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POSITIVE SOLUTIONS FOR A NONLOCAL PROBLEM WITH SINGULARITY

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Abstract. In this article we study a nonlocal problem involving singular nonlinearity. Based on the variational and perturbation methods, we obtain the existence of two positive solutions for this problem.

1. Introduction and statement of main result

In recent years, the problem

$$\begin{gathered} -\Big(a+b\int_\Omega |\nabla u|^2dx\Big)\Delta u = h(x,u), \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered}$$

has received considerable attention, we refer to [2]–[6]. In particular, if $h(x,u) = \lambda u^3 + \mu u^{-\gamma}$ $(0<\gamma<1)$, in [10], the existence and multiplicity of solutions for problem have been considered for this case by using the variational method and the Nehari manifold. When $h(x,u) = f(x)u^{-\gamma} - \lambda u^p$, in [9], we have studied the uniqueness of positive solution via the minima method. In addition, in [5], the existence and multiplicity of positive solutions have been obtained in the cases when $h(x,u) = \lambda u^{-\gamma} + u^5$.

In particular, Yin and Liu [17] considered the nonlocal problem

$$\begin{gathered} -\Big(a-b\int_\Omega |\nabla u|^2dx\Big)\Delta u = |u|^{p-2}u, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered}$$

where $2<p<\frac{2N}{N-2}$. By employing the mountain pass lemma, two nontrivial solutions were obtained.

Recently, in [4], we investigate the existence and multiplicity of positive solutions to problem

$$\begin{gathered} -\Big(a-b\int_\Omega |\nabla u|^2dx\Big)\Delta u = f_\lambda(x)|u|^{q-2}u, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered}$$

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where f_λ is possibly sign-changing on $\overline{\Omega}$, $1 < q < 2$. Under the previous assumptions, we obtain two positive solutions via the variational methods.

Based on our previous work [4, 5, 9], we shall give some multiplicity results for the nonlocal problem

$$\begin{gathered} -\Big(a - b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \frac{\lambda}{u^\gamma}, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $a, b > 0$, and λ is positive parameter. Now we state our main result.

Theorem 1.1. *Assume $a, b > 0$, $0 < \gamma < 1$, there exists $\lambda_* > 0$ such that $0 < \lambda < \lambda_*$, then* (1.1) *has at least two positive solutions.*

2. Proof of main theorem

Let $H_0^1(\Omega)$ be the usual Sobolev space equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$, denote by B_r (respectively, ∂B_r) the closed ball (respectively, the sphere) of center zero and radius r, i.e. $B_r = \{u \in H_0^1(\Omega) : \|u\| \le r\}$, $\partial B_r = \{u \in H_0^1(\Omega) : \|u\| = r\}$ and C be various positive constant. Let S be the best Sobolev constant, i.e.,

$$S = \inf\big\{\|u\|^2 : u \in H_0^1(\Omega), \int_\Omega |u|^6 dx = 1\big\}.$$

Consider the energy functional $I_0 : H_0^1(\Omega) \to \mathbb{R}$ given by

$$I_0(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma} dx.$$

It is well known that the singular term leads to the non-differentiability of the functional I_0 on $H_0^1(\Omega)$, therefore problem (1.1) cannot be considered by using critical point theory directly. Now, we consider the perturbed equation

$$\begin{gathered} -\Big(a - b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = \frac{\lambda}{(|u| + \alpha)^\gamma}, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.1}$$

where $\alpha > 0$, the functional associated with (2.1) is

$$I_\alpha = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{1-\gamma}\int_\Omega [(|u| + \alpha)^{1-\gamma} - \alpha^{1-\gamma}] dx.$$

Lemma 2.1. *Assume $a, b > 0$, $0 < \gamma < 1$, then I_α satisfies the $(PS)_c$ condition with $c < \frac{a^2}{4b} - D\lambda$, where $D = \frac{1}{1-\gamma} S^{-\frac{1-\gamma}{2}} |\Omega|^{\frac{5+\gamma}{6}} \left(\frac{a+1}{b}\right)^{\frac{1-\gamma}{2}}$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a nonnegative ($I_\alpha(u_n) = I_\alpha(|u_n|)$) $(PS)_c$ sequence for I_α, i. e.,

$$I_\alpha(u_n) \to c, \quad I'_\alpha(u_n) \to 0, \quad \text{as } n \to \infty. \tag{2.2}$$

It follows from (2.2) that

$$b\|u_n\|^4 = a\|u_n\|^2 - \int_\Omega \frac{u_n}{(u_n + \alpha)^\gamma} dx + o(1) \le a\|u_n\|^2 + o(1),$$

so that

$$\|u_n\|^2 \le \frac{a+1}{b},$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Therefore, there exist a subsequence (still denoted by $\{u_n\}$) and $u_* \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_*$ weakly in $H_0^1(\Omega)$ as $n \to \infty$. It follows easily from the Vitali Convergence Theorem that

$$\lim_{n\to\infty} \int_\Omega \frac{u_n}{(u_n+\alpha)^\gamma} dx = \int_\Omega \frac{u_*}{(u_*+\alpha)^\gamma} dx.$$

Set $w_n = u_n - u_*$, then $\|w_n\| \to 0$. Otherwise, there exists a subsequence (still denoted by w_n) such that $\lim_{n\to\infty} \|w_n\| = l > 0$. From (2.2), letting $n \to \infty$, it holds

$$(a - bl^2 - b\|u_*\|^2) \int_\Omega (\nabla u_*, \nabla \phi) dx - \lambda \int_\Omega \frac{\phi}{(u_*+\alpha)^\gamma} dx = 0, \quad \forall \phi \in H_0^1(\Omega). \tag{2.3}$$

Taking the test function $\phi = u_*$ in (2.3), it follows

$$(a - bl^2 - b\|u_*\|^2)\|u_*\|^2 - \lambda \int_\Omega \frac{u_*}{(u_*+\alpha)^\gamma} dx = 0. \tag{2.4}$$

Note that $\langle I'_\alpha(u_n), u_n \rangle \to 0$ as $n \to \infty$, it holds

$$a\|w_n\|^2 + a\|u_*\|^2 - b\|w_n\|^4 - 2b\|w_n\|^2\|u_*\|^2 - b\|u_*\|^4 - \lambda \int_\Omega \frac{u_*}{(u_*+\alpha)^\gamma} dx = o(1).$$

From this and (2.4), it follows

$$a\|w_n\|^2 - b\|w_n\|^4 - b\|w_n\|^2\|u_*\|^2 = o(1). \tag{2.5}$$

Consequently,

$$l^2 = \frac{a}{b} - \|u_*\|^2, \quad l > 0.$$

Note that the subadditivity of $t^{1-\gamma}$, namely

$$(|v| + \alpha)^{1-\gamma} - \alpha^{1-\gamma} \le |v|^{1-\gamma}. \tag{2.6}$$

On one hand, recall that $\|u_n\|^2 \le \frac{a}{b}$, then using (2.4) and (2.6), it follows

$$\begin{aligned} I_\alpha(u_*) &= \frac{a}{2}\|u_*\|^2 - \frac{b}{4}\|u_*\|^4 - \frac{\lambda}{1-\gamma} \int_\Omega [(u_*+\alpha)^{1-\gamma} - \alpha^{1-\gamma}] dx \\ &\ge \frac{a}{4}\|u_*\|^2 + \frac{b}{4} l^2 \|u_*\|^2 - \frac{\lambda}{1-\gamma} S^{-\frac{1-\gamma}{2}} |\Omega|^{\frac{5+\gamma}{6}} \Big(\frac{a+1}{b}\Big)^{\frac{1-\gamma}{2}} \\ &= \frac{a}{4}\|u_*\|^2 + \frac{b}{4} l^2 \|u_*\|^2 - D\lambda, \end{aligned}$$

where

$$D = \frac{1}{1-\gamma} S^{-\frac{1-\gamma}{2}} |\Omega|^{\frac{5+\gamma}{6}} \Big(\frac{a+1}{b}\Big)^{\frac{1-\gamma}{2}}.$$

On the other hand, from (2.2) and (2.5), it holds

$$\begin{aligned} I_\alpha(u_*) &= I_\alpha(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u_*\|^2 + o(1) \\ &< \frac{a^2}{4b} - D\lambda - \frac{a}{4}\Big(\frac{a}{b} - \|u_*\|^2\Big) + \frac{b}{4} l^2 \|u_*\|^2 \\ &= \frac{a}{4}\|u_*\|^2 + \frac{b}{4} l^2 \|u_*\|^2 - D\lambda. \end{aligned}$$

This is a contradiction. Therefore, $l = 0$, it implies that $u_n \to u_*$ in $H_0^1(\Omega)$. The proof is complete. $\square$

Lemma 2.2. *Assume $a, b > 0$, there exist $\Lambda_0 > 0$ and $\rho > 0$ such that for any $\lambda \in (0, \Lambda_0)$, it holds*
$$I_\alpha\big|_{u\in\partial\overline{B_\rho}} > 0, \quad \inf_{u\in\overline{B_\rho}} I_\alpha(u) < 0.$$

Proof. By Hölder's inequality and (2.6), one has
$$\begin{aligned} I_\alpha(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{1-\gamma}\int_\Omega [(|u|+\alpha)^{1-\gamma} - \alpha^{1-\gamma}]dx \\ &\geq \|u\|^{1-\gamma}\Big(\frac{a}{2}\|u\|^{1+\gamma} - \frac{b}{4}\|u\|^{3+\gamma} - \frac{\lambda}{1-\gamma}|\Omega|^{\frac{5+\gamma}{6}} S^{-\frac{1-\gamma}{2}}\Big), \end{aligned}$$
set $h(t) = \frac{a}{2}t^{1+\gamma} - \frac{b}{4}t^{3+\gamma}$, we see that there exists a constant $\rho = \sqrt{\frac{2a(1+\gamma)}{b(3+\gamma)}}$ such that $\max_{t>0} h(t) = h(\rho) > 0$. Let
$$\Lambda_0 = \frac{(1-\gamma)S^{\frac{1-\gamma}{2}}}{2|\Omega|^{\frac{5+\gamma}{6}}} h(\rho).$$
Consequently, $I_\alpha|_{\|u\|=\rho} \geq \frac{h(\rho)}{2}\rho^{1-\gamma}$ for any $\lambda \in (0, \Lambda_0)$. Moreover, for $u \in H_0^1(\Omega)\backslash\{0\}$ it holds
$$\begin{aligned} \lim_{t\to 0^+} \frac{I_\alpha(tu)}{t} &= -\frac{\lambda}{1-\gamma}\lim_{t\to 0^+}\frac{1}{t}\int_\Omega [(t|u|+\alpha)^{1-\gamma} - \alpha^{1-\gamma}]dx \\ &= -\frac{\lambda}{1-\gamma}\lim_{t\to 0^+}\int_\Omega \frac{(1-\gamma)\xi^{-\gamma}t|u|}{t}dx \quad (\alpha < \xi < t|u|+\alpha) \\ &= -\lambda\int_\Omega \frac{|u|}{\alpha^\gamma}dx \quad (\text{as } t\to 0^+,\ \xi\to\alpha) \\ &< 0. \end{aligned}$$
Thus there exists u small enough such that $I_\alpha(u) < 0$.
$$m = \inf_{u\in\overline{B_\rho}} I_\alpha(u) < 0 < \inf_{u\in\partial\overline{B_\rho}} I_\alpha(u).$$
∎

Lemma 2.3. *Assume $a, b > 0$, $0 < \lambda < \Lambda_0$. Then problem* (2.1) *has a positive solution $u_\alpha \in H_0^1(\Omega)$, enjoying $I_\alpha(u_\alpha) < 0$.*

Proof. By Lemmas 2.1 and 2.2, similarly to the paper [6], we can prove that problem (2.1) has a nonzero nonnegative solution $u_\alpha \in \overline{B_\rho} \subset H_0^1(\Omega)$ such that $I_\alpha(u_\alpha) = m < 0$. Note that $u_\alpha \in \overline{B_\rho}$, it holds
$$\|u_\alpha\|^2 \leq \frac{2a(1+\gamma)}{b(3+\gamma)} < \frac{a}{b},$$
which implies that $a - b\|u_\alpha\|^2 > 0$. Therefore, by using the strong maximum principle, we obtain $u_\alpha > 0$ in Ω. The proof is complete. ∎

Remark 2.4. Assume $(U_{1/n})$ is a positive solution of (2.1), then for every $K \Subset \Omega$, there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that
$$U_{1/n}(x) \geq \delta, \quad \forall x \in K \text{ and } n \geq n_0.$$
Indeed, consider $\Psi_n \in H_0^1(\Omega)$ a weak solution of the problem
$$-\Delta\Psi_n = \frac{\lambda}{a(|\Psi_n|+1)^\gamma}, \quad \text{in } \Omega,$$

$$\Psi_n = 0, \quad \text{on } \partial\Omega.$$

It is easy to prove that (Ψ_n) is a bounded sequence in $H_0^1(\Omega)$, thus there is $\Psi \in H_0^1(\Omega)$ such that for some subsequence, still denoted with the same symbol,

$$\begin{gathered}\Psi_n \rightharpoonup \Psi \quad \text{in } H_0^1(\Omega), \\ \Psi_n(x) \to \Psi(x) \quad \text{a.e. in } \Omega.\end{gathered}$$

Setting

$$h_n(x) = \frac{\lambda}{a(|\Psi_n(x)| + 1)^\gamma},$$

we see that (h_n) is bounded in $L^\infty(\Omega)$, and so, it is bounded in $L^2(\Omega)$. Then, for some subsequence, we also have

$$\begin{gathered}h_n(x) \to h(x) = \frac{\lambda}{a(|\Psi(x)| + 1)^\gamma} \quad \text{a.e. in } \Omega, \\ h_n \rightharpoonup h \quad \text{in } L^2(\Omega).\end{gathered}$$

The above information yield

$$\begin{gathered}-\Delta\Psi = \frac{\lambda}{a(|\Psi| + 1)^\gamma}, \quad \text{in } \Omega, \\ \Psi = 0, \quad \text{on } \partial\Omega,\end{gathered}$$

from where it follows that $\Psi \in C(\overline{\Omega})$ and $\Psi(x) > 0$ for all $x \in \Omega$. Moreover, the elliptic regularity gives

$$\Psi_n \to \Psi \quad \text{in } C(\overline{\Omega}).$$

Thereby, fixed a compact set $K \subset \Omega$, there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\Psi_n(x) \geq \delta, \quad \forall x \in K \text{ and } n \geq n_0.$$

On the other hand, let $U_{1/n}$ be a positive solution of (2.1), we know that

$$\begin{gathered}-\Delta U_{1/n} \geq= -\Delta\Psi_n, \quad \text{in } \Omega, \\ U_{1/n} = \Psi_n = 0, \quad \text{on } \partial\Omega,\end{gathered}$$

and so, by maximum principle,

$$U_{1/n}(x) \geq \Psi_n(x), \quad \forall x \in \Omega \text{ and all } n \in \mathbb{N}.$$

As a byproduct of above arguments, for each compact set $K \subset \Omega$, there are $n_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$U_{1/n}(x) \geq \delta, \quad \forall x \in K\text{and all } n \geq n_0.$$

Now, we show that the functional I_α satisfies the mountain-pass lemma.

Lemma 2.5. *The functional I_α satisfies the following conditions for any $\lambda \in (0, \Lambda_0)$*
- (i) *$I_\alpha(u) > 0$ if $\|u\| = \rho$;*
- (ii) *There exists $\zeta \in H_0^1(\Omega)$ such that $I_\alpha(\zeta) < 0$.*

Proof. Conclusion (i) follows from Lemma 2.2. To prove (ii), let $u \in H_0^1(\Omega)\backslash\{0\}$ and $t > 0$, it follows that

$$\begin{aligned}I_\alpha(tu) &\leq \frac{at^2}{2}\|u\|^2 - \frac{bt^4}{4}\|u\|^4 - \frac{\lambda t^{1-\gamma}}{1-\gamma}\int_\Omega [(|u| + \alpha)^{1-\gamma} - \alpha^{1-\gamma}]dx \\ &\to -\infty\end{aligned}$$

as $t \to +\infty$. Therefore we can easily find $\zeta \in H_0^1(\Omega)$ with $\|\zeta\| > \rho$, such that $I_\alpha(\zeta) < 0$. The proof is complete. □

Now, it is well known that the function

$$w_\varepsilon(x) = \frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2 + |x|^2)^{1/2}}, \quad x \in \mathbb{R}^3,\ \varepsilon > 0$$

satisfies

$$-\Delta w_\varepsilon = w_\varepsilon^5 \quad]textin\mathbb{R}^3.$$

Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \eta \le 1$, $|\nabla\eta| \le C$ and $\eta(x) = 1$ for $|x| < R$ and $\eta(x) = 0$ for $|x| > 2R$, we set $u_\varepsilon(x) = \eta(x) w_\varepsilon(x)$. Then

$$\|u_\varepsilon\|^2 = S^{\frac{3}{2}} + O(\varepsilon), \quad |u_\varepsilon|_6^6 = S^{\frac{3}{2}} + O(\varepsilon^3).$$

Lemma 2.6. *Assume $a, b > 0$ and $0 < \gamma < 1$. Then*

$$\sup_{t \ge 0} I_\alpha(u_\alpha + t u_\varepsilon) < \frac{a^2}{4b} - D\lambda.$$

Proof. As u_α is a positive solution of (2.1), for each $\varphi \in H_0^1(\Omega)$, it holds

$$(a - b\|u_\alpha\|^2) \int_\Omega (\nabla u_\alpha, \nabla \varphi) dx = \lambda \int_\Omega \frac{\varphi}{(u_\alpha + \alpha)^\gamma} dx.$$

In particular, it holds

$$(a - b\|u_\alpha\|^2) \int_\Omega (\nabla u_\alpha, \nabla u_\varepsilon) dx = \lambda \int_\Omega \frac{u_\varepsilon}{(u_\alpha + \alpha)^\gamma} dx.$$

Recalling that $a - b\|u_\alpha\|^2 > 0$, we have

$$\int_\Omega (\nabla u_\alpha, \nabla u_\varepsilon) dx \ge 0.$$

As $I_\alpha(u_\alpha) < 0$, by Remark 2.4, we have

$$\begin{aligned}
I_\alpha(u_\alpha + t u_\varepsilon) &= \frac{a}{2}\|u_\alpha\|^2 + at \int_\Omega (\nabla u_\alpha, \nabla u_\varepsilon) dx + \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{b}{4}\|u_\alpha\|^4 \\
&\quad - \frac{bt^4}{4}\|u_\varepsilon\|^4 - bt\|u_\alpha\|^2 \int_\Omega (\nabla u_\alpha, \nabla u_\varepsilon) dx - \frac{bt^2}{2}\|u_\alpha\|^2 \|u_\varepsilon\|^2 \\
&\quad - bt^2 \Big(\int_\Omega (\nabla u_\alpha, \nabla u_\varepsilon) dx \Big)^2 - bt^3 \|u_\varepsilon\|^2 \int_\Omega (\nabla u_\alpha, \nabla u_\varepsilon) dx \\
&\quad - \frac{\lambda}{1-\gamma} \int_\Omega [(u_\alpha + t u_\varepsilon + \alpha)^{1-\gamma} - \alpha^{1-\gamma}] dx \\
&\le I_\alpha(u_\alpha) + \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\alpha\|^2 \|u_\varepsilon\|^2 \\
&\quad - \frac{\lambda}{1-\gamma} \int_\Omega [(u_\alpha + t u_\varepsilon + \alpha)^{1-\gamma} - (u_\alpha + \alpha)^{1-\gamma}] dx \\
&\quad + \lambda t \int_\Omega \frac{u_\varepsilon}{(u_\alpha + \alpha)^\gamma} dx \\
&\le \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\alpha\|^2 \|u_\varepsilon\|^2 + \delta\lambda t \int_\Omega u_\varepsilon dx.
\end{aligned}$$

Set

$$g(t)=\frac{at^2}{2}\|u_\varepsilon\|^2-\frac{bt^4}{4}\|u_\varepsilon\|^4-\frac{bt^2}{2}\|u_\alpha\|^2\|u_\varepsilon\|^2+\delta\lambda t\int_\Omega u_\varepsilon dx.$$

It is similar to the paper [6] that there exist $t_\varepsilon>0$ and positive constants t_1,t_2 independent of ε,λ, such that $\sup_{t\geq 0}g(t)=g(t_\varepsilon)$ and $0<t_1\leq t_\varepsilon\leq t_2<\infty$.

Note that $\int_\Omega u_\varepsilon dx\leq O(\varepsilon^{1/2})$, by Remark 2.4, there exists positive constant $c>0$ (independent of λ) such that $\|u_\alpha\|^2\geq c$. Then, it holds

$$\begin{aligned}
\sup_{t\geq 0}I_\alpha(u_\alpha+tu_\varepsilon)&\leq\sup_{t\geq 0}g(t)\\
&\leq\sup_{t\geq 0}\Big\{\frac{at^2}{2}\|u_\varepsilon\|^2-\frac{bt^4}{4}\|u_\varepsilon\|^4\Big\}-c\|u_\varepsilon\|^2+\lambda O(\varepsilon^{1/2})\\
&\leq\frac{a^2}{4b}+c_1\varepsilon^{1/2}-c_2S^{\frac{3}{2}},\quad(0<\lambda<1)
\end{aligned}$$

where $c_1,c_2>0$. Let $\varepsilon=\lambda^2$, when $0<\lambda<\Lambda_1\triangleq\frac{c_2S^{\frac{3}{2}}}{c_1+D}$, it holds

$$c_1\lambda-c_2S^{\frac{3}{2}}<c_1\lambda-(c_1+D)\lambda=-D\lambda.$$

Consequently, $\sup_{t\geq 0}I_\alpha(u_\alpha+tu_\varepsilon)<\frac{a^2}{4b}-D\lambda$. The proof is complete. □

Lemma 2.7. *Assume $a,b>0$ and $\lambda>0$ is sufficiently small, problem* (2.1) *admits a solution v_α with $I_\alpha(v_\alpha)>0$.*

Proof. Set $\lambda^*=\min\{\Lambda_0,\Lambda_1,\frac{a^2}{4bD},1\}$. Then applying the mountain-pass lemma [3], there exists a sequence $\{v_n\}\subset H_0^1(\Omega)$, such that

$$I_\alpha(v_n)\to c>\frac{h(\rho)}{2}\rho^{1-\gamma},\quad\text{and}\quad I'_\alpha(v_n)\to 0,\tag{2.7}$$

where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\alpha(\gamma(t)),$$

$$\Gamma=\big\{\gamma\in C([0,1],H_0^1(\Omega)):\gamma(0)=u_\alpha,\gamma(1)=\zeta\big\}.$$

By Lemmas 2.1 and 2.6, $\{v_n\}\subset H_0^1(\Omega)$ has a convergent subsequence, say $\{v_n\}$, we may assume that $v_n\to v_\alpha$ in $H_0^1(\Omega)$ as $n\to\infty$. Hence, from (2.7), it holds

$$I_\alpha(v_\alpha)=\lim_{n\to\infty}I_\alpha(v_n)=c>\frac{h(\rho)}{2}\rho^{1-\gamma}>0,$$

this implies that $v_\alpha\not\equiv 0$. Furthermore, from the continuity of I'_α, we obtain that v_α is a nonzero nonnegative solution of (2.1). The proof is complete. □

Proof of Theorem 1.1. Let $(U_{1/n})$ be a solution of (2.1), then we can prove that $(U_{1/n})$ is bounded in $H_0^1(\Omega)$, then up to a subsequence, there exists $u\in H_0^1(\Omega)$ such that

$$U_{1/n}\rightharpoonup u\text{ weakly in }H_0^1(\Omega),\quad U_{1/n}(x)\to u(x)\text{ a.e. in }\Omega\text{ as }n\to\infty.$$

By Remark 2.4, and similar to [5], for each $\phi\in H_0^1(\Omega)$, it holds

$$(a-b\lim_{n\to\infty}\|U_{1/n}\|^2)\int_\Omega(\nabla u,\nabla\phi)dx-\lambda\int_\Omega\frac{\phi}{u^\gamma}dx=0.\tag{2.8}$$

If $U_{1/n}=u_\alpha$, by Lemma 2.1, Lemma 2.3 and (2.8), we conclude that $U_{1/n}\to u$ in $H_0^1(\Omega)$, and u is a positive solution of (1.1) with $I_0(u)=\lim_{n\to\infty}I_{1/n}(U_{1/n})<0$.

If $U_{1/n} = v_a$, combining Lemma 2.1, Lemma 2.6 and (2.8), we also deduce that $U_{1/n} \to u$ in $H_0^1(\Omega)$, and u is a positive solution of (1.1) with $I_0(u) = \lim_{n\to\infty} I_{1/n}(U_{1/n}) > 0$. Therefore problem (1.1) has at least two different positive solutions. The proof is complete. $\square$

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