)\Big|_{x=0}$$

$$+ B(x)\Big(I_{q,0^+}\frac{1}{p(\cdot)}I^{\alpha}_{q,a^-}Y_y(\cdot)\Big)(x)\Big|_{x=a} + C(x)\Big(I_{q,0^+}\frac{1}{p(\cdot)}I^{\alpha}_{q,a^-}Y_y(\cdot)\Big)(x)\Big|_{x=0},$$

where

$$A(x) = \frac{\beta_2}{\Delta}\Big[x^{\alpha-1}(\gamma_1\psi_\alpha(a)+\gamma_2) - \gamma_1\,\psi_\alpha(x)\Gamma_q(\alpha)\Big],$$
$$B(x) = \frac{\gamma_1}{\Delta}\Big[\beta_1\,\psi_\alpha(x)\Gamma_q(\alpha) - x^{\alpha-1}(\beta_1\psi_\alpha(0)+\beta_2)\Big],$$
$$C(x) = \frac{\beta_1\,A(x)}{\beta_2}.$$

Proof. Since Y_y is defined by

$$Y_y(x) := r(x)y(x) - \lambda w_\alpha(x)y(x),$$

equation (1.1) takes the form

$${}^cD^{\alpha}_{q,a^-}p(x)D^{\alpha}_{q,0^+}y(x) + Y_y(x) = 0.$$

Using (2.9), we can rewrite Y_y as

$$Y_y(x) := \Big({}^cD^{\alpha}_{q,a^-}pD^{\alpha}_{q,0^+}I^{\alpha}_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_y\Big)(x) + \frac{a^{-\alpha}}{\Gamma_q(1-\alpha)}(qx/a;q)_{-\alpha}\Big(I^{1-\alpha}_{q,a^-}Y_y\Big)(\frac{a}{q}).$$

This implies

$${}^cD^{\alpha}_{q,a^-}p(x)D^{\alpha}_{q,0^+}\Big[y(\cdot) + I^{\alpha}_{q,0^+}\frac{1}{p(\cdot)}I^{\alpha}_{q,a^-}Y_y(\cdot)\Big](x) = \frac{c\,a^{-\alpha}}{\Gamma_q(1-\alpha)}(qx/a;q)_{-\alpha},$$

where $c = \Big(I^{1-\alpha}_{q,a^-}Y_y(\cdot)\Big)(a/q)$. Now, set

$$\phi_0 = y(x) + I^{\alpha}_{q,0^+}\Big(\frac{1}{p(\cdot)}I^{\alpha}_{q,a^-}Y_y(\cdot)\Big)(x),$$

and using Lemma 4.1, we obtain

$$y(x) + I^{\alpha}_{q,0^+}\frac{1}{p(x)}I^{\alpha}_{q,a^-}Y_y(x) = c_1x^{\alpha-1} + c_2\psi_\alpha(x). \tag{4.4}$$

This implies the following equalities

$$\Big(I^{1-\alpha}_{q,0^+}y\Big)(x) + \Big(I_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_y\Big)(x) = c_1\,\Gamma_q(\alpha) + c_2\,I^{\alpha}_{q,0^+}\frac{1}{p(x)}, \tag{4.5}$$

$$\Big(pD^{\alpha}_{q,0^+}y\Big)(x) + I^{\alpha}_{q,a^-}Y_y(x) = c_2. \tag{4.6}$$

Using (4.5) and (4.6), we obtain

$$\Big(I^{1-\alpha}_{q,0^+}y\Big)(0) + \Big(I^{\alpha}_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_y\Big)(0) = c_1\Gamma_q(\alpha) + c_2\,\Big(I^{\alpha}_{q,0^+}\frac{1}{p}\Big)(0), \tag{4.7}$$

$$\Big(pD^{\alpha}_{q,0^+}y\Big)(0) + \Big(I^{\alpha}_{q,a^-}Y_y\Big)(0) = c_2, \tag{4.8}$$

$$\Big(I^{1-\alpha}_{q,0^+}y\Big)(a) + \Big(I^{\alpha}_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_y\Big)(a) = c_1\,\Gamma_q(\alpha) + c_2\Big(I^{\alpha}_{q,0^+}\frac{1}{p}\Big)(a), \tag{4.9}$$

$$\Big(pD^{\alpha}_{q,0^+}y\Big)(a/q) = c_2. \tag{4.10}$$

Substituting from (4.7) and (4.8) into (1.2) and from (4.9) and (4.10) in (1.3), we obtain the system

$$c_1(\beta_1\Gamma_q(\alpha)) + c_2\Big[\beta_1 I^{\alpha}_{q,0^+}\frac{1}{p(0)} + \beta_2\Big] = \beta_1X(0) + \beta_2Z$$

$$c_1(\gamma_1\Gamma_q(\alpha)) + c_2\Big[\gamma_1 I^{\alpha}_{q,0^+}\frac{1}{p(a)} + \gamma_2\Big] = \gamma_1 X(a),$$

where $X := I^{\alpha}_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_y$ and $Z = I^{\alpha}_{q,a^-}Y_y(0)$.

Since $\Delta \neq 0$, the solution for coefficients c_1 and $\tilde{c}_2$ is unique, and is given by

$$c_1 = \frac{1}{\Delta}\Big[(\beta_1 X(0) + \beta_2 Z)(\gamma_1\psi_\alpha(a) + \gamma_2) - \gamma_1 X(a)(\beta_1\psi_\alpha(0) + \beta_2)\Big],$$

$$c_2 = \frac{\gamma_1\Gamma_q(\alpha)}{\Delta}\Big[\beta_1 X(a) - (\beta_1 X + \beta_2 Z)(0)\Big].$$

Now, substituting the expressions of c_1 and c_2 into (4.4), we obtain the desired result. □

Note that by using Lemma 4.2, we can verify that the regular qFSLP (1.1) can be interpreted as a fixed point for the mapping $T : C(A^*_{q,a}) \to C(A^*_{q,a})$ which defined by

$$\begin{aligned} Tf(x) = &-\Big(I^{\alpha}_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_f\Big)(x) + A(x)\Big(I^{\alpha}_{q,a^-}Y_f\Big)(x)\Big|_{x=0} \\ &+ B(x)\Big(I_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_f\Big)(x)\Big|_{x=a} + C(x)\Big(I_{q,0^+}\frac{1}{p}I^{\alpha}_{q,a^-}Y_f\Big)(x)\Big|_{x=0}. \end{aligned}$$

Set

$$Y_f(x) := r(x)y(x) - \lambda w_\alpha(x)y(x),$$

we obtain

$$\|Y_g - Y_h\| \le \|g - h\|\,\|r - \lambda w_\alpha\|, \quad g, h \in C(A^*_{q,a}).$$

Now, denoting

$$A = \|A(x)\|, \quad B = \|B(x)\|, \quad m_p = \inf_{x\in A^*_{q,a}}|p(x)|, \quad M_\phi := \|\phi\|, \quad \tilde{M} := \|\tilde{\phi}\|,$$

where $\phi := I^{\alpha}_{q,0^+}I^{\alpha}_{q,a^-}$ and $\tilde{\phi} := I^{\alpha}_{q,a^-}$, it follows that

$$\|T_g - T_h\| \le \|g - h\|\,L, \quad L := \|r - \lambda w_\alpha\|\Big(\frac{M_\phi}{m_p} + A\,\tilde{\phi}(0) + \frac{Ba}{m_p}\tilde{\phi}(a)\Big).$$

Therefore, if

$$\|r - \lambda w_\alpha\| < \frac{m_p}{M_\phi + m_p A\tilde{\phi}(0) + Ba\tilde{\phi}(a)}, \tag{4.11}$$

we conclude that there is a unique fixed point $f_\lambda \in C(A^*_{q,a})$ which satisfies the regular qFSLP (1.1)–(1.3). Hence we have the following result.

Theorem 4.3. *Let $\alpha \in (0,1)$. If $\Delta \neq 0$, then unique q-regular at zero function f_λ for the regular qFSLP* (1.1)–(1.3) *corresponding to each eigenvalue obeying* (4.11) *exists, and such eigenvalue is simple.*

Note that if r and w_α are $L^2_q(A^*_{q,a})$ functions, then we have the following version of Theorem 4.3.

Theorem 4.4. *Let $\alpha \in (\frac{1}{4}, 1)$. Assume that the functions r and w_α are $L^2_q(A^*_{q,a})$ functions, and p is a function satisfying $\inf_{x\in A^*_{q,a}} p(x) > 0$. If $\Delta \neq 0$, then there exists a unique q-regular at zero function y_λ for the regular qFSLP* (1.1)–(1.3) *corresponding to each eigenvalue obeying*

$$\|r - \lambda w_\alpha\|_2 \le \frac{\sigma_\alpha m_p}{\sqrt{a}\big(Ba^{\frac{1}{2}-\alpha} + B_q(\alpha, \alpha + \frac{1}{2})\big)},$$

where

$$\sigma_\alpha = \Gamma_q(\alpha)(q^\alpha;q)_\infty\sqrt{\frac{1-q^{1-2\alpha}}{1-q}}, \quad \text{for } \frac{1}{4}<\alpha<\frac{1}{2},$$

and satisfying

$$\|r-\lambda w_\alpha\|_2 \le \frac{\mu_\alpha\, m_p}{a^\alpha\big(\Gamma_q(\alpha)a^{\alpha-\frac{1}{2}}+B(1-q)^{1-\alpha}\big)},$$

where

$$\mu_\alpha = \frac{\Gamma_q(\alpha)(q;q)_\infty\sqrt{1-q^{2\alpha-1}}}{(1-q)^{\alpha-\frac{1}{2}}}, \quad \text{for } \frac{1}{2}<\alpha<1.$$

Proof. As in the proof of Theorem 4.3, the regular qFSLP (1.1) can be interpreted as a fixed point for the mapping $T: C(A^*_{q,a}) \to C(A^*_{q,a})$ which is defined by

$$\begin{aligned} Tf(x) = &-\Big(I^\alpha_{q,0^+}\frac{1}{p}I^\alpha_{q,a^-}Y_f\Big)(x) + A(x)\Big(I^\alpha_{q,a^-}Y_f\Big)(x)\Big|_{x=0} \\ &+ B(x)\Big(I_{q,0^+}\frac{1}{p}I^\alpha_{q,a^-}Y_f\Big)(x)\Big|_{x=a} + C(x)\Big(I_{q,0^+}\frac{1}{p}I^\alpha_{q,a^-}Y_f\Big)(x)\Big|_{x=0}. \end{aligned} \tag{4.12}$$

We will use the estimate

$$\begin{aligned} &\|I^\alpha_{q,a^-}(Y_g-Y_h)(x)\|_2 \\ &\le \|g-h\|\|r-\lambda w_\alpha\|_2\frac{1}{\Gamma_q(\alpha)}\Big(\int_{qx}^a t^{2\alpha-2}(qx/t;q)^2_{\alpha-1}\,d_qt\Big)^{1/2}, \end{aligned} \tag{4.13}$$

and the following inequalities (see [15, Theorem 3.8]):

$$\begin{aligned} &\|I^\alpha_{q,0^+}\Big(\frac{1}{p}I^\alpha_{q,a^-}(Y_g-Y_h)\Big)(x)\| \\ &\le \begin{cases} \|g-h\|\|r-\lambda w_\alpha\|_2\frac{\sigma_{1\alpha}\sqrt{a}}{m_p}, & \frac{1}{4}<\alpha<1/2, \\ \|g-h\|\|r-\lambda w_\alpha\|_2\frac{\sigma_{2\alpha}a^{2\alpha-\frac{1}{2}}}{m_p}, & \frac{1}{2}<\alpha<1, \end{cases} \end{aligned} \tag{4.14}$$

where

$$\sigma_{1\alpha} = \frac{\Gamma_q(\alpha+\frac{1}{2})}{(q^\alpha;q)_\infty\Gamma_q(2\alpha+\frac{1}{2})}\sqrt{\frac{1-q}{1-q^{1-2\alpha}}}, \quad \sigma_{2\alpha} = \frac{(1-q)^{\alpha-\frac{1}{2}}}{(q;q)_\infty\sqrt{1-q^{2\alpha-1}}}.$$

For the first case ($\frac{1}{4}<\alpha<\frac{1}{2}$), we have

$$\int_{qx}^a t^{2\alpha-2}(qx/t;q)^2_{\alpha-1}\,d_qt \le \frac{x^{1-2\alpha}}{(q^\alpha;q)^2_\infty}\frac{(1-q)}{1-q^{1-2\alpha}}. \tag{4.15}$$

From (4.13) and (4.15), we obtain

$$\|I^\alpha_{q,a^-}(Y_g-Y_h)(x)\|_2 \le \|g-h\|\|r-\lambda w_\alpha\|_2\frac{\sigma_{1\alpha}x^{\frac{1}{2}-\alpha}}{B_q(\alpha,\alpha+\frac{1}{2})}. \tag{4.16}$$

Using (4.12), (4.14) and (4.16), we obtain

$$\begin{aligned} \|T_g-T_h\|_2 &\le \|g-h\|\,\|r-\lambda w_\alpha\|_2\Big[\frac{\sigma_{1\alpha}\sqrt{a}}{m_p}\big(1+\frac{Ba^{\frac{1}{2}-\alpha}}{B_q(\alpha,\alpha+\frac{1}{2})}\big)\Big] \\ &= L_1\|g-h\|, \end{aligned}$$

where

$$L_1 = \|r-\lambda w_\alpha\|_2\Big[\frac{\sigma_{1\alpha}\sqrt{a}}{m_p}\big(1+\frac{Ba^{\frac{1}{2}-\alpha}}{B_q(\alpha,\alpha+\frac{1}{2})}\big)\Big].$$

Using the assumption of the theorem, we conclude that there is a unique fixed point $y_\lambda \in C(A^*_{q,a})$ which satisfies the regular qFSLP (1.1)–(1.3). Therefore, such eigenvalue is simple.

For the second case ($\frac{1}{2} < \alpha < 1$), we have

$$\int_{qx}^{a} t^{2\alpha-2}(qx/t;q)^2_{\alpha-1}\, d_q t \le \frac{a^{2\alpha-1}}{(q^\alpha;q)^2_\infty}\frac{(1-q)}{1-q^{2\alpha-1}},$$

$$\|I^\alpha_{q,a^-}(Y_g - Y_h)(x)\|_2 \le \|g-h\|\|r-\lambda w_\alpha\|_2 \frac{\sigma_{2\alpha}(1-q)^{1-\alpha}}{\Gamma_q(\alpha)} x^{\alpha-\frac{1}{2}}.$$

This implies

$$\begin{aligned} \|T_g - T_h\|_2 &\le \|g-h\|\,\|r-\lambda w_\alpha\|_2 \Big[\frac{\sigma_{2\alpha} a^\alpha}{m_p}\big(a^{\alpha-\frac{1}{2}} + \frac{B}{\Gamma_q(\alpha)}(1-q)^{1-\alpha}\big)\Big] \\ &= L_2\|g-h\|, \end{aligned}$$

where

$$L_2 = \|r-\lambda w_\alpha\|_2 \Big[\frac{\sigma_{2\alpha} a^\alpha}{m_p}\big(a^{\alpha-\frac{1}{2}} + \frac{B}{\Gamma_q(\alpha)}(1-q)^{1-\alpha}\big)\Big].$$

Using the assumption of the theorem, we conclude that there is a unique fixed point $y_\lambda \in C(A^*_{q,a})$ which satisfies the regular qFSLP (1.1)–(1.3). Therefore, such eigenvalue is simple, The proof is complete. □

Theorem 4.5. *Let $0 < \alpha < 1$ and k_0, k_1 be real numbers. Assume that the functions p, r and w_α are $C(A^*_{q,a})$ functions such that $\inf_{x\in A^*_{q,a}} p(x) > 0$. Then, the regular qFSLP* (1.1)–(1.3) *with the initial conditions*

$$\Big(I^{1-\alpha}_{q,0^+}\, y\Big)(0) = k_0, \quad \Big(pD^\alpha_{q,0^+}\, y\Big)(0) = k_1, \tag{4.17}$$

has a unique solution in $C(A^*_{q,a})$.

Proof. Assume that y_1 and y_2 are two solutions of (1.1) satisfying the initial conditions (4.17). Then $z = y_1 - y_2$ is a solution of (1.1) with the conditions

$$\Big(I^{1-\alpha}_{q,0^+}\, z\Big)(0) = \Big(pD^\alpha_{q,0^+}\, z\Big)(0) = 0. \tag{4.18}$$

From Lemma 4.2, we have

$$\begin{gathered} z(x) + \Big(I^\alpha_{q,0^+}\frac{1}{p} I^\alpha_{q,a^-} Y_z\Big)(x) = c_1\, x^{\alpha-1} + c_2\psi_\alpha(x), \\ \Big(I^{1-\alpha}_{q,0^+} z\Big)(x) + \Big(I_{q,0^+}\frac{1}{p} I^\alpha_{q,a^-} Y_z\Big)(x) = c_1\,\Gamma_q(\alpha) + c_2 I^\alpha_{q,0^+}\frac{1}{p(x)}, \\ \Big(pD^\alpha_{q,0^+} z\Big)(x) + I^\alpha_{q,a^-} Y_z(x) = c_2. \end{gathered}$$

Thus, we can verify that the regular qFSLP (1.1) can be interpreted as a fixed point for the mapping $T : C(A^*_{q,a}) \to C(A^*_{q,a})$ which defined by

$$\begin{aligned} Tf(x) = &-\Big(I^\alpha_{q,0^+}\frac{1}{p} I^\alpha_{q,a^-} Y_f\Big)(x) + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\Big(I_{q,0^+}\frac{1}{p} I^\alpha_{q,a^-} Y_f\Big)(0) \\ &+ \psi_\alpha(x)\Big(I^\alpha_{q,a^-} Y_f\Big)(0). \end{aligned} \tag{4.19}$$

Using the inequality (see [15])

$$\|I^\alpha_{q,0^+} f\| \le \frac{a^\alpha}{\Gamma_q(\alpha+1)}\|f(x)\|, \tag{4.20}$$

we obtain $\|\psi_\alpha(x)\| \le \frac{a^\alpha}{m_p\Gamma_q(\alpha+1)}\|f(x)\|$, and using the estimate
$$\|Y_g - Y_h\| \le \|g-h\|\,\|r-\lambda w_\alpha\|, \quad g,h \in C(A^*_{q,a}),$$
we have
$$\|T_g - T_h\| \le \|g-h\|\,\|r-\lambda w_\alpha\|\Big(\frac{M_\phi}{m_p} + \frac{a^\alpha}{m_p\Gamma_q(\alpha+1)}\tilde{\phi}(0)\Big).$$
So, if
$$\frac{\|r-\lambda w_\alpha\|}{m_p\Gamma_q(\alpha+1)}\Big(\Gamma_q(\alpha+1)M_\phi + a^\alpha\tilde{\phi}(0)\Big) < 1,$$
then $T : C(A^*_{q,a}) \to C(A^*_{q,a})$ is a contraction mapping and z is a unique fixed point of (4.19). Therefore, $z \equiv 0$, i.e., $y_1 = y_2$ on $A^*_{q,a}$. □

5. An application

The little q-Legendre polynomials $p_n(x|q)$, cf. ([13, 17]), are defined by
$$\begin{aligned} p_n(x|q) &= {}_2\phi_1(q^{-n},\, q^{n+1};\, q;\, q,\, qx) \\ &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k(q^{n+1};q)_k}{(q;q)_k(q;q)_k} q^k x^k. \end{aligned}$$
Recall that the little q-Legendre polynomials are the little q-Jacobi polynomials $p_n(x;q^\alpha,q^\beta|q)$ with $q^\alpha = q^\beta = 1$. These polynomials satisfy the orthogonality relation
$$\sum_{k=0}^{\infty} q^k p_m(q^k|q)p_n(q^k|q) = \frac{q^n}{(1-q^{2n+1})}\,\delta_{mn}.$$
They also satisfy the second-order q-differential equation
$$\frac{-1}{q}D_q\Big((x(1-x))D_q^{-1}y(x)\Big) + q^{-n}[n]_q[n+1]_q\,y(x) = 0,$$
where
$$[n]_q = \frac{1-q^n}{1-q}, \quad n \in \mathbb{R}.$$

In this section, we prove that the little q-Legendre polynomials satisfy a fractional q-Sturm-Liouville problem. Consider the q-fractional differential equation
$$ {}^cD^\mu_{q,1^-}(x^\mu(qx;q)_\mu)D^\mu_{q,0^+}y(x) = \lambda y(x), \quad x \in A^*_q, \mu \in (0,1), \tag{5.1}$$
subject to the boundary conditions
$$(I^{1-\mu}_{q,0^+}y)(0) = (x^\mu(qx;q)_\mu D^\alpha_{q,0^+}y)(\frac{1}{q}) = 0. \tag{5.2}$$
We shall prove that Problem (5.1)–(5.2) has a discrete spectrum $\{\phi_n,\lambda_n\}$, where ϕ_n is a little q-Legendre polynomials and the eigenvalues $\{\lambda_n\}$ has no finite limit points. The main result reads as follows.

Theorem 5.1. *For $\mu \in (0,1)$ and $\beta > -1$, the little q-Legendre polynomials*
$$\phi_n(x) = p_n(x;1,1|q), \quad n \in \mathbb{N}_0$$
are eigenfunctions of the qFSLP (5.1)–(5.2) associated to the eigenvalues
$$\lambda_n = q^{-n\mu}\frac{\Gamma_q(1+n+\mu)}{\Gamma_q(1+n-\mu)}.$$

To prove Theorem 5.1, we need the following results from [15].

Lemma 5.2.

$$I^{\mu}_{q,0^+}\Big((\cdot)^{\alpha}\, p_n(.;q^{\alpha},q^{\beta}|q)\Big)(x)=\frac{\Gamma_q(1+\alpha)}{\Gamma_q(1+\alpha+\mu)}\,x^{\alpha+\mu}\,p_n(x;q^{\alpha+\mu},\,q^{\beta-\mu}|q).$$

Lemma 5.3. *If α, β and μ are real numbers satisfying $\alpha>-1$, $\beta>-1$ and $\beta-1<\mu<\alpha+1$, then*

$$I^{\mu}_{q,1^-}\Big((qt;q)_{\beta}p_m(t;q^{\alpha},q^{\beta}|q)\Big)=$$
$$q^{m\mu}\frac{\Gamma_q(\beta+m+1)\Gamma_q(1+\alpha+m-\mu)\Gamma_q(1+\alpha)}{\Gamma_q(1+m+\beta+\mu)\Gamma_q(1+\alpha+m)\Gamma_q(1+\alpha-\mu)}(qt;q)_{\beta+\mu}p_m(t;q^{\alpha-\mu},q^{\beta+\mu}|q).$$

The following equation follows immediately from Lemma 5.2 and (2.12),

$$D^{\mu}_{q,0^+}p_n(x;1,q^{\beta-\mu}|q)=\frac{1}{\Gamma_q(1-\mu)}x^{-\mu}[p_n(x;q^{-\mu},q^{\beta}|q)-1]. \tag{5.3}$$

Also, from Lemma 5.3 and (2.9) we obtain

$$\begin{aligned}&{}^cD^{\mu}_{q,1^-}(qx;q)_{\beta+\mu}p_n(x;q^{\alpha-\mu},q^{\beta+\mu}|q)\\&=q^{-m\mu}\frac{\Gamma_q(1+n+\beta+\mu)\Gamma_q(1+\alpha+n)\Gamma_q(1+\alpha-\mu)}{\Gamma_q(\beta+n+1)\Gamma_q(1+\alpha+n-\mu)\Gamma_q(1+\alpha)}(qx;q)_{\beta}p_n(x;q^{\alpha},q^{\beta}|q)\\&\quad-\frac{(qx;q)_{-\mu}}{\Gamma_q(1-\mu)}\Big(I^{\mu}_{q,1^-}(q(\cdot);q)_{\beta}p_n(.\,;q^{\alpha},q^{\beta}|q)\Big)(\frac{1}{q}).\end{aligned} \tag{5.4}$$

Proof of Theorem 5.1. Setting $\beta=\mu$ in (5.3) we obtain

$$D^{\mu}_{q,0^+}p_n(x;1,1|q)=\frac{x^{-\mu}}{\Gamma_q(1-\mu)}[p_n(x;q^{-\mu},q^{\beta}|q)-1]. \tag{5.5}$$

Using (5.2), (5.4) and (5.5), it follows that

$$\begin{aligned}&{}^cD^{\mu}_{q,1^-}(x^{\mu}(qx;q)_{\mu})D^{\mu}_{q,0^+}p_n(x;1,1|q)\\&=\frac{{}^cD^{\mu}_{q,1^-}(qx;q)_{\mu}}{\Gamma_q(1-\mu)}[p_n(x;q^{-\mu},q^{\mu}|q)-1]\\&=q^{-m\mu}\frac{\Gamma_q(1+n+\mu)}{\Gamma_q(1+n-\mu)}(qx;q)_{\beta}p_n(x;1,1|q).\end{aligned} \tag{5.6}$$

Now, combining (5.1) and (5.6) gives the required result. □

Remark 5.4. Theorem 5.1 is a q-analogue of the following classical eigenvalue problem for the Legendre polynomials (see [9])

$$((1-x^2)y')'+\lambda y=0,\quad -1\le x\le 1.$$

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