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CENTER CONDITIONS AND LIMIT CYCLES FOR BILIÉNARD SYSTEMS

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Abstract. In this article we study the center problem for polynomial BiLiénard systems of degree n. Computing the focal values and using Gröbner bases we find the center conditions for such systems for $n = 6$. We also establish a conjecture about the center conditions for polynomial BiLiénard systems of arbitrary degree.

1. Introduction and statement of main results

The so-called Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ with where $f(x)$ and $g(x)$ are polynomials, which we rewrite as a differential system in the plane

$$\dot{x} = y, \quad \dot{y} = -g(x) - yf(x), \tag{1.1}$$

arises frequently in the study of various mathematical models of physical, chemical, biology and other areas. We assume that the singular point is at the origin $g(0) = 0$ and which is nondegenerate $g'(0) > 0$. By means of the Liénard transformation $y \mapsto y + F(x)$, where $F(x) = \int_0^x f(x)dx$, system (1.1) becomes

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{1.2}$$

The centers of system (1.2) are orbitally reversible, that is, are symmetric with respect to an analytic invertible transformation and a scaling of time followed by a reversion of time, see [1, 3, 9]. We recall that system (1.2) has a center at the origin if all its solutions in a neighborhood of the origin are closed. The center problem consists in finding necessary and sufficient conditions over F and g to have a center at the origin. In fact the original system studied by Liénard was with $g(x) = x$, see [15]. Liénard equations were intensely studied as they can be used to model oscillating circuits in vacuum tube technology, see for instance [9]. Moreover other equations may be reduced to Liénard equations, see [12].

In this work we study a family of polynomial systems which is a generalization of the original Liénard system, and corresponds to systems of the form

$$\dot{x} = -y + F(x), \quad \dot{y} = x + G(y), \tag{1.3}$$

where $F(x)$ and $G(y)$ are polynomials without constant and linear terms. These systems are called *BiLiénard systems*, see [8]. In [13] the center problem has been studied when $F(x)$ and $G(y)$ are polynomials of fourth degree and it was shown

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that all the centers are time-reversible. We recall that a system is time-reversible if it is invariant under the symmetry $(x, y, t) \mapsto (-x, y, -t)$ or $(x, y, t) \mapsto (x, -y, -t)$.

Furthermore, there are families of centers for $F(x)$ and $G(y)$ of arbitrary degree, see [8]. In [11] the authors classify all centers of the family of the BiLiénard systems of degree five and find the maximum number of limit cycles which can bifurcate from a fine focus for such systems.

In the following theorem we classify all centers of system (1.3) when $F(x)$ and $G(y)$ are polynomials of degree six.

Theorem 1.1. *Consider the differential system*

$$
\begin{aligned}
\dot{x} &= -y + F(x) = -y + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6, \\
\dot{y} &= \ \ x + G(y) = x + b_2y^2 + b_3y^3 + b_4y^4 + b_5y^5 + b_6y^6,
\end{aligned}
\tag{1.4}
$$

where a_i and b_i are real numbers. The origin is a center if, and only if, one of the following cases holds:

(a) $a_2 = a_3 = a_4 = a_5 = a_6 = b_3 = b_5 = 0$;
(b) $b_3 = -a_3$, $b_2 = \pm a_2$, $b_4 = \pm a_4$, $b_5 = -a_5$ *and* $b_6 = \pm a_6$;
(c) $a_3 = a_5 = b_2 = b_3 = b_4 = b_5 = b_6 = 0$.

Moreover, all centers at the origin are time-reversible.

The determination of the center conditions allows to study the small-amplitude limit cycles which can bifurcate from the origin of perturbations of such systems, see for instance [5, 10] and references therein. For system (1.4) we have the following result.

Proposition 1.2. *The maximum number of small–amplitude limit cycles which can bifurcate from the origin of system* (1.4) *is at least eight.*

Theorem 1.1 and Proposition 1.2 are proved in section 2 and 3 respectively. From the results presented in this work we can establish the following conjecture

Conjecture 1.3. *All the centers of system* (1.3) *are time-reversible and given by the following families*

(i) $F \equiv 0$ *and* $G(x) = G(-x)$;
(ii) $G \equiv 0$ *and* $F(x) = F(-x)$;
(iii) $F(x) = -G(x)$;
(iv) $F(x) = G(-x)$.

Moreover the result should carry over to the case where F and G are analytic functions. In the first case system (1.3) is invariant by the symmetry $(x, y, t) \mapsto (x, -y, -t)$. In the second case system (1.3) is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$. In fact these first two cases are classical Liénard families with a center. The last two cases are centers because they are invariant by the symmetry $(x, y, t) \mapsto (y, x, -t)$.

Cases (a) and (c) of Theorem 1.1 correspond to case (i) and (ii) of Conjecture 1.3, respectively. Case (b) of Theorem 1.1 corresponds to the cases (iii) and (iv) of Conjecture 1.3.

2. Proof of Theorem 1.1

First we determine the necessary conditions for having a center. These necessary conditions can be determined by different methods, see [13, 17]. We use here the method developed by Poincaré of construction of a formal first integral. To construct this first integral we will use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. So we transform system (1.4) through this change of variables and we propose the Poincaré series

$$H(r,\theta) = \sum_{m=2}^{\infty} H_m(\theta) r^m,$$

where $H_2(\theta) = 1/2$ and $H_m(\theta)$ are homogeneous trigonometric polynomials in θ of degree m. We suppose that the transformed system (1.4) has this power series as a formal first integral, i.e.,

$$\dot H(r,\theta) = \frac{\partial H}{\partial r}\dot r + \frac{\partial H}{\partial \theta}\dot\theta = \sum_{k=2}^{\infty} V_{2k} r^{2k}.$$

Here V_{2k} are the *focal values* which are polynomials in the parameters of system (1.4). The first nonzero focal value is $V_4 = a_3 + b_3$. The next nonzero focal value is

$$\begin{aligned} V_6 = & -195a_2^2a_3 + 30a_5 + 12a_2^3b_2 + 44a_4b_2 - 133a_3b_2^2 \\ & - 12a_2b_2^3 - 205a_2^2b_3 - 123b_2^2b_3 - 44a_2b_4 + 30b_5. \end{aligned}$$

The size of the next focal values increases greatly hence we do not present them explicitly here. The reader can easily compute these next focal values. The Hilbert Basis theorem assures that the ideal $J = \langle V_4, V_6, \dots\rangle$ generated by the focal values is finitely generated. This implies the existence of $v_1, v_2, \dots, v_k$ such that $J = \langle v_1, v_2, \dots, v_k\rangle$. This set of generators is a basis of J and the conditions $v_j = 0$ for $j = 1, \dots, k$ provide a finite set of necessary conditions to have a center for system (1.4). In practice we compute a certain number of focal values thinking that inside this number there is the set of generators. Let J_i be the ideal generated only by the first $i-1$ focal values, i.e., $J_i = \langle V_4, \dots, V_{2i}\rangle$.

Next we decompose this algebraic set into its irreducible components using the computer algebra system Singular [14]. The computational tool used is the routine minAssGTZ [4] which is based on the Gianni-Trager-Zacharias algorithm [6]. Note that if for system (1.4) $a_6 \neq 0$, then by a linear transformation we can take $a_6 = 1$. Using this observation and in order to simplify calculations, we split system (1.4) into two system considering separately the cases:

$$(\alpha) : a_6 = 1, \quad (\beta) : a_6 = 0\,.$$

For the case (α) the decomposition of the ideal J_9 given by $J_9 = \langle V_4, V_6, \dots, V_{18}\rangle$ consist of 3 components defined by the following prime ideals:

(1) $\langle a_3, a_5, b_2, b_3, b_4, b_5, b_6\rangle$,
(2) $\langle a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, 1 + b_6\rangle$,
(3) $\langle a_2 - b_2, a_3 + b_3, a_4 - b_4, a_5 + b_5, 1 - b_6\rangle$,

We were not able to compute the decomposition over the field of rational numbers because of the complexity of the computations. Hence we use modular arithmetics. In fact the decomposition is obtained over the field of characteristic 32003. We have chosen this prime number because the computations are relatively fast using this prime.

As we have used modular arithmetics we must check if the decomposition is complete and no component is lost. To do that we use the algorithm developed in [16]. Let P_i denote the polynomials defining each component. Using the instruction *intersect* of Singular we compute the intersection $P = \cap_i P_i = \langle p_1, \dots, p_m\rangle$. By the

Strong Hilbert Nullstellensatz (see for instance [17]) to check whether $V(J_j) = V(P)$ it is sufficient to check if the radicals of the ideals are the same, that is, if $\sqrt{J_j} = \sqrt{P}$. Computing over characteristic 0 reducing Gröbner bases of ideals $\langle 1 - wV_{2k}, P : V_{2k} \in J_j\rangle$ we find that each of them is $\{1\}$. By the Radical Membership Test this implies that $\sqrt{J_j} \subseteq \sqrt{P}$. To check the opposite inclusion, $\sqrt{P} \subseteq \sqrt{J_j}$ it is sufficient to check that

$$\langle 1 - wp_k, J_j : p_k \text{ for } k = 1, \dots, m\rangle = \langle 1\rangle. \tag{2.1}$$

Using the Radical Membership Test to check if (2.1) is true, we were able to complete computations working in the field of characteristic zero so we know that the decomposition of the center variety is complete.

For the case (β) the obtained decomposition of the ideal J_9 consist of 4 components defined by the following prime ideals:

(1) $\langle a_3, a_5, b_2, b_3, b_4, b_5, b_6\rangle$,
(2) $\langle a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, b_6\rangle$,
(3) $\langle a_2 - b_2, a_3 + b_3, a_4 - b_4, a_5 + b_5, b_6\rangle$,
(4) $\langle a_2, a_3, a_4, a_5, b_3, b_5\rangle$,

This decomposition is also obtained using modular arithmetics so proceeding as in the previous case we can check that this decomposition is complete. In this case this is also true.

The sufficiency is derived from the results presented in the previous section.

3. Proof of Proposition 1.2

To find the maximum number of small-amplitude limit cycles which can bifurcate from the origin we use the method of finding a fine focus of maximum order, see for instance [13]. From our calculations it is easy to see that if $a_2 = b_2 = a_3 + b_3 = a_5 + b_5 = a_6 + b_6 = 0$ then $V_4 = V_6 = V_8 = 0$ and V_{10} takes the form

$$V_{10} = (a_4 + b_4)(379a_3a_4 + 398a_6 - 379a_3b_4).$$

We vanish this focal value taking $a_6 - 379a_3(a_4 - b_4)/398$ and V_{12} becomes

$$V_{12} = (a_4 - b_4)(a_4 + b_4)(445561a_5 - 3104010a_3^2).$$

Taking $a_5 = 3104010a_3^2/445561$ we have $V_{12} = 0$ and V_{14} reads for

$$V_{14} = (a_4 - b_4)(a_4 + b_4)(10770211123227a_3^3 - 775833091250a_4b_4).$$

Now we made the reparametrization $a_3 = z^{1/3}$ and we can vanish V_{14} taking $z = 775833091250a_4b_4/\ 10770211123227$. In this case V_{16} and V_{18} take the form

$$\begin{aligned} V_{16} = {}& (a_4 - b_4)(a_4b_4)^{1/3}(a_4 + b_4)(68732087591790148677a_4^2 \\ & - 298114693011794424032a_4b_4 + 68732087591790148677b_4^2), \\ V_{18} = {}& (a_4 - b_4)(a_4b_4)^{2/3}(a_4 + b_4)(7226530034982884356352004477a_4^2 \\ & + 13348721106142735246693837622a_4b_4 \\ & + 7226530034982884356352004477b_4^2). \end{aligned}$$

We can vanish V_{16} taking one of the two reals roots of the quadratic polynomial and under this assumption V_{18} is different from zero if $a_4b_4 \neq 0$ and $a_4 \neq \pm b_4$, and therefore we obtain a fine focus of order eight for the BiLiénard system (1.4).

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