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MULTIPLE SOLUTIONS OF A p-KIRCHHOFF EQUATION WITH SINGULAR AND CRITICAL NONLINEARITIES

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Abstract. In this article, we explore the existence of multiple solutions for a p-Kirchhoff equation with the nonlinearity containing both singular and critical terms. By means of the concentration compactness principle and Ekeland's variational principle, we obtain two positive weak solutions.

1. Introduction

Consider the p-Kirchhoff equation

$$\begin{gathered} -M(\|u\|^p)\Delta_p u = \lambda u^{p^*-1} + \rho(x)u^{-\gamma}, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $M(s) = a + bs^m$, $\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < N$, and $\lambda > 0$ is a real parameter. Here, $\gamma \in (0, 1)$ is a constant, $\rho(x) : \Omega \to \mathbb{R}$ is a given non-negative function in $L^p(\Omega)$, and $p^* = Np/(N - p)$ is the critical Sobolev exponent.

Problem (1.1) displays some meaningful features. It is nonlocal due to the presence of the Kirchhoff-type coefficient M which makes the equation no longer a pointwise identity. Moreover, it involves singular and critical terms. To the best of our knowledge, not much has been known on the Kirchhoff nonlocal structure with the presence of singular and critical nonlinearities in quasilinear elliptic problems.

Recently, considerable attention has been given to the existence of positive solutions by variational methods for the problem [3, 2, 6, 1]:

$$\begin{gathered} -M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x, u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.2}$$

and the stationary analogue of the Kirchhoff equation [4]:

$$u_{tt} - M\Big(\int_\Omega |\nabla_x u|^2 dx\Big)\Delta_x u = f(x, t), \tag{1.3}$$

where $M(s) = a + bs$, $a > 0$ and $b > 0$. Equation (1.3) was proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Sun et al [8, 9] considered the existence of solutions to a related

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singular elliptic problem. By using the concentration compactness principle [5] and Ekeland's variational principle [7], the existence of two positive weak solutions was presented when the parameter λ is small enough.

Since problem (1.1) contains a critical term, it becomes difficult for us to apply variational methods directly and does not have the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. It is also noted that the singular term leads to the non-differentiability of the associated functional I_λ on $W_0^{1,p}(\Omega)$, so the critical point theory becomes invalid. Based on this fact, in this study we attempt to use the concentration compactness principle, Vitali's theorem as well as Ekeland's variational principle to explore the existence of multiple solutions of (1.1).

Following the traditional notation, we let $X = W_0^{1,p}(\Omega)$ be the standard Sobolev space endowed with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}$$

and $\|u\|_\sigma$ denotes the norm in $L^\sigma(\Omega)$ by

$$\|u\|_\sigma = \Big(\int_\Omega |u|^\sigma dx\Big)^{1/\sigma}.$$

Let S be the best Sobolev constant as

$$S = \inf\Big\{\frac{\|u\|^p}{\|u\|_{p^*}^p},\ u \in X \text{ and } u \neq 0\Big\}. \tag{1.4}$$

Then, the infimum is never achieved if $\Omega \neq \mathbb{R}^N$.

For $u \in X$, we define I_λ: $X \to \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{\lambda}{p^*}\int_\Omega |u|^{p^*} dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u|^{1-\gamma} dx,$$

where $\widehat{M}(s) = \int_0^s M(t)dt = as + \frac{b}{m+1}s^{m+1}$. By analyzing the associated minimization problems for the functional I_λ, one can study weak solutions for (1.1).

Note that if u is a weak solution of (1.1), then u satisfies

$$M(\|u\|^p)\int_\Omega |\nabla u|^p dx - \lambda\int_\Omega |u|^{p^*} dx - \int_\Omega \rho(x)|u|^{1-\gamma} dx = 0.$$

So we define a set

$$\Lambda = \Big\{u \in X | M(\|u\|^p)\int_\Omega |\nabla u|^p dx - \lambda\int_\Omega |u|^{p^*} dx - \int_\Omega \rho(x)|u|^{1-\gamma} dx = 0\Big\}.$$

We consider

$$h_u(t) = \frac{1}{p}\widehat{M}(t^p\|u\|^p) - \frac{\lambda t^{p^*}}{p^*}\int_\Omega |u|^{p^*} dx - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega \rho(x)|u|^{1-\gamma} dx.$$

A straightforward calculation gives

$$h_u'(t) = M(t^p\|u\|^p)t^{p-1}||u||^p - \lambda t^{p^*-1}\int_\Omega |u|^{p^*} dx - t^{-\gamma}\int_\Omega \rho(x)|u|^{1-\gamma} dx$$

and

$$\begin{aligned} h_u''(t) &= a(p-1)t^{p-2}\|u\|^p + b[p(m+1)-1]t^{p(m+1)-2}\|u\|^{p(m+1)} \\ &\quad - \lambda(p^*-1)t^{p^*-2}\int_\Omega |u|^{p^*} dx + \gamma t^{-\gamma-1}\int_\Omega \rho(x)|u|^{1-\gamma} dx. \end{aligned}$$

So we have

$$h_u''(1) = a(p+\gamma-1)\|u\|^p + b[p(m+1)+\gamma-1]\|u\|^{p(m+1)} - \lambda(p^*+\gamma-1)\|u\|_{p^*}^{p^*}.$$

It is natural to split Λ into three parts corresponding to the local minima, the local maxima and the point of inflection. Accordingly, we define

$$\begin{aligned}\Lambda^+ &= \{u \in \Lambda | h''(u)(1) > 0\},\\ \Lambda^0 &= \{u \in \Lambda | h''(u)(1) = 0\},\\ \Lambda^- &= \{u \in \Lambda | h''(u)(1) < 0\}.\end{aligned}$$

Throughout this paper, we make the following assumptions:

- (A1) $M(s) = a + bs^m$, where a, b, $m > 0$;
- (A2) $1 < p < p(m+1) < p^*$ and $0 < \gamma < 1$;
- (A3) $\rho : \Omega \to \mathbb{R}$ is a given non-negative and nontrivial function in $L^p(\Omega)$, and there exists some $\Theta > 0$ such that if $\|\rho\|_p \le \Theta$, then $\Lambda^- \ne \emptyset$.

We summarize our main results as follows.

Theorem 1.1. *Assume that conditions (A1)–(A3) hold. Then there exists $\lambda^* > 0$ small enough such that for any $\lambda \in (0, \lambda^*)$, there exist at least two weak positive solutions u^1, $u^2 \in X$ to problem (1.1). Moreover, u^1 is a local minimizer of I_λ in X with $I_\lambda(u^1) < 0$, and u^2 is a minimizer of I_λ on Λ^- with $I_\lambda(u^2) \ge 0$.*

The remainder of this article is organized as follows. In Section 2, we present some preliminary results, and Section 3 is dedicated to the proof of main results.

2. Preliminaries

Lemma 2.1. *The energy functional I_λ has a local minimum c in X with $c < 0$.*

Proof. By Hölder's and Sobolev inequalities, there exist positive constants C_0 and C_1 such that

$$\begin{aligned}\int_\Omega \rho(x)|u|^{1-\gamma}dx &\le \|\rho\|_p \|u\|_{p^*}^{1-\gamma} |\Omega|^{\frac{(p-1)p^*-p(1-\gamma)}{pp^*}}\\ &\le C_0\|\rho\|_p\|u\|_{p^*}^{1-\gamma}\\ &\le C_1\|\rho\|_p\|u\|^{1-\gamma}.\end{aligned}$$

From (1.4), we have

$$\int_\Omega |u|^{p^*}dx \le S^{-\frac{p^*}{p}}\Big(\int_\Omega |\nabla u|^p dx\Big)^{p^*/p}, \quad u \in X. \tag{2.1}$$

Thus, it gives

$$\begin{aligned}I_\lambda(u) &\ge \frac{a}{p}\|u\|^p + \frac{b}{p(m+1)}\|u\|^{p(m+1)} - \frac{\lambda}{p^*}S^{-\frac{p^*}{p}}\|u\|^{p^*} - C_2\|\rho\|_p\|u\|^{1-\gamma}\\ &\ge \frac{2}{p}\sqrt{\frac{ab}{m+1}}\|u\|^{\frac{p(m+2)}{2}} - \frac{\lambda}{p^*}S^{-\frac{p^*}{p}}\|u\|^{p^*} - C_2\|\rho\|_p\|u\|^{1-\gamma}.\end{aligned}$$

Since $1-\gamma < \frac{p(m+2)}{2} < p^*$, there exists $\lambda_1 > 0$ such that for any $\lambda \in (0, \lambda_1)$, there are R, $\xi > 0$ satisfying $I_\lambda(u) \ge \xi$ for all $u \in X$ with $\|u\| = R$ and $I_\lambda(u)$ is bounded from below on $B_R = \{u \in X | \|u\| \le R\}$. Then, $c = \inf_{u\in B_R} I_\lambda(u)$ is well-defined for the fixed $\lambda \in (0, \lambda_1)$. Since $0 < 1-\gamma < 1$, we have $I_\lambda(t\sigma) < 0$ for all $\sigma \ne 0$ and small $t > 0$. Thus, we arrive at $c = \inf_{u\in B_R} I_\lambda(u) < 0$. $\square$

Lemma 2.2. *There exists $u^1 \in B_R$ satisfying $I_\lambda(u^1) = c$.*

Proof. From Lemma 2.1, there exists a minimizing sequence $\{u_k\} \subset B_R$ such that $I_\lambda(u_k) \to c < 0$. Since $I_\lambda(u_k) = I_\lambda(|u_k|)$, we can assume $u_k \geq 0$. Due to $||u|| \leq R$, there exists a subsequence (still denoted by $\{u_k\}$) satisfying

$$u_k \rightharpoonup u^1 \quad \text{in } X.$$

From (2.1) we know that u_k is bounded in $L^{p^*}(\Omega)$. Since X is self-reflexive, and B_R is closed and convex, we see $u^1 \in B_R$.

By the concentration-compactness principle [5], there exist non-negative bounded measures η and μ such that

$$|u_k|^{p^*} \rightharpoonup \eta \quad \text{and} \quad |\nabla u_k|^p \rightharpoonup \mu$$

weakly in the sense of measures. Furthermore, there exists a countable index set J, a collection of points $\{x_j\}_{j\in J} \subset \overline{\Omega}$ and two numbers μ_j, $\eta_j > 0$ such that

$$\eta = |u^1|^{p^*} + \sum_{j\in J} \eta_j \delta_{x_j} \quad \text{and} \quad \mu \geq |\nabla u^1|^p + \sum_{j\in J} \mu_j \delta_{x_j},$$

where δ_{x_j} is the Dirac measure concentrated at x_j, and η_j and μ_j satisfy

$$S\eta_j^{p/p^*} \leq \mu_j.$$

Letting $k \to \infty$ leads to

$$\int_\Omega |\nabla u_k|^p dx \to \int_\Omega d\mu \geq \int_\Omega |\nabla u^1|^p dx + \sum_{j\in J} S\eta_j^{p/p^*},$$

and

$$\int_\Omega |u_k|^{p^*} dx \to \int_\Omega d\eta = \int_\Omega |u^1|^{p^*} dx + \sum_{j\in J} \eta_j. \tag{2.2}$$

By Vitali's theorem, we find

$$\lim_{k\to\infty} \int_\Omega \rho(x)|u_k|^{1-\gamma} dx = \int_\Omega \rho(x)|u^1|^{1-\gamma} dx.$$

Then, we deduce that

$$\begin{aligned}
c &= \lim_{k\to\infty} \Big\{ \frac{1}{p}\widehat{M}(\|u_k\|^p) - \frac{\lambda}{p^*}\int_\Omega |u_k|^{p^*} dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u_k|^{1-\gamma} dx \Big\} \\
&= \lim_{k\to\infty} \Big\{ \frac{a}{p}\|u_k\|^p + \frac{b}{p(m+1)}\|u_k\|^{p(m+1)} - \frac{\lambda}{p^*}\int_\Omega |u_k|^{p^*} dx \\
&\quad - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u_k|^{1-\gamma} dx \Big\} \\
&\geq \frac{a}{p}\Big(\int_\Omega |\nabla u^1|^p dx + \sum_{j\in J} S\eta_j^{p/p^*} \Big) + \frac{b}{p(m+1)}\Big(\int_\Omega |\nabla u^1|^p dx + \sum_{j\in J} S\eta_j^{p/p^*} \Big)^{m+1} \\
&\quad - \frac{\lambda}{p^*}\Big(\int_\Omega |u^1|^{p^*} dx + \sum_{j\in J} \eta_j \Big) - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u^1|^{1-\gamma} dx \\
&\geq \frac{1}{p}\widehat{M}(\|u^1\|^p) - \frac{\lambda}{p^*}\int_\Omega |u^1|^{p^*} dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u^1|^{1-\gamma} dx \\
&\quad + \frac{a}{p}\sum_{j\in J} S\eta_j^{p/p^*} + \frac{b}{p(m+1)}\Big(\sum_{j\in J} S\eta_j^{p/p^*} \Big)^{m+1} - \frac{\lambda}{p^*}\sum_{j\in J} \eta_j.
\end{aligned}$$

That is,

$$c \geq \frac{1}{p}\widehat{M}(\|u^1\|^p) - \frac{\lambda}{p^*}\int_\Omega |u^1|^{p^*}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u^1|^{1-\gamma}dx.$$

From the definition of c, it gives

$$c \leq \frac{1}{p}\widehat{M}(\|u^1\|^p) - \frac{\lambda}{p^*}\int_\Omega |u^1|^{p^*}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u^1|^{1-\gamma}dx.$$

Thus, we have

$$c = \frac{1}{p}\widehat{M}(\|u^1\|^p) - \frac{\lambda}{p^*}\int_\Omega |u^1|^{p^*}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u^1|^{1-\gamma}dx.$$

Suppose that $J \neq \emptyset$. By way of contradiction, from (2.2) we obtain

$$\int_\Omega d\eta > \int_\Omega |u^1|^{p^*}dx,$$
$$\sum_{j\in J}\eta_j = \int_\Omega d\eta - \int_\Omega |u^1|^{p^*}dx > 0.$$

On the other hand, one can find that

$$\begin{aligned} c &\leq \frac{1}{p}\widehat{M}(\|u^1\|^p) - \frac{\lambda}{p^*}\int_\Omega |u^1|^{p^*}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u^1|^{1-\gamma}dx \\ &\leq c - \frac{a}{p}\sum_{j\in J} S\eta_j^{p/p^*} - \frac{b}{p(m+1)}\Big(\sum_{j\in J} S\eta_j^{p/p^*}\Big)^{m+1} + \frac{\lambda}{p^*}\sum_{j\in J}\eta_j \\ &\leq c - \frac{a}{p}\sum_{j\in J} S\eta_j^{p/p^*} + \frac{\lambda}{p^*}\sum_{j\in J}\eta_j. \end{aligned}$$

If for all $j \in J$ and $0 < \eta_j < 1$, we have $\eta_j^{p/p^*} > \eta_j$ and

$$c \leq c - \frac{a}{p}\sum_{j\in J} S\eta_j^{p/p^*} + \frac{\lambda}{p^*}\sum_{j\in J}\eta_j \leq c + \Big(\frac{\lambda}{p^*} - \frac{aS}{p}\Big)\sum_{j\in J}\eta_j.$$

This yields a contradiction when $\lambda < ap^*S/p$.

If there exists a subsequence $\{\eta_j\}$ ($j \in K = \{1, 2, \dots\}$) such that $\eta_j \geq 1$, where K is a finite set, we choose $\lambda < \widetilde{\lambda_2}$, and let $\widetilde{\lambda_2}$ satisfy

$$\Big(\frac{\widetilde{\lambda_2}}{p^*} - \frac{aS}{p}\Big)\sum_{j\in J}\eta_j + \frac{\widetilde{\lambda_2}}{p^*}\sum_{\eta_j\geq 1}\eta_j - \frac{a}{p}\sum_{\eta_j\geq 1} S\eta_j^{p/p^*} < 0.$$

Then, we see that

$$\begin{aligned} c &\leq c - \frac{a}{p}\sum_{j\in J} S\eta_j^{p/p^*} + \frac{\lambda}{p^*}\sum_{j\in J}\eta_j \\ &\leq c + \Big(\frac{\lambda}{p^*} - \frac{aS}{p}\Big)\sum_{j\in J}\eta_j + \frac{\lambda}{p^*}\sum_{\eta_j\geq 1}\eta_j - \frac{a}{p}\sum_{\eta_j\geq 1} S\eta_j^{p/p^*} < c. \end{aligned}$$

This leads to another contradiction. Consequently, $J = \emptyset$ by choosing $\lambda < \lambda_2 = \min\{\frac{ap^*S}{p}, \widetilde{\lambda_2}\}$. □

Lemma 2.3. *The functional I_λ is coercive on Λ.*

Proof. For any $u \in \Lambda$, it holds
$$M(\|u\|^p)\|u\|^p - \lambda \int_\Omega |u|^{p^*} dx - \int_\Omega \rho(x)|u|^{1-\gamma} dx = 0.$$
Then, we have
$$\begin{aligned} I_\lambda(u) &= \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{\lambda}{p^*}\int_\Omega |u|^{p^*} dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|u|^{1-\gamma} dx \\ &= \frac{1}{p}\widehat{M}(\|u\|^p) - \frac{1}{p^*}M(\|u\|^p)||u||^p + \Big(\frac{1}{p^*} - \frac{1}{1-\gamma}\Big)\int_\Omega \rho(x)|u|^{1-\gamma} dx \\ &\geq a\Big(\frac{1}{p} - \frac{1}{p^*}\Big)\|u\|^p + b\big[\frac{1}{p(m+1)} - \frac{1}{p^*}\big]\|u\|^{p(m+1)} \\ &\quad + \Big(\frac{1}{p^*} - \frac{1}{1-\gamma}\Big)C_1||\rho||_p\|u\|^{1-\gamma}. \end{aligned}$$
Since $0 < 1-\gamma < 1 < p < p(m+1) < p^*$, we deduce that $\lim_{||u||\to\infty} I_\lambda(u) = +\infty$. $\square$

Lemma 2.4. *There exists $\lambda_3 > 0$, such that $\Lambda^0 = \{0\}$ for all $\lambda \in (0,\lambda_3)$.*

Proof. By contradiction, we suppose that there exists some $u \in \Lambda^0\backslash\{0\}$ satisfying
$$a(p+\gamma-1)\|u\|^p + b[p(m+1)+\gamma-1]\|u\|^{p(m+1)} = \lambda(p^*+\gamma-1)\|u\|_{p^*}^{p^*} \tag{2.3}$$
and
$$a\frac{p^*-p}{p^*+\gamma-1}\|u\|^p + b\frac{p^*-p(m+1)}{p^*+\gamma-1}\|u\|^{p(m+1)} = \int_\Omega \rho(x)|u|^{1-\gamma} dx. \tag{2.4}$$
By (2.1) and (2.3), it follows Young's inequality that
$$2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}\|u\|^{\frac{p(m+2)}{2}} \leq \lambda(p^*+\gamma-1)S^{-\frac{p^*}{p}}\|u\|^{p^*}.$$
Since $\frac{p(m+2)}{2} < p^*$, it follows that
$$\|u\| \geq \Big\{\frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}S^{p^*/p}\Big\}^{\frac{2}{2p^*-p(m+2)}}.$$
By (2.1) and (2.4), and using Young's inequality again, we obtain
$$\frac{2\sqrt{ab(p^*-p)[p^*-p(m+1)]}}{p^*+\gamma-1}\|u\|^{\frac{p(m+2)}{2}} \leq C_1\|\rho\|_p\|u\|^{1-\gamma}.$$
When $\frac{p(m+2)}{2} > 1 > 1-\gamma > 0$, we see that
$$\|u\| \leq \Big\{\frac{C_1(p^*+\gamma-1)\|\rho\|_p}{2\sqrt{ab(p^*-p)[p^*-p(m+1)]}}\Big\}^{\frac{2}{p(m+2)-2(1-\gamma)}}.$$
This yields a contradiction if we choose
$$\begin{aligned} \lambda < \lambda_3 = &\frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}S^{p^*/p}}{p^*+\gamma-1} \\ &\times \Big\{\frac{2\sqrt{ab(p^*-p)[p^*-p(m+1)]}}{C_1(p^*+\gamma-1)\|\rho\|_p}\Big\}^{\frac{2p^*-p(m+2)}{p(m+2)-2(1-\gamma)}}. \end{aligned}$$
Consequently, for all $\lambda \in (0,\lambda_3)$, it holds $\Lambda^0 = \{0\}$. $\square$

Lemma 2.5. Λ^- *is closed in X.*

Proof. Let $\{u_n\} \subset \Lambda^-$ satisfy $u_n \to u$ in X. There exists a subsequence (still denoted by $\{u_n\}$) such that $u_n \to u$ a.e. in Ω, and $\lim_{n\to\infty} \|u_n\|_{p^*} = \|u\|_{p^*}$. By the definition of Λ^-, it gives

$$a(p+\gamma-1)\|u_n\|^p + b[p(m+1)+\gamma-1]\|u_n\|^{p(m+1)} - \lambda(p^*+\gamma-1)\|u_n\|_{p^*}^{p^*} < 0.$$

So we have

$$\begin{aligned}&\lim_{n\to\infty}\Big\{a(p+\gamma-1)\int_\Omega |\nabla u_n|^p dx + b[p(m+1)+\gamma-1]\Big(\int_\Omega |\nabla u_n|^p dx\Big)^{m+1}\\ &- \lambda(p^*+\gamma-1)\int_\Omega |u_n|^{p^*}dx\Big\} \le 0.\end{aligned}$$

Clearly, we see that $u \in \Lambda^0 \cup \Lambda^-$. If Λ^- is not closed, then $u \in \Lambda^0$. By Lemma 2.4, we obtain $u \equiv 0$.

On the other hand, for any $\{u_n\} \subset \Lambda^-$ we have

$$\begin{aligned}&\int_\Omega |u_n|^{p^*}dx\\ &> \frac{a(p+\gamma-1)}{\lambda(p^*+\gamma-1)}\int_\Omega |\nabla u_n|^p dx + \frac{b[p(m+1)+\gamma-1]}{\lambda(p^*+\gamma-1)}\Big(\int_\Omega |\nabla u_n|^p dx\Big)^{m+1}\\ &\ge \frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}\Big(\int_\Omega |\nabla u_n|^p dx\Big)^{\frac{m+2}{2}}\\ &\ge \frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}\Big[S\Big(\int_\Omega |u_n|^{p^*}dx\Big)^{p/p^*}\Big]^{\frac{m+2}{2}}\\ &= \frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}S^{\frac{m+2}{2}}\Big(\int_\Omega |u_n|^{p^*}dx\Big)^{\frac{p(m+2)}{2p^*}}.\end{aligned}$$

That is,

$$\Big(\int_\Omega |u_n|^{p^*}dx\Big)^{1/p^*} > \Big\{\frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}S^{\frac{m+2}{2}}\Big\}^{\frac{2}{2p^*-p(m+2)}}.$$

As $n \to \infty$, one can see that

$$\|u\|_{p^*} \ge \Big\{\frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}\Big\}^{\frac{2}{2p^*-p(m+2)}} S^{\frac{m+2}{2p^*-p(m+2)}} > 0. \quad (2.5)$$

This yields a contradiction to the fact $u = 0$. Thus, $u \in \Lambda^-$ and Λ^- is closed in X. □

Lemma 2.6. *There exists $\lambda_4 > 0$ such that for any $u \in \Lambda^-$ and any $\lambda \in (0, \lambda_4)$, $I_\lambda(u) \ge 0$ holds.*

Proof. By contradiction, we suppose that there exists $\widetilde{u} \in \Lambda^-$ satisfying $I_\lambda(\widetilde{u}) < 0$. That is,

$$\frac{1}{p}\widehat{M}(\|\widetilde{u}\|^p) - \frac{\lambda}{p^*}\int_\Omega |\widetilde{u}|^{p^*}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|\widetilde{u}|^{1-\gamma}dx < 0.$$

Note that

$$\frac{1}{p}\widehat{M}(\|\widetilde{u}\|^p) > \frac{1}{p(m+1)}[a\|\widetilde{u}\|^p + b\|\widetilde{u}\|^{p(m+1)}] = \frac{1}{p(m+1)}M(\|\widetilde{u}\|^p)\|\widetilde{u}\|^p.$$

So we have

$$\frac{1}{p(m+1)}M(\|\widetilde{u}\|^p)\|\widetilde{u}\|^p - \frac{\lambda}{p^*}\int_\Omega |\widetilde{u}|^{p^*}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|\widetilde{u}|^{1-\gamma}dx < 0$$

and

$$\lambda\big[\frac{1}{p(m+1)} - \frac{1}{p^*}\big]\int_\Omega |\widetilde{u}|^{p^*}dx - \big[\frac{1}{1-\gamma} - \frac{1}{p(m+1)}\big]\int_\Omega \rho(x)|\widetilde{u}|^{1-\gamma}dx < 0.$$

By (2.1), we obtain

$$\lambda\big[\frac{1}{p(m+1)} - \frac{1}{p^*}\big]\int_\Omega |\widetilde{u}|^{p^*}dx < \big[\frac{1}{1-\gamma} - \frac{1}{p(m+1)}\big]C_0\|\rho(x)\|_p\|\widetilde{u}\|_{p^*}^{1-\gamma}.$$

This leads to

$$\|\widetilde{u}\|_{p^*}^{p^*+\gamma-1} < \frac{p^*[p(m+1)+\gamma-1]C_0\|\rho\|_p}{\lambda[p^*-p(m+1)](1-\gamma)}.$$

By choosing

$$\begin{aligned}\lambda_4 = &\Big\{\frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{p^*+\gamma-1}S^{\frac{m+2}{2}}\Big\}^{\frac{2(p^*+\gamma-1)}{p(m+2)-2(1-\gamma)}}\\ &\times\Big\{\frac{[p^*-p(m+1)](1-\gamma)}{p^*[p(m+1)+\gamma-1]C_0\|\rho\|_p}\Big\}^{\frac{2p^*-p(m+2)}{p(m+2)-2(1-\gamma)}},\end{aligned}$$

for all $\lambda < \lambda_4$ we have

$$\|u\|_{p^*} < \Big\{\frac{2\sqrt{ab(p+\gamma-1)[p(m+1)+\gamma-1]}}{\lambda(p^*+\gamma-1)}\Big\}^{\frac{2}{2p^*-p(m+2)}}S^{\frac{m+2}{2p^*-p(m+2)}}.$$

This yields a contradiction to inequality (2.5). Hence, the proof of Lemma 2.6 is complete. $\square$

Lemma 2.7. *If $u \in \Lambda^-$, then there exist an $\epsilon > 0$ and a differentiable function $f = f(w) > 0$, where $w \in W_0^{1,p}(\Omega)$ and $\|w\| < \epsilon$ such that $f(0) = 1$ and $f(w)(u + w) \in \Lambda^-$ for all $w \in W_0^{1,p}(\Omega)$.*

Proof. Define F: $\mathbb{R} \times W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned}F(t,w) = &at^{p+\gamma-1}\int_\Omega |\nabla(u+w)|^p dx + bt^{p(m+1)+\gamma-1}\Big(\int_\Omega |\nabla(u+w)|^p dx\Big)^{m+1}\\ &- \lambda t^{p^*+\gamma-1}\int_\Omega |u+w|^{p^*}dx - \int_\Omega \rho(x)|u+w|^{1-\gamma}dx.\end{aligned}$$

Since $u \in \Lambda^- \subset \Lambda$, we have $F(1,0) = 0$, and

$$\begin{aligned}F_t(1,0) = &a(p+\gamma-1)\|u\|^p + b[p(m+1)+\gamma-1]\|u\|^{p(m+1)}\\ &- \lambda(p^*+\gamma-1)\|u\|_{p^*}^{p^*} < 0.\end{aligned}$$

According to the implicit function theorem at the point $(1,0)$, there exist an $\overline{\epsilon} > 0$ and a continuous function $f = f(w) > 0$, where $w \in W_0^{1,p}(\Omega)$ and $\|w\| < \overline{\epsilon}$, such that

$$f(0) = 1 \text{ and } f(w)(u+w) \in \Lambda \text{ for all } w \in W_0^{1,p}(\Omega).$$

Clearly, we can take $\epsilon > 0$ sufficiently small ($< \overline{\epsilon}$) satisfying

$$f(w)(u+w) \in \Lambda^- \quad \forall w \in W_0^{1,p}(\Omega) \text{ and } |w\| < \epsilon.$$

$\square$

Lemma 2.8. *For all $\lambda > 0$, problem* (1.1) *has a weak solution u^2 in X.*

Proof. From (A3) and Lemma 2.3, we know $\Lambda^- \neq \emptyset$ and $c_- = \inf_{u\in\Lambda^-} I_\lambda(u) > -\infty$ is well-defined. By Ekeland's variational principle, there exists a minimizing sequence $\{v_k\} \subset \Lambda^-$ satisfying

$$I_\lambda(v_k) < c_- + \frac{1}{k} \quad \text{and} \quad I_\lambda(v_k) \le I_\lambda(v) + \frac{1}{k}\|v - v_k\| \quad \forall v \in \Lambda^-.$$

Since $I_\lambda(v_k) = I_\lambda(|v_k|)$, we assume $v_k \ge 0$ in Ω and (up to a subsequence if necessary) it converges to a function, denoted by $u^2 \ge 0$. Then, we have

$$v_k \rightharpoonup u^2 \text{in } X \quad \text{and} \quad v_k \to u^2 \text{ a.e. in } \Omega.$$

Let $u = v_k \in \Lambda^-$, $w = t\varphi$, $\varphi \in W_0^{1,p}(\Omega)$, and $t > 0$ be small enough. There exists a differentiable function $f_k(t) = f_k(t\varphi)$ satisfying

$$f_k(0) = 1 \quad \text{and} \quad f_k(t)(v_k + t\varphi) \in \Lambda^-.$$

Since $\Lambda^- \subset \Lambda$, it follows that

$$\begin{aligned}
&f_k^p(t) M(f_k^p(t)\|v_k + t\varphi\|^p)\|v_k + t\varphi\|^p - \lambda f_k^{p^*}(t)\int_\Omega |v_k + t\varphi|^{p^*}dx \\
&- f_k^{1-\gamma}(t)\int_\Omega \rho(x)|v_k + t\varphi|^{1-\gamma}dx = 0
\end{aligned}$$

and

$$M(\|v_k\|^p)\|v_k\|^p - \lambda\int_\Omega |v_k|^{p^*}dx - \int_\Omega \rho(x)|v_k|^{1-\gamma}dx = 0.$$

By Ekeland's variational principle, we have

$$\begin{aligned}
&\frac{1}{k}[|f_k(t) - 1|\|v_k\| + tf_k(t)\|\varphi\|] \\
&\ge \frac{1}{k}[f_k(t)(v_k + t\varphi) - v_k] \\
&\ge I_\lambda(v_k) - I_\lambda[f_k(t)(v_k + t\varphi)] \\
&= \frac{1}{p}\widehat{M}(\|v_k\|^p) - \frac{1}{p}\widehat{M}(f_k^p(t)\|v_k + t\varphi\|^p) + \frac{\lambda f_k^{p^*}(t)}{p^*}\int_\Omega |v_k + t\varphi|^{p^*}dx \\
&\quad - \frac{\lambda}{p^*}\int_\Omega |v_k|^{p^*}dx + \frac{f_k^{1-\gamma}(t)}{1-\gamma}\int_\Omega \rho(x)|v_k + t\varphi|^{1-\gamma}dx - \frac{1}{1-\gamma}\int_\Omega \rho(x)|v_k|^{1-\gamma}dx \\
&= \frac{1}{p}\widehat{M}(\|v_k\|^p) - \frac{1}{p}\widehat{M}(f_k^p(t)\|v_k + t\varphi\|^p) + \Big(\frac{\lambda f_k^{p^*}(t)}{p^*} - \frac{\lambda}{p^*}\Big)\int_\Omega |v_k + t\varphi|^{p^*}dx \\
&\quad + \frac{\lambda}{p^*}\Big[\int_\Omega |v_k + t\varphi|^{p^*}dx - \int_\Omega |v_k|^{p^*}dx\Big] \\
&\quad + \frac{1}{1-\gamma}[f_k^{1-\gamma}(t) - 1]\int_\Omega \rho(x)|v_k + t\varphi|^{1-\gamma}dx \\
&\quad + \frac{1}{1-\gamma}\Big[\int_\Omega \rho(x)|v_k + t\varphi|^{1-\gamma}dx - \int_\Omega \rho(x)|v_k|^{1-\gamma}dx\Big].
\end{aligned}$$

Dividing it by $t > 0$ and letting $t \to 0$, we find

$$\begin{aligned}
&\frac{1}{k}[|f_k'(0)|\|v_k\| + \|\varphi\|] \\
&\ge -M(\|v_k\|^p)f_k'(0)\|v_k\|^p - M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}\nabla v_k \cdot \nabla\varphi dx
\end{aligned}$$

$$
\begin{aligned}
&+\lambda f_k'(0)\int_\Omega |v_k|^{p^*}dx+\lambda\int_\Omega v_k^{p^*-1}\varphi dx+f_k'(0)\int_\Omega \rho(x)|v_k|^{1-\gamma}dx\\
&+\int_\Omega \rho(x)v_k^{-\gamma}\varphi dx\\
=&-f_k'(0)[M(\|v_k\|^p)\|v_k\|^p-\lambda\int_\Omega |v_k|^{p^*}dx-\int_\Omega \rho(x)|v_k|^{1-\gamma}dx]\\
&-M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}\nabla v_k\cdot\nabla\varphi dx+\lambda\int_\Omega v_k^{p^*-1}\varphi dx+\int_\Omega \rho(x)v_k^{-\gamma}\varphi dx\\
=&-M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}\nabla v_k\cdot\nabla\varphi dx+\lambda\int_\Omega v_k^{p^*-1}\varphi dx+\int_\Omega \rho(x)v_k^{-\gamma}\varphi dx.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\int_\Omega \rho(x)v_k^{-\gamma}\varphi dx\\
&\le \frac{1}{k}[|f_k'(0)|\|v_k\|+\|\varphi\|]+M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}\nabla v_k\cdot\nabla\varphi dx-\lambda\int_\Omega v_k^{p^*-1}\varphi dx.
\end{aligned}
$$

Since there exists $C_3>0$ such that $|f_k'(0)|\le C_3$, as $k\to\infty$ it follows from Fatou's Lemma that

$$
\begin{aligned}
\int_\Omega \rho(x)(u^2)^{-\gamma}\varphi dx &\le \liminf_{k\to\infty}\int_\Omega \rho(x)v_k^{-\gamma}\varphi dx\\
&\le M(\|u^2\|^p)\int_\Omega |\nabla u^2|^{p-2}\nabla u^2\cdot\nabla\varphi dx-\lambda\int_\Omega (u^2)^{p^*-1}\varphi dx.
\end{aligned}
$$

Note that φ is arbitrary. The above inequality also holds for $-\varphi$:

$$
\int_\Omega \rho(x)(u^2)^{-\gamma}\varphi dx \ge M(\|u^2\|^p)\int_\Omega |\nabla u^2|^{p-2}\nabla u^2\cdot\nabla\varphi dx-\lambda\int_\Omega (u^2)^{p^*-1}\varphi dx.
$$

Thus, we see that

$$
M(\|u^2\|^p)\int_\Omega |\nabla u^2|^{p-2}\nabla u^2\cdot\nabla\varphi dx-\lambda\int_\Omega (u^2)^{p^*-1}\varphi dx-\int_\Omega \rho(x)(u^2)^{-\gamma}\varphi dx=0,
$$

where $\varphi\in W_0^{1,p}(\Omega)$. This implies that u^2 is a weak solution of (1.1). □

Lemma 2.9. *There exists $\lambda_5>0$ such that $u^2\in\Lambda^-$ for any $\lambda\in(0,\lambda_5)$.*

Proof. For any $u\in\Lambda^-\subset\Lambda$, we have

$$
\begin{aligned}
I_\lambda(u)&=\frac{1}{p}\widehat{M}(\|u\|^p)-\frac{\lambda}{p^*}\int_\Omega |u|^{p^*}dx-\frac{1}{1-\gamma}\int_\Omega \rho(x)|u|^{1-\gamma}dx\\
&=\frac{1}{p}\widehat{M}(\|u\|^p)-\frac{\lambda}{p^*}\int_\Omega |u|^{p^*}dx\\
&\quad-\frac{1}{1-\gamma}\Big[M(\|u\|^p)\|u\|^p-\lambda\int_\Omega |u|^{p^*}dx\Big]\\
&=\frac{1}{p}\widehat{M}(\|u\|^p)-\frac{1}{1-\gamma}M(\|u\|^p)\|u\|^p-\lambda\Big(\frac{1}{p^*}-\frac{1}{1-\gamma}\Big)\int_\Omega |u|^{p^*}dx\\
&=-\frac{a(p+\gamma-1)\|u\|^p}{p(1-\gamma)}-\frac{b[p(m+1)+\gamma-1]\|u\|^{p(m+1)}}{p(m+1)(1-\gamma)}\\
&\quad-\lambda\Big(\frac{1}{p^*}-\frac{1}{1-\gamma}\Big)\int_\Omega |u|^{p^*}dx
\end{aligned}
$$

$$
\begin{aligned}
&< -\frac{1}{p(m+1)(1-\gamma)}\big[a(p+\gamma-1)\|u\|^p + b[p(m+1)+\gamma-1]\|u\|^{p(m+1)}\big] \\
&\quad - \lambda\Big(\frac{1}{p^*} - \frac{1}{1-\gamma}\Big)\int_\Omega |u|^{p^*}dx \\
&< \frac{1}{p(m+1)(1-\gamma)}\lambda(p^*+\gamma-1)\int_\Omega |u|^{p^*}dx - \lambda\Big(\frac{1}{p^*} - \frac{1}{1-\gamma}\Big)\int_\Omega |u|^{p^*}dx \\
&= \lambda\Big[\frac{p^*+\gamma-1}{p(m+1)(1-\gamma)} - \frac{1}{p^*} + \frac{1}{1-\gamma}\Big]\int_\Omega |u|^{p^*}dx \\
&= \lambda(p^*+\gamma-1)\Big[\frac{1}{p(m+1)(1-\gamma)} + \frac{1}{p^*(1-\gamma)}\Big]\int_\Omega |u|^{p^*}dx.
\end{aligned}
$$

One can see $c_- = \inf_{u\in\Lambda^-} I_\lambda(u) < \frac{1}{N}(aS)^{N/p}$ if we choose $\lambda < \widetilde{\lambda_5}$, where $\widetilde{\lambda_5}$ satisfies

$$
\widetilde{\lambda_5}(p^*+\gamma-1)\Big[\frac{1}{p(m+1)(1-\gamma)} + \frac{1}{p^*(1-\gamma)}\Big]\int_\Omega |u|^{p^*}dx < \frac{1}{N}(aS)^{N/p}.
$$

Now, we show that $u^2 \in \Lambda^-$. Since Λ^- is closed and $v_k \rightharpoonup u^2$ in X, we only need to prove $\|v_k\| \to \|u^2\|$. Similar to the proof of the existence of u^1, we assume that

$$
\begin{gathered}
|\nabla v_k|^p \rightharpoonup \mu, \quad |v_k|^{p^*} \rightharpoonup \eta, \\
\eta = |u^2|^{p^*} + \sum_{j\in J}\eta_j\delta_{x_j}, \quad \mu \geq |\nabla u^2|^p + \sum_{j\in J}\mu_j\delta_{x_j},
\end{gathered}
$$

where η and μ are non-negative bounded measures on $\overline{\Omega}$, and numbers $\mu_j > 0$ and $\eta_j > 0$ satisfy $\mu_j \geq S\eta_j^{p/p^*}$. So we have

$$
\int_\Omega |\nabla v_k|^p dx \to \int_\Omega d\mu < +\infty \tag{2.6}
$$

and

$$
\int_\Omega |v_k|^{p^*}dx \to \int_\Omega d\eta < +\infty.
$$

Choose $\psi \in C_0^\infty(\mathbb{R}^N)$ with $0 \leq \psi \leq 1$, and take $\psi = 1$ if $|x| < 1$ and $\psi = 0$ if $|x| \geq 2$ and $\|\nabla\psi\|_\infty \leq 2$.

We fix $\epsilon > 0$ and $j \in J$ and set

$$
\psi_{\epsilon,j} = \psi\big(\frac{x-x_j}{\epsilon}\big).
$$

Lemma (2.8) leads to

$$
M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}\nabla v_k\cdot\nabla\varphi dx - \lambda\int_\Omega v_k^{p^*-1}\varphi dx - \int_\Omega \rho(x)v_k^{-\gamma}\varphi dx = o_k(1)
$$

as $k \to \infty$, for all $\varphi \in W_0^{1,p}(\Omega)$. Since $\psi_{\epsilon,j}v_k \in W_0^{1,p}(\Omega)$, we have

$$
\begin{aligned}
&M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}\nabla v_k\cdot\nabla(\psi_{\epsilon,j}v_k)dx \\
&- \lambda\int_\Omega v_k^{p^*-1}(\psi_{\epsilon,j}v_k)dx - \int_\Omega \rho(x)v_k^{-\gamma}(\psi_{\epsilon,j}v_k)dx = o_k(1).
\end{aligned}
$$

A direct calculation gives

$$\begin{aligned}
&M(\|v_k\|^p)\int_\Omega |\nabla v_k|^p\psi_{\epsilon,j}dx \\
&= -M(\|v_k\|^p)\int_\Omega |\nabla v_k|^{p-2}v_k(\nabla v_k\cdot\nabla\psi_{\epsilon,j})dx \\
&\quad + \lambda\int_\Omega v_k^{p^*}\psi_{\epsilon,j}dx + \int_\Omega \rho(x)v_k^{1-\gamma}\psi_{\epsilon,j}dx + o_k(1).
\end{aligned} \tag{2.7}$$

From (2.6) and (A1), we obtain

$$\lim_{k\to\infty}\sup M(\|v_k\|^p) < +\infty.$$

By Hölder's inequality,

$$\begin{aligned}
&\lim_{k\to\infty}\sup |M(\|v_k\|^p)\int_{B(x_j,2\epsilon)} |\nabla v_k|^{p-2}v_k(\nabla v_k\cdot\nabla\psi_{\epsilon,j})dx| \\
&\le \lim_{k\to\infty} C_4\int_{B(x_j,2\epsilon)} |\nabla v_k|^{p-1}|v_k\nabla\psi_{\epsilon,j}|dx \\
&\le \lim_{k\to\infty} C_4\Big(\int_{B(x_j,2\epsilon)} |\nabla v_k|^p dx\Big)^{\frac{p-1}{p}}\Big(\int_{B(x_j,2\epsilon)} |v_k\nabla\psi_{\epsilon,j}|^p dx\Big)^{1/p} \\
&\le C_5\Big(\int_{B(x_j,2\epsilon)} |u^2|^{p^*}dx\Big)^{1/p^*}\Big(\int_{B(x_j,2\epsilon)} |\nabla\psi_{\epsilon,j}|^N dx\Big)^{1/N} \\
&\le C_6(\int_{B(x_j,2\epsilon)} |u^2|^{p^*}dx)^{1/p^*}.
\end{aligned}$$

As $k\to\infty$, from (2.7) it follows that

$$\begin{aligned}
&a\int_{B(x_j,2\epsilon)}\psi_{\epsilon,j}d\mu \\
&\le M(\|v_k\|^p)\int_{B(x_j,2\epsilon)}\psi_{\epsilon,j}d\mu \\
&\le C_6(\int_{B(x_j,2\epsilon)} |u^2|^{p^*}dx)^{1/p^*} + \lambda\int_{B(x_j,2\epsilon)}\psi_{\epsilon,j}d\eta + \int_{B(x_j,2\epsilon)}\rho(x)(u^2)^{1-\gamma}\psi_{\epsilon,j}dx.
\end{aligned}$$

Letting $\epsilon\to 0$, we have $a\mu_j\le\lambda\eta_j$ and $\lambda\eta_j\ge aS\eta_j^{p/p^*}$. So, $\eta_j = 0$ or $\eta_j\ge(\frac{aS}{\lambda})^{N/p}$.

Next, we show that $\eta_j\ge(\frac{aS}{\lambda})^{N/p}$ is impossible. By contradiction, we suppose that there exists some j_0 satisfying $\eta_{j_0}\ge(\frac{aS}{\lambda})^{N/p}$. Then

$$\begin{aligned}
c_- &= \lim_{k\to\infty} I_\lambda(v_k) \\
&= \lim_{k\to\infty}\Big\{I_\lambda(v_k) - \frac{1}{p(m+1)}\Big[M(\|v_k\|^p)\|v_k\|^p - \lambda\int_\Omega |v_k|^{p^*}dx \\
&\quad - \int_\Omega \rho(x)|v_k|^{1-\gamma}dx\Big]\Big\} \\
&= \lim_{k\to\infty}\Big\{a\big[\frac{1}{p} - \frac{1}{p(m+1)}\big]\|v_k\|^p + \lambda\big[\frac{1}{p(m+1)} - \frac{1}{p^*}\big]\int_\Omega |v_k|^{p^*}dx \\
&\quad + \big[\frac{1}{p(m+1)} - \frac{1}{1-\gamma}\big]\int_\Omega \rho(x)|v_k|^{1-\gamma}dx\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\geq a\big[\frac{1}{p}-\frac{1}{p(m+1)}\big]\int_\Omega |\nabla u^2|^p dx + a\big[\frac{1}{p}-\frac{1}{p(m+1)}\big]\sum_{j\in J}\mu_j \\
&\quad + \lambda\big[\frac{1}{p(m+1)}-\frac{1}{p^*}\big]\int_\Omega |u^2|^{p^*}dx + \lambda\big[\frac{1}{p(m+1)}-\frac{1}{p^*}\big]\sum_{j\in J}\eta_j \\
&\quad + \big[\frac{1}{p(m+1)}-\frac{1}{1-\gamma}\big]\int_\Omega \rho(x)|u^2|^{1-\gamma}dx \\
&> a\big[\frac{1}{p}-\frac{1}{p(m+1)}\big]S\big(\frac{aS}{\lambda}\big)^{N/p^*} + \lambda\big[\frac{1}{p(m+1)}-\frac{1}{p^*}\big]\big(\frac{aS}{\lambda}\big)^{N/p} \\
&\quad + \lambda\big[\frac{1}{p(m+1)}-\frac{1}{p^*}\big]\|u^2\|_{p^*}^{p^*} + \big[\frac{1}{p(m+1)}-\frac{1}{1-\gamma}\big]C_0\|\rho\|_p\|u^2\|_{p^*}^{1-\gamma} \\
&= \frac{1}{N}(aS)^{N/p}\lambda^{1-\frac{N}{p}} + \lambda\big[\frac{1}{p(m+1)}-\frac{1}{p^*}\big]\|u^2\|_{p^*}^{p^*} \\
&\quad + \big[\frac{1}{p(m+1)}-\frac{1}{1-\gamma}\big]C_0\|\rho\|_p\|u^2\|_{p^*}^{1-\gamma} \\
&> \frac{1}{N}(aS)^{N/p}.
\end{aligned}
$$

Take $\widetilde{\widetilde{\lambda_5}} > 0$ such that the last inequality holds for $\lambda < \widetilde{\widetilde{\lambda_5}}$. This yields a contradiction with the fact $c_- < \frac{1}{N}(aS)^{N/p}$. Consequently, choosing $\lambda_5 = \min\{\widetilde{\lambda_5},\ \widetilde{\widetilde{\lambda_5}}\}$, we find $\eta_j = 0$, $|v_k|^{p^*} \rightharpoonup \eta = |u^2|^{p^*}$, and $u^2 \in \Lambda^-$ for all $\lambda \in (0, \lambda_5)$. $\square$

3. Proof of Theorem 1.1

Let $\lambda^* = \min\{\lambda_i\}$ $(i = 1, 2, 3, 4, 5)$. It is easy to see that Lemmas 2.1-2.9 hold for all $\lambda \in (0, \lambda^*)$. We only need to prove that u^1 is a weak positive solution of (1.1) and $u^2 > 0$ in Ω.

From Lemma 2.2, we see that

$$
\min_{t\in\mathbb{R}} I_\lambda(u^1+t\varphi) = I_\lambda(u^1+t\varphi)|_{t=0} = I_\lambda(u^1), \quad \forall\varphi \in W_0^{1,p}(\Omega).
$$

This implies that

$$
M(\|u^1\|^p)\int_\Omega |\nabla u^1|^{p-2}\nabla u^1\cdot\nabla\varphi dx - \lambda\int_\Omega (u^1)^{p^*-1}\varphi dx - \int_\Omega \rho(x)(u^1)^{-\gamma}\varphi dx = 0,
$$

for all $\varphi \in W_0^{1,p}(\Omega)$. Thus, u^1 is a weak solution of (1.1).

Since $u^1 \geq 0$ for any $\varphi \in W_0^{1,p}(\Omega)$, $\varphi \geq 0$ and $t > 0$, we have

$$
\begin{aligned}
0 &\leq I_\lambda\left(u^1+t\varphi\right) - I_\lambda\left(u^1\right) \\
&= \frac{1}{p}\widehat{M}\left(\|u^1+t\varphi\|^p\right) - \frac{1}{p}\widehat{M}\big(\|u^1\|^p\big) + \frac{\lambda}{p^*}\Big[\int_\Omega |u^1|^{p^*}dx - \int_\Omega |u^1+t\varphi|^{p^*}dx\Big] \\
&\quad + \frac{1}{1-\gamma}\Big[\int_\Omega \rho(x)|u^1|^{1-\gamma}dx - \int_\Omega \rho(x)|u^1+t\varphi|^{1-\gamma}dx\Big] \\
&\leq \frac{1}{p}\widehat{M}\left(\|u^1+t\varphi\|^p\right) - \frac{1}{p}\widehat{M}\big(\|u^1\|^p\big).
\end{aligned}
$$

That is,

$$
\frac{1}{p}\widehat{M}\Big(\|u^1+t\varphi\|^p\Big) - \frac{1}{p}\widehat{M}\Big(\|u^1\|^p\Big) \geq 0 \quad \forall\varphi \in W_0^{1,p}(\Omega) \text{ and all } \varphi \geq 0.
$$

Dividing by $t > 0$ and letting $t \to 0$ yields

$$M(\|u^1\|^p) \int_\Omega |\nabla u^1|^{p-2} \nabla u^1 \cdot \nabla \varphi dx \geq 0.$$

This implies that $u^1 \in W_0^{1,p}(\Omega)$ and

$$-M(\|u^1\|^p)\Delta_p u \geq 0 \quad \text{in } \Omega.$$

By the strong maximum principle, we deduce that $u^1 > 0$ in Ω. From Lemmas 2.8 and 2.9, we see that the solution u^2 is the minimizer of I_λ in Λ^-. Then, one can see that $u^2 > 0$ in Ω by the same arguments as the proof of positivity of u^1. Consequently, the proof of Theorem 1.1 is complete.

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