Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 80, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

GROUND STATE SOLUTIONS FOR NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS INVOLVING CRITICAL GROWTH

HUA JIN, WENBIN LIU

Communicated by Marco Squassina

Abstract. This article concerns the ground state solutions of nonlinear fractional Schrödinger equations involving critical growth. We obtain the existence of ground state solutions when the potential is not a constant and not radial. We do not use the Ambrosetti-Rabinowitz condition, or the monotonicity condition on the nonlinearity.

1. Introduction

The fractional Laplacian $(-\Delta)^s$ is a classical linear integro-differential operator of order s. The main feature, and also its main difficulty, is that it is a non-local operator. Recently, a great deal of attention has been devoted to the fractional Laplacian and non-local operators of elliptic type, both for their interesting theoretical structure and concrete applications. The fractional Laplacian $(-\Delta)^s$ arises in the description of various phenomena in the applied science, such as the thin obstacle problem [8, 29], phase transition [1, 30], Markov processes [16] and fractional quantum mechanics [23] and the references therein for more details.

The fractional Schrödinger equation formulated by Laskin [21, 22, 23] has the form

$$i\varphi_t - (-\Delta)^s\varphi - V(x)\varphi + f(\varphi) = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$, φ is the wavefunction and $V(x)$ is the potential energy. The fractional quantum mechanics has been discovered as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths. Since we are concerned with the standing wave solutions of the form

$$\varphi(x,t) = e^{-iwt}u(x), \quad w \in \mathbb{R},$$

then (1.1) can be converted into

$$(-\Delta)^s u + V(x)u = f(u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

When $s = 1$, equation (1.2) gives back to the classical nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 35A15, 35B33, 35Q55.
Key words and phrases. Fractional Schrödinger equations; ground state solutions; critical growth; Pohozǎev identity.

Submitted December 2, 2016. Published March 24, 2017.

which has been studied theoretically and numerically in the last decades. We should emphasize that the potential $V(x)$ plays a crucial role concerning the existence of nontrivial solutions and the existence of ground state solutions. If the potential $V(x)$ is a constant, namely (1.3) is autonomous, in the celebrated paper [5], Berestycki and Lions first proposed the Berestycki-Lions conditions which are almost optimal for the existence of ground state solutions in the subcritical case. The authors investigated the constraint minimization problem and use the Schwarz symmetrization in $H^1_r(\mathbb{R}^N)$. For the critical nonlinearity f, because of the lack of compactness of $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, the existence of ground state solutions of problem (1.3) becomes rather more complicated. In [38], the critical case was considered by modifying the minimization methods with constrains. Since the radial symmetry plays a crucial role, the method is invalid for the non-radial case.

In the non-autonomous case, that is $V(x) \not\equiv V$, where V is a constant, the main obstacle to get the existence of solutions or ground state solutions is the boundedness of the Palais-Smale (PS for short) sequence because of no some global conditions on f, such as the Ambrosetti-Rabinowitz (A-R for short) condition. Moreover, the lack of compactness due to the unboundedness of the domain prevents us from checking the (PS) condition. To avoid the difficulties mentioned above, in the seminal paper [19], Jeanjean and Tanaka used an indirect approach developed in [18] to get a bounded (PS) (BPS for short) sequence for the energy functional I, then the existence of positive solutions and moreover ground state solutions is obtained in the subcritical case when the nonlinearity f and potential $V(x)$ satisfy the following assumptions:

- (A1) $f \in C(\mathbb{R}^+, \mathbb{R})$, $f(0) = 0$ and $f'(0)$ defined as $\lim_{t\to 0^+} f(t)/t$ exists,
- (A2) there is $p < \infty$ if $N = 2$, $p < 2^* - 1$ if $N \geq 3$ such that $\lim_{t\to\infty} f(t)/t^p = 0$,
- (A3) $\lim_{t\to\infty} f(t)/t = +\infty$,
- (A4) $f'(0) < \inf \sigma(-\Delta + V(x))$, where $\sigma(-\Delta + V(x))$ denotes the spectrum of the self-adjoint operator $-\Delta + V(x)$,
- (A5) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x) \to V(\infty) \in \mathbb{R}$ as $|x| \to \infty$,
- (A6) $V(x) \leq V(\infty)$,
- (A7) there exists a function $\phi \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that

$$|x||\nabla V(x)| \leq \phi^2(x), \forall x \in \mathbb{R}^N.$$

Here, the decay condition (A7) is crucial to derive the boundedness of the (PS)-sequence. For the critical case, the problem is different and more difficulty. In [35], by use of the indirect approach developed in [18], the authors completed the proof of existence of ground state solutions in the critical case with the same conditions on $V(x)$. As for the nonlinearity f, the following conditions are satisfied

- (A8) $f \in C(\mathbb{R}^+, \mathbb{R})$, $f(t) = o(t)$ as $t \to o^+$,
- (A9) $\lim_{t\to+\infty} f(t)/t^{2^*-1} = K > 0$, where $2^* = \frac{2N}{N-2}$,
- (A10) there exist $D > 0$ and $2 < q < 2^*$ such that $f(t) \geq Kt^{2*-1} + Dt^{q-1}, \forall t \geq 0$,
- (A11) $f \in C^1(\mathbb{R}^+, \mathbb{R}), |f'(t)| \leq C(1 + |t|^{\frac{4}{N-2}})$.

Now, we return our attention to the fractional and non-local problems. With the aid of the extended techniques developed by Caffarelli and Silvestre [9], some existence and nonexistence of Dirichlet problems involving the fractional Laplacian on bounded domains have been established, see [4, 7, 32] and so on. For the general fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), x \in \mathbb{R}^N$$

in the subcritical or critical case, many results have been obtained on the existence of ground state solutions, positive solutions, the multiplicity of standing wave solutions, the symmetry of solutions and so forth, under the different conditions on $V(x)$ and f, for example the monotonicity condition, (A-R) condition, see [3, 6, 9, 10, 12, 33, 34, 40] and the references therein.

As is well known, the existence and concentration phenomena of solutions on the singularly perturbed fractional Schrödinger equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(u), x \in \mathbb{R}^N$$

is also a hot topic. For this subject we refer, for example, to [2, 14, 15, 17, 27, 28] and the references therein.

Now, let us say more about the existence of ground state solutions of a class of fractional scalar field equations

$$(-\Delta)^s u + V(x)u = f(u), x \in \mathbb{R}^N.$$

When $f(u) - V(x)u = g(u)$, the authors [11] obtained the existence of radial positive ground state solutions under the general Berestycki-Lions type assumptions in the case of subcritical growth. By using the fractional Pohozǎev identity and the monotonicity trick of Struwe-Jeanjean, they showed that the compactness still holds under their assumptions without the Strauss type radial lemma in $H^s_r(\mathbb{R}^N)$. In [37], the existence of radial ground state solutions was obtained when $V(x) \equiv V$ involving the critical growth by means of the constraint variational argument, where $V > 0$ is a constant. When $V(x) = V(|x|)$, Secchi [31] proved the existence of radially symmetric solutions for equation (1.2) in $H^s_r(\mathbb{R}^N)$ by the fractional Pohozǎev identity and the monotonicity trick in subcritical case. The conditions on f and $V(x)$ are as follows

- (A12) $f \in C(\mathbb{R},\mathbb{R})$ is of class $C^{1,\gamma}$ for some $\gamma > \max\{0, 1-2s\}$, and odd,
- (A13) $-\infty < \liminf_{t\to 0^+} f(t)/t \le \limsup_{t\to 0^+} f(t)/t = -m < 0$,
- (A14) $-\infty < \limsup_{t\to+\infty} f(t)/t^{2^*_s-1} \le 0$, where $2^*_s = \frac{2N}{N-2s}$,
- (A15) for some $\zeta > 0$, there holds $F(\zeta) = \int_0^\zeta f(t)dt > 0$,
- (A16) $V \in C^1(\mathbb{R}^N,\mathbb{R}), V(x) \ge 0$ for every $x \in \mathbb{R}^N$ and this inequality is strict at some point,
- (A17) $\|\max\{\langle \nabla V(x), x\rangle, 0\}\|_{L^{N/2s}(\mathbb{R}^N)} < 2sS_s$,
- (A18) $\lim_{|x|\to+\infty} V(x) = 0$,
- (A19) $V(x)$ is radially symmetric,

where S_s is the best Sobolev constant for the critical embedding, that is

$$S_s = \inf_{u\in H^s(\mathbb{R}^N), u\neq 0} \frac{\|(-\Delta)^{s/2}u\|^2_{L^2}}{\|u\|^2_{L^{2^*_s}}},$$

here $H^s(\mathbb{R}^N)$ is the fractional Sobolev space with respect to the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + |u|^2.$$

Where (A13)-(A15) are called Berestycki-Lions type conditions, and (A17) is used to get the boundedness of the (PS)-sequence by use of the monotonicity trick. The condition $f \in C^1$ ensures that the fractional Pohozǎev identity can be used.

Now, the problem is how about the existence of ground state solutions when $V(x)$ is non-radial in the critical case. As we all know, for the critical case, the loss of the compactness for the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is the main difficulty. What's more, (PS) condition, in general, fails. Since $V(x)$ is non-radial, the method introduced in [37] can not be used here since the fractional space they used is $H_r^s(\mathbb{R}^N)$. With critical growth, the authors [17] proved the existence of solutions for equation (1.2) under the monotonicity condition on $f(t)/t$ and $0 < \mu F(t) = \mu \int_0^t f(t)dt \le tf(t), \mu \in (2, 2_s^*)$ for all $t > 0$.

Motivated by the seminal papers above, we use the indirect approach developed in [18] to investigate the existence of ground state solutions for nonlinear fractional Schrödinger equation (1.2) involving the critical nonlinearity, where the potential $V(x)$ depends on x non-radially. More precisely, on the nonlinearity f, we assume

- (A20) $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\lim_{t\to 0} f(t)/t = 0$,
- (A21) $\lim_{t\to\infty} f(t)/t^{2_s^*-1} = 1$,
- (A22) There exist $D > 0$ and $p < 2_s^*$ such that $f(t) \ge t^{2_s^*-1} + Dt^{p-1}, t \ge 0$.

We assume $f(t) \equiv 0$ for $t \le 0$ throughout the paper since we are concerned with the positive solutions.

On potential $V \in C^1(\mathbb{R}^N, \mathbb{R})$, we assume

- (A23) There exists $V_0 > 0$ such that $\inf_{x\in\mathbb{R}^N} V(x) \ge V_0$,
- (A24) $V(x) \le V(\infty) := \lim_{|x|\to\infty} V(x) < \infty$ for all $x \in \mathbb{R}^N$ and $V(x) \not\equiv V(\infty)$,
- (A25) $\| \max\{\langle \nabla V(x), x\rangle, 0\}\|_{L^{N/2s}(\mathbb{R}^N)} < 2sS_s$.

In contrast to the conditions in [17], our conditions are more weaker. The main result is the following.

Theorem 1.1. *Assume $N > 2s, s \in (0, 1)$, if $\max\{2, 2_s^*-2\} < p < 2_s^*$,* (A20)–(A25) *hold, then problem* (1.2) *has a ground state solution.*

The proof of Theorem 1.1 is inspired by the ideas in [19] and [35].

Firstly, we show the existence of positive solutions of (1.2). For this purpose, we look for a special BPS sequence for the energy functional I associated with (1.2) by use of the Struwe's monotonicity trick. Precisely, with the help of the auxiliary energy functional I_λ satisfying

$$I(u_{\lambda_j}) = I_{\lambda_j}(u_{\lambda_j}) + (\lambda_j - 1)\int_{\mathbb{R}^N} F(u_{\lambda_j}), \lambda_j \to 1, \quad j \to \infty,$$

we prove the existence of positive critical points denoted by u_{λ_j} of I_{λ_j}. Thanks to the decomposition of BPS sequence, the properties of $\{u_{\lambda_j}\}$ and the energy estimation of $I_{\lambda_j}(u_{\lambda_j})$ are obtained. Consequently, we show that $\{u_{\lambda_j}\}$ is a BPS sequence for I at some level value.

Secondly, for the proof of the existence of ground state solutions, we construct a minimizing sequence $\{u_n\}$ which is composed of the critical points of I. We show that $\{u_n\}$ is a BPS sequence for I at m, here m denotes the least energy. Then, making use of the decomposition of BPS sequence and the relationship of I and I^∞, we prove that m is attained at some $\tilde{u} \ne 0$.

Remark 1.2. In the proof of our main results, the estimations of the Mountain Pass (MP for short) values, Pohožaev identity and the decomposition of BPS all play crucial roles.

This article is organized as follows. In section 2, we introduce a variational setting of our problem and present some preliminary results. In section 3, we are concerned with the decomposition of BPS and the existence of nontrivial critical points for the auxiliary energy functional. Section 4 is devoted to the completion of the proof of Theorem 1.1.

In the following, the letters C, δ, δ_0 are indiscriminately used to denote various positive constants whose exact values are irrelevant.

2. Preliminaries and functional setting

To establish the variational setting for (1.2), we give some useful facts of the fractional Sobolev space [25] and some preliminary lemmas.

The fractional Laplacian operator $(-\Delta)^s$ with $s \in (0,1)$ of a function $u : \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ is the Fourier transform. For $s \in (0,1)$, the fractional order Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2 d\xi < \infty\},$$

endowed with the norm $\|u\|_{H^s(\mathbb{R}^N)} = (\int_{\mathbb{R}^N}(|\xi|^{2s}|\hat{u}|^2 + |\hat{u}|^2)d\xi)^{1/2}$, where $\hat{u} \doteq \mathcal{F}(u)$. By Plancherel's theorem, we have $\|u\|_{L^2(\mathbb{R}^N)} = \|\hat{u}\|_{L^2(\mathbb{R}^N)}$ and

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2 dx = \int_{\mathbb{R}^N} (|\xi|^s|\hat{u}|)^2 d\xi.$$

It follows that $\|u\|_{H^s(\mathbb{R}^N)} = \big(\int_{\mathbb{R}^N}(|(-\Delta)^{s/2}u(x)|^2 + |u|^2)dx\big)^{1/2}$, $u \in H^s(\mathbb{R}^N)$. If u is smooth enough, $(-\Delta)^s u$ can be computed by the following singular integral

$$(-\Delta)^s u(x) = c_{N,s}\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy.$$

Here $c_{N,s}$ is the normalization constant and P.V. is the principal value. So, one can get an alternative definition of the fractional Sobolev space $H^s(\mathbb{R}^N)$ as follows,

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\}$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |u|^2 + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\Big)^{1/2}.$$

The space $D^s(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo norm

$$\|u\|_{D^s(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2 d\xi\Big)^{1/2} = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\Big)^{1/2}.$$

Since we investigate the existence of solutions of problem (1.2), we need the fractional Sobolev space $H^s_V(\mathbb{R}^N)$ which is a Hilbert subspace of $H^s(\mathbb{R}^N)$ with the norm

$$\|u\|_{H^s_V(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} \Big(|(-\Delta)^{s/2}u|^2 + V(x)|u|^2\Big)dx\Big)^{1/2} < \infty. \tag{2.1}$$

It is easy to check that $H^s_V(\mathbb{R}^N) \equiv H^s(\mathbb{R}^N)$ if (A23) and (A24) hold. In our paper, we shall work on $H^s(\mathbb{R}^N)$ with norm (2.1) and we denote $\|u\|_{H^s(\mathbb{R}^N)}$ by $\|u\|$ for simplicity.

Associated with problem (1.2), is the energy functional $I : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V(x)|u|^2 - \int_{\mathbb{R}^N} F(u), u \in H^s(\mathbb{R}^N),$$

where $F(u) = \int_0^u f(t)dt$. Conditions (A20)–(A22) imply that $I \in C^1(H^s(\mathbb{R}^N), \mathbb{R})$.

Definition 2.1. u is said to be a solution of (1.2) if u is a critical point of the energy functional I and satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\varphi + \int_{\mathbb{R}^N} V(x)u\varphi = \int_{\mathbb{R}^N} f(u)\varphi, u \in H^s(\mathbb{R}^N), \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

u is said to be a ground state solution of (1.2) if u is a solution with the least energy among all nontrivial solutions of (1.2).

In this article, we use the embedding lemma and Lions lemma as follows.

Lemma 2.2. *([24]) For any $s \in (0,1)$, $H^s(\mathbb{R}^N)$ is continuously embedded into $L^r(\mathbb{R}^N)$ for $r \in [2, 2^*_s]$ and compactly embedded into $L^r_{loc}(\mathbb{R}^N)$ for $r \in [2, 2^*_s)$.*

Lemma 2.3. *([26]) Suppose that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |u_n|^2 \to 0.$$

Then $\|u_n\|_{L^r} \to 0$ for $r \in (2, 2^*_s)$ when $N \geq 3$ and for $r \in (2, +\infty)$ when $N = 1, 2$. Here $B_1(z) = \{y \in \mathbb{R}^N, |y - z| \leq 1\}$.

3. Solutions for auxiliary problems

In this section, we consider the family of functionals $I_\lambda(u) : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V(x)|u|^2 - \lambda \int_{\mathbb{R}^N} F(u).$$

The corresponding auxiliary problems are

$$(-\Delta)^s u + V(x)u = \lambda f(u). \tag{3.1}$$

The main aim of this section is to prove that for almost every $\lambda \in [1/2, 1]$, I_λ has a nontrivial critical point u_λ such that $I_\lambda(u_\lambda) \leq c_\lambda$, where

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0,1], H^s(\mathbb{R}^N)), \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\}.$$

Before we prove the existence of solutions for the auxiliary problems (3.1), we give some propositions and lemmas.

Proposition 3.1. *Let $u(x)$ be a critical point of I_λ with $\lambda \in [1/2, 1]$, then $u(x)$ satisfies*

$$\begin{aligned} &\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + \frac{N}{2}\int_{\mathbb{R}^N} V(x)|u|^2 \\ &+ \frac{1}{2}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u|^2 - N\lambda \int_{\mathbb{R}^N} F(u) = 0. \end{aligned} \tag{3.2}$$

As we all know, (3.2) is named Pohozǎev identity. The proof is similar as that in [11] and we omit it here.

Lemma 3.2. *Assume* (A20) *and* (A21) *hold. Let* $\{u_n\} \subset H^s(\mathbb{R}^N)$ *be such that* $u_n \to u$ *weakly in* $H^s(\mathbb{R}^N)$. *Then up to a subsequence,*

$$\int_{\mathbb{R}^N} (f(u_n) - f(u) - f(u_n - u))\phi = o_n(1)\|\phi\|.$$

where $o_n(1) \to 0$ *uniformly for* $\phi \in C_0^\infty(\mathbb{R}^N)$ *as* $n \to \infty$.

The proof of the above lemma is similar to that in [39]. So we omit it. Similar the proof of Brezis-Lieb Lemma in [36], we can give the following lemma.

Lemma 3.3. *For* $s \in (0,1)$, *assume* (A20) *and* (A21). *Let* $\{u_n\} \subset H^s(\mathbb{R}^N)$ *such that* $u_n \to u$ *weakly in* $H^s(\mathbb{R}^N)$ *and a.e. in* $\mathbb{R}^N$ *as* $n \to \infty$, *then*

$$\int_{\mathbb{R}^N} F(u_n) = \int_{\mathbb{R}^N} F(u_n - u) + \int_{\mathbb{R}^N} F(u) + o_n(1),$$

where $o_n(1) \to 0$ *as* $n \to \infty$.

To obtain the existence of critical points for I_λ, the following abstract result is needed from [18], which shows that for almost every $\lambda \in [1/2, 1]$, I_λ possesses a BPS sequence at the level c_λ.

Theorem 3.4. *Let* X *be a Banach space equipped with a norm* $\|\cdot\|_X$ *and let* $J \subset \mathbb{R}^+$ *be an interval. For a family* $(I_\lambda)_{\lambda \in J}$ *of* C^1*-functionals on* X *of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \forall \lambda \in J,$$

where $B(u) \geq 0, \forall u \in X$ *and such that either* $A(u) \to +\infty$ *or* $B(u) \to +\infty$ *as* $\|u\|_X \to \infty$. *If there are two points* v_1, v_2 *in* X *such that*

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}, \quad \forall \lambda \in J,$$

where

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, *there is a sequence* $\{v_n\} \subset X$ *such that*

- (i) $\{v_n\}$ *is bounded,*
- (ii) $I_\lambda(v_n) \to c_\lambda$,
- (iii) $I'_\lambda(v_n) \to 0$ *in the dual* X^{-1} *of* X.

In the following, we use Theorem 3.4 to seek nontrival critical points of I_λ for almost every $\lambda \in J$. In what follows, let $X = H^s(\mathbb{R}^N)$ and

$$A(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x)|u|^2, \quad B(u) = \int_{\mathbb{R}^N} F(u).$$

Obviously, $A(u) \to +\infty$ as $\|u\| \to \infty$ and $B(u) \geq 0$ for any $u \in H^s(\mathbb{R}^N)$ by (A22). Now, we give the following lemma to ensure that I_λ has the MP geometry. Consequently, we obtain a BPS for I_λ by Theorem 3.4.

Lemma 3.5. *Assume* (A20)–(A24) *hold. Then*

- (i) *there exists a* $v \in H^s(\mathbb{R}^N) \setminus \{0\}$ *with* $I_\lambda(v) \leq 0$ *for all* $\lambda \in [1/2, 1]$;
- (ii) $c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(v)\} > 0$ *for all* $\lambda \in [1/2, 1]$, *where*

$$\Gamma = \{\gamma \in C([0,1], H^s(\mathbb{R}^N)), \gamma(0) = 0, \gamma(1) = v\};$$

(iii) *there exists a BPS sequence* $\{u_n\}$ *at the MP level* c_λ *for* I_λ, *where* $u_n \geq 0$.

Proof. Since (A20) and (A21) hold, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N} F(u) \leq \varepsilon \int_{\mathbb{R}^N} |u|^2 + C(\varepsilon) \int_{\mathbb{R}^N} |u|^{2_s^*}, \forall u \in H^s(\mathbb{R}^N).$$

Thus

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} F(u) \\ &\geq \frac{1}{2}\|u\|^2 - \varepsilon\|u\|_{L^2}^2 - C(\varepsilon)\|u\|_{L^{2_s^*}}^{2_s^*} \end{aligned}$$

From Lemma 2.2, there exist constants $\rho > 0$ and $\delta > 0$ independent of λ such that for $\|u\| = \rho$, $I_\lambda(u) \geq \delta$. On the other hand, (A22) implies

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 - \frac{1}{2}\|u\|_{L^{2_s^*}}^{2_s^*} - \frac{D}{2p}\|u\|_{L^p}^p.$$

Set $v_0 \in H^s(\mathbb{R}^N)$ such that $v_0 \geq 0, v_0 \neq 0$. Since $I_\lambda(tv_0) \to -\infty$ as $t \to +\infty$, then there exists t_0 such that $I_\lambda(t_0 v_0) < 0$ as $\|t_0 v_0\| > \rho$. Set $v = t_0 v_0$, then (i) and (ii) hold. So, the conditions of Theorem 3.4 are satisfied. Therefore, for almost every $\lambda \in [1/2, 1]$, there exists a BPS sequence $\{u_n\}$ for I_λ at the MP value c_λ. Now, we show $u_n \geq 0$. Let $u_n = u_n^+ + u_n^-$. Using u_n^- as a test function, since $f(t) \equiv 0$ for all $t \leq 0$, we have

$$\begin{aligned} (I_\lambda'(u_n), u_n^-) &= \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2} u_n^- + \int_{\mathbb{R}^N} V(x)(u_n u_n^-) - \lambda \int_{\mathbb{R}^N} f(u_n) u_n^- \\ &= \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2} u_n^- + \int_{\mathbb{R}^N} V(x)|u_n^-|^2. \end{aligned}$$

Since for every $x, y \in \mathbb{R}^N$, we have $(u_n^+(x) - u_n^+(y))(u_n^-(x) - u_n^-(y)) \geq 0$, it follows that

$$\begin{aligned} &(u_n(x) - u_n(y))\big(u_n^-(x) - u_n^-(y)\big) \\ &= \big(u_n^+(x) - u_n^+(y)\big)\big(u_n^-(x) - u_n^-(y)\big) + \big(u_n^-(x) - u_n^-(y)\big)^2 \\ &\geq \big(u_n^-(x) - u_n^-(y)\big)^2. \end{aligned}$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2} u_n^- &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u_n(x) - u_n(y))(u_n^-(x) - u_n^-(y))}{|x-y|^{N+2s}} \\ &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u_n^-(x) - u_n^-(y)\big)^2}{|x-y|^{N+2s}} \\ &= \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n^-|^2. \end{aligned}$$

Therefore, from $(I_\lambda'(u_n), u_n^-) \to 0$, we have $\|u_n^-\| \to 0$. The proof is complete. □

From the argument above, we obtain a BPS for I_λ at the level c_λ. To get the convergence of the BPS sequence, we give some lemmas and propositions.

Lemma 3.6. *Assume* (A20)–(A24) *hold. If* $\max\{2, 2_s^* - 2\} < p < 2_s^*$, *then*

$$c_\lambda < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ is a cut-off function with support B_2 such that $\varphi \equiv 1$ on B_1 and $0 \le \varphi \le 1$ on B_2, where B_r denotes the ball in $\mathbb{R}^N$ of center at origin and radius r. For $\varepsilon > 0$, we define $\psi_\varepsilon(x) = \varphi(x) U_\varepsilon(x)$, where

$$U_\varepsilon(x) = \kappa \varepsilon^{-\frac{N-2s}{2}} \Big(\mu^2 + \big|\frac{x}{\varepsilon S_s^{\frac{1}{2s}}}\big|^2\Big)^{-\frac{N-2s}{2}}.$$

By [13], S_s can be achieved by $U_\varepsilon(x)$. Let $v_\varepsilon = \frac{\psi_\varepsilon}{\|\psi_\varepsilon\|_{L^{2_s^*}}}$, then $\|(-\Delta)^{s/2} v_\varepsilon\|_{L^2}^2 \le S_s + O(\varepsilon^{N-2s})$. From [17], we have the estimates

$$\|v_\varepsilon\|_{L^2}^2 = \begin{cases} O(\varepsilon^{2s}), & N > 4s, \\ O(\varepsilon^{2s} \ln \frac{1}{\varepsilon}), & N = 4s, \\ O(\varepsilon^{N-2s}), & N < 4s, \end{cases}$$

and

$$\|v_\varepsilon\|_{L^p}^p = \begin{cases} O(\varepsilon^{\frac{2N-(N-2s)p}{2}}), & p > \frac{N}{N-2s}, \\ O(\varepsilon^{\frac{(N-2s)p}{2}}), & p < \frac{N}{N-2s}. \end{cases}$$

By (A22), for any $t > 0$,

$$\begin{aligned} I_\lambda(tv_\varepsilon) &= \frac{t^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_\varepsilon|^2 + V(x)|v_\varepsilon|^2 - \lambda \int_{\mathbb{R}^N} F(tv_\varepsilon) \\ &= \frac{t^2}{2} \|v_\varepsilon\|^2 - \lambda \int_{\mathbb{R}^N} F(tv_\varepsilon) \\ &\le \frac{t^2}{2} \|v_\varepsilon\|^2 - \frac{\lambda}{2_s^*} t^{2_s^*} - \frac{Dt^p}{2p} \|v_\varepsilon\|_{L^p}^p. \end{aligned}$$

Obviously, $I_\lambda(tv_\varepsilon) \to -\infty$ as $t \to +\infty$ and $I_\lambda(tv_\varepsilon) > 0$ for $t > 0$ small. Let $g(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{\lambda}{2_s^*} t^{2_s^*}$. Then $t_\varepsilon = \big(\frac{\|v_\varepsilon\|^2}{\lambda}\big)^{\frac{1}{2_s^*-2}}$ is the maximum point of $g(t)$.

For $\varepsilon < 1$, by the definition of v_ε, there exists $t_1 > 0$ small enough such that

$$\max_{t \in (0,t_1)} I_\lambda(tv_\varepsilon) \le \frac{t^2}{2} \|v_\varepsilon\|^2 < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}.$$

Since $I_\lambda(tv_\varepsilon) \to -\infty$ as $t \to +\infty$, it is easy to obtain that there exists $t_2 > 0$ such that

$$\max_{t \in (t_2,+\infty)} I_\lambda(tv_\varepsilon) < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}.$$

If $t \in [t_1, t_2]$,

$$\max_{t \in [t_1,t_2]} I_\lambda(tv_\varepsilon) \le \max_{t \in [t_1,t_2]} \{g(t) - \frac{Dt_1^p}{2p} \|v_\varepsilon\|_{L^p}^p\} \le g(t_\varepsilon) - \frac{Dt_1^p}{2p} \|v_\varepsilon\|_{L^p}^p$$

For $g(t_\varepsilon)$, we have

$$\begin{aligned} g(t_\varepsilon) &= \frac{s}{N\lambda^{\frac{N-2s}{2s}}} (\|v_\varepsilon\|^2)^{\frac{N}{2s}} \\ &= \frac{s}{N\lambda^{\frac{N-2s}{2s}}} \Big(\|(-\Delta)^{s/2} v_\varepsilon\|_{L^2}^2 + \int_{\mathbb{R}^N} V(x)|v_\varepsilon|^2\Big)^{\frac{N}{2s}} \\ &\le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} \Big(S_s + O(\varepsilon^{N-2s}) + C\|v_\varepsilon\|_{L^2}^2\Big)^{\frac{N}{2s}}. \end{aligned}$$

By $(a+b)^q \le a^q + q(a+b)^{q-1}b$, where $a>0, b>0, q>1$, we have

$$\begin{aligned} g(t_\varepsilon) &\le \frac{s}{N\lambda^{\frac{N-2s}{2s}}}\Big(S_s^{\frac{N}{2s}} + \frac{N}{2s}\Big(S_s + O(\varepsilon^{N-2s}) \\ &\quad + C\|v_\varepsilon\|_{L^2}^2\Big)^{\frac{N-2s}{2s}} \big(O(\varepsilon^{N-2s}) + C\|v_\varepsilon\|_{L^2}^2\big)\Big) \\ &\le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + C\|v_\varepsilon\|_{L^2}^2. \end{aligned}$$

Thus

$$\max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon) \le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + C\|v_\varepsilon\|_{L^2}^2 - \frac{Dt_1^p}{2p}\|v_\varepsilon\|_{L^p}^p.$$

Next, we estimate $\max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon)$ in three cases.

Case 1: If $N>4s$, then $\frac{N}{N-2s}<2$, with $p>\max\{2, 2_s^*-2\}$, we have $p>\frac{N}{N-2s}$. So

$$\max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon) \le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + O(\varepsilon^{2s}) - O(\varepsilon^{\frac{2N-(N-2s)p}{2}}).$$

From $p>2, N>4s$, then $\frac{2N-(N-2s)p}{2} < 2s < N-2s$. Thus, for $\varepsilon>0$ small enough, we obtain

$$\max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon) < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}.$$

Case 2: If $N=4s$, then $2<p<4$. For $\varepsilon>0$ small enough, we obtain

$$\begin{aligned} \max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon) &\le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) + O(\varepsilon^{2s}\ln\frac{1}{\varepsilon}) - O(\varepsilon^{4s-sp}) \\ &\le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}} + O\Big(\varepsilon^{2s}(1+\ln\frac{1}{\varepsilon})\Big) - O(\varepsilon^{4s-sp}) \\ &< \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}, \end{aligned}$$

since

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon^{4s-sp}}{\varepsilon^{2s}(1+\ln\frac{1}{\varepsilon})} \to +\infty.$$

Case 3: If $2s<N<4s$, then $\frac{N}{N-2s}>2$, with $p>\max\{2, 2_s^*-2\}$, we have $p>\frac{N}{N-2s}$. So

$$\max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon) \le \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}} + O(\varepsilon^{N-2s}) - O(\varepsilon^{\frac{2N-(N-2s)p}{2}}).$$

From $p>\frac{4s}{N-2s}$, then $\frac{2N-(N-2s)p}{2} < N-2s$. For $\varepsilon>0$ small enough, we obtain

$$\max_{t\in[t_1,t_2]} I_\lambda(tv_\varepsilon) < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}.$$

The proof is complete. □

In (1.2), if $V(x)\equiv V(\infty)$, for $\lambda\in[1/2,1]$, the family of functionals $I_\lambda^\infty : H^s(\mathbb{R}^N)\mapsto\mathbb{R}$, defined as

$$I_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V(\infty)|u|^2 - \lambda\int_{\mathbb{R}^N} F(u),$$

plays an important role in our paper. Similar as that in [20, 38], we can derive the following result.

Lemma 3.7. *For $\lambda \in [1/2,1]$, if $w_\lambda \in H^s(\mathbb{R}^N)$ is a nontrivial critical point of I_λ^∞, then there exists $\gamma_\lambda \in C([0,1],H^s(\mathbb{R}^N))$ such that $\gamma_\lambda(0)=0, I_\lambda^\infty(\gamma_\lambda(1))<0, w_\lambda \in \gamma_\lambda[0,1]$ and $\max_{t\in[0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = I_\lambda^\infty(w_\lambda)$.*

Lemma 3.8 ([37])**.** *If f satisfies* (A20)–(A22) *and* $\max\{2, 2_s^*-2\} < p < 2_s^*$, *then for almost every $\lambda \in [1/2,1]$, I_λ^∞ has a positive ground state solution.*

Lemma 3.9. *If $V(x)\equiv V(\infty)>0$ and* (A20) *and* (A21) *hold, then there exists a constant $\delta>0$ independent of λ such that any nontrivial critical point u of I_λ^∞ satisfies $I_\lambda^\infty(u)\ge\delta$.*

Proof. Letting u be a nontrivial critical point of I_λ^∞, from the Pohozǎev identity (3.2), we have

$$I_\lambda^\infty(u) = \frac{s}{N}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2.$$

Since (A20) and (A21) hold, for any $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V|u|^2 \le \varepsilon \int_{\mathbb{R}^N} |u|^2 + C(\varepsilon)\int_{\mathbb{R}^N} |u|^{2_s^*}.$$

Thus, $\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \le C\int_{\mathbb{R}^N} |u|^{2_s^*}$. On the other hand, by the Sobolev embedding theorem, we have $\int_{\mathbb{R}^N} |u|^{2_s^*} \le \tilde{C}(\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2)^{\frac{2_s*}{2}}$. Since $u\ne 0$, there exists a constant $\delta_0>0$ such that $\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \ge \delta_0$ and so $I_\lambda^\infty(u)\ge \delta := s\delta_0/N$. The proof is complete. □

Now, we give the decomposition of a BPS sequence.

Proposition 3.10. *Assume* (A20)–(A25) *hold. If* $\max\{2,2_s^*-2\}<p<2_s^*$, *for almost every $\lambda\in[1/2,1]$, $\{u_n\}$ given in Lemma 3.5 is the BPS sequence at the MP value c_λ. Moreover, $c_\lambda < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}$. Then there exist a subsequence, still denoted by $\{u_n\}$, an integer $k\in\mathbb{N}\cup\{0\}$ and $v_\lambda^j \in H^s(\mathbb{R}^N)$ for $1\le j\le k$, such that*

- (i) $u_n \to u_\lambda$ *weakly in* $H^s(\mathbb{R}^N)$ *and* $I'_\lambda(u_\lambda)=0$,
- (ii) $v_\lambda^j \ne 0, v_\lambda^j \ge 0$ *and* ${I_\lambda^\infty}'(v_\lambda^j)=0$ *for* $1\le j\le k$,
- (iii) $c_\lambda = I_\lambda(u_\lambda) + \sum_{j=1}^k I_\lambda^\infty(v_\lambda^j)$,
- (iv) $\|u_n - u_0 - \sum_{j=1}^k v_\lambda^j(\cdot - y_n^j)\| \to 0$.

where $|y_n^j|\to\infty$ and $|y_n^i - y_n^j|\to\infty$ as $n\to\infty$ for any $i\ne j$.

Proof. For $\lambda\in[1/2,1]$, let $\{u_n\}\subset H^s(\mathbb{R}^N), u_n\ge 0$ be given in Lemma 3.5. Since $\{u_n\}$ is bounded, there exist a subsequence denoted by $\{u_n\}$ and $u_\lambda \in H^s(\mathbb{R}^N)$ satisfying $u_n\to u_\lambda$ weakly in $H^s(\mathbb{R}^N)$ and $u_n \to u_\lambda$ a.e. in $\mathbb{R}^N$. It is not hard to verify that $I'_\lambda(u_\lambda)=0$.

Step 1. Let $v_n^1 = u_n - u_\lambda$. If $v_n^1\to 0$ strongly in $H^s(\mathbb{R}^N)$, the Proposition holds with $k=0$.

Step 2. We claim that if $v_n^1 \not\to 0$ strongly, then $\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)} |v_n^1|^2>0$. Since $I_\lambda(u_n)\to c_\lambda$, by Lemma 3.3, we have

$$c_\lambda - I_\lambda(u_\lambda) = I_\lambda(v_n^1) + o(1). \tag{3.3}$$

Since $v_n^1 \rightharpoonup 0$, by (A24) and Lemma 2.2, we have

$$I_\lambda^\infty(v_n^1) - I_\lambda(v_n^1) = \int_{\mathbb{R}^N} (V(\infty)-V(x))|v_n^1|^2$$

$$
\begin{aligned}
&= \int_{B_R(0)} (V(\infty)-V(x))|v_n^1|^2 + \int_{\mathbb{R}^N\setminus B_R(0)} (V(\infty)-V(x))|v_n^1|^2 \\
&\to 0.
\end{aligned}
$$

Consequently,

$$
c_\lambda - I_\lambda(u_\lambda) = I_\lambda^\infty(v_n^1) + o(1). \tag{3.4}
$$

Suppose $\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|v_n^1|^2 = 0$. By Lemma 2.3, we have

$$
v_n^1 \to 0 \quad \text{in } L^t(\mathbb{R}^N), \quad \forall t\in(2,2_s^*). \tag{3.5}
$$

Let $f(t) = h(t) + (t^+)^{2_s^*-1}$, from (A20) and (A21), for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$
\Big|\int_{\mathbb{R}^N} H(v_n^1)\Big| \le \varepsilon\Big(\int_{\mathbb{R}^N}|v_n^1|^2 + |v_n^1|^{2_s^*}\Big) + C(\varepsilon)\int_{\mathbb{R}^N}|v_n^1|^r,
$$

where $r < 2_s^*$. Since $v_n^1 \in H^s(\mathbb{R}^N)$, from (3.5), we obtain

$$
\Big|\int_{\mathbb{R}^N} H(v_n^1)\Big| \le \varepsilon C + o(1),
$$

which implies $\int_{\mathbb{R}^N} H(v_n^1) = o(1)$ since ε is small enough. Furthermore, by the Brezis-Lieb lemma, we have

$$
\int_{\mathbb{R}^N}|v_n^1|^{2_s^*} = \int_{\mathbb{R}^N}|u_n|^{2_s^*} - \int_{\mathbb{R}^N}|u_\lambda|^{2_s^*} + o(1).
$$

Thus, (3.3) reduces to

$$
c_\lambda - I_\lambda(u_\lambda) = \frac{1}{2}\|v_n^1\|^2 - \frac{\lambda}{2_s^*}\int_{\mathbb{R}^N}|v_n^1|^{2_s^*} + o(1). \tag{3.6}
$$

Noting that $I'_\lambda(u_n)v_n^1 \to 0$ and $I'_\lambda(u_\lambda)v_n^1 = 0$, by direct calculation, we obtain

$$
\|v_n^1\|^2 - \lambda\int_{\mathbb{R}^N}\left(f(u_n)-f(u_\lambda)\right)v_n^1 = I'_\lambda(u_n)v_n^1 - I'_\lambda(u_\lambda)v_n^1 \to 0.
$$

By Lemma 3.2,

$$
\begin{aligned}
\int_{\mathbb{R}^N}(f(u_n)-f(u))v_n^1 &= \int_{\mathbb{R}^N} f(v_n^1)v_n^1 + o(1)\|v_n^1\| \\
&= \int_{\mathbb{R}^N} h(v_n^1)v_n^1 + \int_{\mathbb{R}^N}|v_n^1|^{2_s^*} + o(1)\|v_n^1\|.
\end{aligned}
$$

By (3.5) and similar argument as above, we have

$$
\int_{\mathbb{R}^N}\left(f(u_n)-f(u_\lambda)\right)v_n^1 = \int_{\mathbb{R}^N}|v_n^1|^{2_s^*} + o(1).
$$

Therefore,

$$
\|v_n^1\|^2 - \lambda\int_{\mathbb{R}^N}|v_n^1|^{2_s^*} = o(1). \tag{3.7}
$$

Combining (3.6) with (3.7), we obtain $c_\lambda - I_\lambda(u_\lambda) = \frac{s}{N}\|v_n^1\|^2 + o(1)$.

Noting that $I'_\lambda(u_\lambda) = 0$, from Pohozăev identity (3.2) and Sobolev embedding theorem, we obtain

$$
\begin{aligned}
&I_\lambda(u_\lambda) \\
&= \frac{s}{N}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\lambda|^2 - \frac{1}{2N}\int_{\mathbb{R}^N}\langle\nabla V(x),x\rangle u_\lambda^2
\end{aligned}
$$

$$\geq \frac{s}{N}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\lambda|^2-\frac{1}{2NS_s}\|\max\{\langle\nabla V(x),x\rangle,0\}\|_{L^{\frac{N}{2s}}}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\lambda|^2.$$

Condition (A25) implies $I_\lambda(u_\lambda)\geq 0$. Thus $c_\lambda-I_\lambda(u_\lambda)<\frac{s}{N\lambda^{\frac{N-2s}{2s}}}S_s^{\frac{N}{2s}}$. On the other hand, since $v_n^1\not\to 0$ strongly, there exists a constant $l>0$ such that $\|v_n^1\|^2\to l$. Set $\|(-\Delta)^{s/2}v_n^1\|_{L^2}^2=\tilde{l}<l$, then

$$S_s=\inf_{u\in H^s(\mathbb{R}^N),u\neq 0}\frac{\|(-\Delta)^{s/2}u\|_{L^2}^2}{\|u\|_{L^{2_s^*}}^2}\leq\frac{\tilde{l}}{(\frac{l}{\lambda})^{\frac{2}{2_s^*}}}\leq l^{\frac{2s}{N}}\lambda^{\frac{N-2s}{N}}.$$

So we have $l\geq\frac{S_s^{\frac{N}{2s}}}{\lambda^{\frac{N-2s}{2s}}}$. Consequently, $c_\lambda-I_\lambda(u_\lambda)\geq\frac{s}{N\lambda^{\frac{N-2s}{2s}}}S_s^{\frac{N}{2s}}$, which is a contradiction. The claim is true.

Step 3. From the argument in step 2, if $v_n\rightharpoonup 0$, then

$$\lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|v_n^1|^2>0.$$

Thus, after extracting a subsequence if necessary, there exist $\{z_n^1\}\subset\mathbb{R}^N$ and $v_\lambda^1\in H^s(\mathbb{R}^N)$ such that $|z_n^1|\to\infty$ and

(i) $\lim_{n\to\infty}\int_{B_1(z_n^1)}|v_n^1|^2>0$,
(ii) $v_n^1(\cdot+z_n^1)\rightharpoonup v_\lambda^1\neq 0$,
(iii) $I_\lambda^{\infty\prime}(v_\lambda^1)=0$.

Clearly (i), (ii) are standard and the point is to show (iii). Set $u_n^1=v_n^1(\cdot+z_n^1)$. To prove $I_\lambda^{\infty\prime}(v_\lambda^1)=0$, it suffices to prove $I_\lambda^{\infty\prime}(u_n^1)\to 0$. For any $\varphi\in C_0^\infty(\mathbb{R}^N)$, from $I_\lambda'(v_n^1)\to 0$, we have

$$\begin{aligned}
&I_\lambda'(v_n^1)\varphi(\cdot-z_n^1)\\
&=\int_{\mathbb{R}^N}(-\Delta)^{s/2}v_n^1(x+z_n^1)(-\Delta)^{s/2}\varphi(x)dx+\int_{\mathbb{R}^N}V(x+z_n^1)v_n^1(x+z_n^1)\varphi(x)dx\\
&\quad-\int_{\mathbb{R}^N}f(v_n^1(x+z_n^1))\varphi(x)dx\\
&=\int_{\mathbb{R}^N}(-\Delta)^{s/2}u_n^1(x)(-\Delta)^{s/2}\varphi(x)dx+\int_{\mathbb{R}^N}V(x+z_n^1)u_n^1(x)\varphi(x)dx\\
&\quad-\int_{\mathbb{R}^N}f(u_n^1(x))\varphi(x)dx\to 0.
\end{aligned}$$

Since $|z_n^1|\to\infty$ and $\varphi\in C_0^\infty(\mathbb{R}^N)$, by (A24), we obtain

$$\int_{\mathbb{R}^N}V(x+z_n^1)u_n^1(x)\varphi(x)dx\to\int_{\mathbb{R}^N}V(\infty)u_n^1(x)\varphi(x)dx.$$

Thus,

$$\begin{aligned}
I_\lambda^{\infty\prime}(u_n^1)&=\int_{\mathbb{R}^N}(-\Delta)^{s/2}u_n^1(x)(-\Delta)^{s/2}\varphi(x)dx+\int_{\mathbb{R}^N}V(\infty)u_n^1(x)\varphi(x)dx\\
&\quad-\int_{\mathbb{R}^N}f(u_n^1(x))\varphi(x)dx\to 0.
\end{aligned}$$

Then we obtain $I_\lambda^{\infty\prime}(v_\lambda^1)=0$ since $u_n^1\rightharpoonup v_\lambda^1$. On the other hand, from (3.4), it is easy to see that $c_\lambda-I_\lambda(u_\lambda)=I_\lambda^\infty(u_n^1)+o(1)$.

So, we obtain a bounded sequence $\{u_n^1\}$ with $u_n^1 \rightharpoonup v_\lambda^1 \neq 0$ satisfying

$$I_\lambda^\infty(u_n^1) \to c_\lambda - I_\lambda(u_\lambda), \quad {I_\lambda^\infty}'(u_n^1) \to 0, \quad {I_\lambda^\infty}'(v_\lambda^1) = 0.$$

Let $v_n^2 = u_n^1 - v_\lambda^1$. Then $u_n = u_\lambda + v_\lambda^1(\cdot - z_n^1) + v_n^2(\cdot - z_n^1)$. If $v_n^2 \to 0$ strongly in $H^s(\mathbb{R}^N)$, we have

$$c_\lambda - I_\lambda(u_\lambda) = I_\lambda^\infty(v_\lambda^1),$$
$$\|u_n - u_\lambda - v_\lambda^1(\cdot - z_n^1)\| \to 0.$$

If $v_n^2 \not\to 0$ strongly, similarly as (3.3) and (3.4), we have

$$c_\lambda - I_\lambda(u_\lambda) - I_\lambda^\infty(v_\lambda^1) = I_\lambda^\infty(v_n^2) + o(1), {I_\lambda^\infty}'(v_n^2) \to 0.$$

By the same argument as step 2, we obtain $\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |v_n^2|^2 > 0$. Then, there exist $\{z_n^2\} \subset \mathbb{R}^N$ and $v_\lambda^2 \neq 0$ such that $|z_n^2| \to \infty$ and

(i) $\lim_{n\to\infty} \int_{B_1(z_n^2)} |v_n^1|^2 > 0$,
(ii) $v_n^2(\cdot + z_n^2) \rightharpoonup v_\lambda^2$,
(iii) ${I_\lambda^\infty}'(v_\lambda^2) = 0$.

Set $u_n^2 = v_n^2(\cdot + z_n^2)$. Then, $\{u_n^2\}$ is a bounded sequence satisfying $u_n^2 \rightharpoonup v_\lambda^2$ and

$$I_\lambda^\infty(u_n^2) \to c_\lambda - I_\lambda(u_\lambda) - I_\lambda^\infty(v_\lambda^1), \quad {I_\lambda^\infty}'(u_n^2) \to 0.$$

Let $v_n^3 = u_n^2 - v_\lambda^2$. Then $u_n = u_\lambda + v_\lambda^1(\cdot - z_n^1) + v_\lambda^2(\cdot - z_n^1 - z_n^2) + v_n^3(\cdot - z_n^1 - z_n^2)$. If $v_n^3 \to 0$ strongly in $H^s(\mathbb{R}^N)$, we have

$$c_\lambda = I_\lambda(u_\lambda) + I_\lambda^\infty(v_\lambda^1) + I_\lambda^\infty(v_\lambda^2),$$
$$\|u_n - u_\lambda - v_\lambda^1(\cdot - z_n^1) - v_\lambda^2(\cdot - z_n^1 - z_n^2)\| \to 0.$$

Otherwise, we repeat the procedure above. From Lemma 3.9, we can terminate our arguments by repeating the above proof by finite k steps. That is, let $y_n^j = \sum_{i=1}^j z_n^i$, then

$$c_\lambda = I_\lambda(u_\lambda) + \sum_{j=1}^k I_\lambda^\infty(v_\lambda^j),$$
$$\|u_n - u_\lambda - \sum_{j=1}^k v_\lambda^j(\cdot - y_n^j)\| \to 0.$$

Step 4. Now, we show that after extracting a subsequence of $\{y_n^j\}$ and redefining $\{v_\lambda^j\}$ if necessary, (iii), (iv) hold for $|y_n^j| \to \infty$ and $|y_n^i - y_n^j| \to \infty$ as $n \to \infty$ for any $i \neq j$. Let $A = \{1, 2, \cdots, k\}$. From $u_n - u_\lambda - \sum_{j=1}^k v_\lambda^j(\cdot - y_n^j) \to 0$ and $u_n \to u_\lambda$ a.e. in $\mathbb{R}^N$, we obtain that $\sum_{j=1}^k v_\lambda^j(\cdot - y_n^j) \to 0$ a.e. in $\mathbb{R}^N$. Since $v_\lambda^j \geq 0$ for any j, it follows that $|y_n^j| \to \infty$. For y_n^i, assume $A_i = \{y_n^j : |y_n^i - y_n^j| \text{ is bounded for } n\}$, then up to a sequence, there exists some $\tilde{v}_\lambda^i \in H^s(\mathbb{R}^N)$ such that $\sum_{j\in A_i} v_\lambda^j(\cdot + y_n^i - y_n^j) \to \tilde{v}_\lambda^i$ strongly in $H^s(\mathbb{R}^N)$. Then $\|u_n - u_\lambda - \tilde{v}_\lambda^i(\cdot - y_n^i) - \sum_{j\in(A\setminus A_i)} v_\lambda^j(\cdot - y_n^j)\| \to 0$. Since $v_\lambda^j (j \in A)$ is the critical point of ${I_\lambda^\infty}'$, we have ${I_\lambda^\infty}'(\tilde{v}_\lambda^i) = 0$. Then we redefine $v_\lambda^i := \tilde{v}_\lambda^i$, and then $\|u_n - u_\lambda - \sum_{j\in(A\setminus A_i)\cup\{i\}} v_\lambda^j(\cdot - y_n^j)\| \to 0$ holds as $n \to \infty$. By repeating the argument above at most $(k-1)$ times and redefining $\{v_\lambda^j\}$ if necessary, there exists $\Lambda \subset A$ such that

$$|y_n^j| \to \infty, \quad |y_n^i - y_n^j| \to \infty, \quad \forall i \neq j, \quad n \to \infty,$$

$$\|u_n - u_\lambda - \sum_{j\in\Lambda} v_\lambda^j(\cdot - y_n^j)\| \to 0.$$

The proof is complete. □

If $V(x) \equiv V > 0$, we can get the similar decomposition of the BPS sequence for the autonomous problem (1.2). Denote the energy functional of autonomous problem (1.2) and auxiliary energy functional by J and J_λ ($\lambda \in [1/2, 1]$) respectively. Let c_λ be the MP value for J_λ, then we have the following result.

Corollary 3.11. *Assume $V(x) \equiv V > 0$ and (A20)–(A22) hold. For $\lambda \in [1/2, 1]$, if $\{u_n\} \subset H^s(\mathbb{R}^N)$ is a sequence such that $u_n \geq 0, \|u_n\| < \infty, J_\lambda(u_n) \to c_\lambda$ and $J'_\lambda(u_n) \to 0$, furthermore $c_\lambda < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}$. Then there exist a subsequence of $\{u_n\}$, an integer $l \in \mathbb{N} \cup \{0\}$ and $w_\lambda^j \in H^s(\mathbb{R}^N)$ for $1 \leq j \leq l$ such that*

- (i) *$u_n \to u_\lambda$ weakly in $H^s(\mathbb{R}^N)$ with $J'_\lambda(u_\lambda) = 0$,*
- (ii) *$w_\lambda^j \neq 0, w_\lambda^j \geq 0$ and $J'_\lambda(w_\lambda^j) = 0$ for $1 \leq j \leq l$,*
- (iii) *$c_\lambda = J_\lambda(u_\lambda) + \sum_{j=1}^l J_\lambda(w_\lambda^j)$,*
- (iv) *$\|u_n - u_0 - \sum_{j=1}^l w_\lambda^j(\cdot - y_n^j)\| \to 0$,*

where $|y_n^j| \to \infty$ and $|y_n^i - y_n^j| \to \infty$ as $n \to \infty$ for any $i \neq j$.

The proof of the above corollary is similar to Proposition 3.10, we omit it here. Now, we complete the proof of the existence of solutions of the auxiliary problems (3.1).

Lemma 3.12. *Assume (A20)–(A25) hold. If $\max\{2, 2_s^* - 2\} < p < 2_s^*$, then for almost every $\lambda \in [1/2, 1]$, I_λ has a positive critical point u_λ satisfying $\|u_\lambda\| \geq \delta$ where $\delta > 0$ independent of λ.*

Proof. From Lemmas 3.5 and 3.6, there exists a bounded sequence $\{u_n\} \subset H^s(\mathbb{R}^N)$, $u_n \geq 0$ and $0 < c_\lambda < \frac{s}{N\lambda^{\frac{N-2s}{2s}}} S_s^{\frac{N}{2s}}$, such that

$$I_\lambda(u_n) \to c_\lambda, \quad I'_\lambda(u_n) \to 0.$$

Then $u_n \to u_\lambda \geq 0$ weakly in $H^s(\mathbb{R}^N)$. It is obvious that u_λ is a critical point of I_λ.

Now, we claim $u_\lambda \neq 0$. If $u_\lambda = 0$, from Proposition 3.10, we can deduce that $k > 0$ since $c > 0$, and

$$c_\lambda = \sum_{j=1}^k I_\lambda^\infty(v_\lambda^j) \geq m_\lambda^\infty := \inf\{I_\lambda^\infty(u) : u \in H^s(\mathbb{R}^N), u \neq 0, {I_\lambda^\infty}'(u) = 0\}, \tag{3.8}$$

where ${I_\lambda^\infty}'(v_\lambda^j) = 0 (j = 1, 2, \dots, k)$. On the other hand, we infer that

$$c_\lambda < m_\lambda^\infty, \tag{3.9}$$

which is contradictory to (3.8) and then the claim is true.

From Lemma 3.8, we let v_λ be the least energy solution of

$$(-\Delta)^s u + V(\infty) u = \lambda f(u).$$

By Lemma 3.7, there exists $\gamma_\lambda(t)$ satisfying $\gamma_\lambda(0) = 0, I_\lambda^\infty(\gamma_\lambda(1)) < 0, v_\lambda \in \gamma_\lambda[0, 1]$ and

$$\max_{t\in[0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = I_\lambda^\infty(v_\lambda) = m_\lambda^\infty.$$

By (A24), we have

$$I_\lambda(\gamma_\lambda(t)) < I_\lambda^\infty(\gamma_\lambda(t)), \quad \forall t \in [0,1],$$

and from the definition of c_λ it follows that

$$c_\lambda \le \max_{t\in[0,1]} I_\lambda(\gamma_\lambda(t)) < \max_{t\in[0,1]} I_\lambda^\infty(\gamma_\lambda(t)) = m_\lambda^\infty.$$

Since (A20) and (A21) hold, by the same argument as that in Lemma 3.9, there exists a constant $\delta_0 > 0$ independent of λ such that $\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2 \ge \delta_0$ since $u_\lambda \neq 0$. Thus, there exists a $\delta > 0$ independent of λ such that $\|u_\lambda\| \ge \delta$. The proof is complete. □

4. Proof of Theorem 1.1

Lemma 3.12 shows that for almost every $\lambda \in [1/2, 1]$, $I_\lambda(u)$ has a positive critical point u_λ. Thus we obtain a critical point sequence $\{u_\lambda\}$ satisfying $I'_\lambda(u_\lambda) = 0$. In the following, we first show that $\{u_\lambda\}$ is a BPS sequence of I and then prove the convergence of $\{u_\lambda\}$ as $\lambda \to 1$. By analyzing the properties of minimizing sequence, we complete the proof of the existence of ground state solutions of (1.2). First, we show the uniform boundedness of $\{u_\lambda\}$.

Proposition 4.1. *Assume* (A20)–(A25) *hold. If* $\max\{2, 2_s^* - 2\} < p < 2_s^*$*, then* $\{u_\lambda\}$ *is bounded uniformly and there exists* $\delta > 0$ *independent of* λ *such that* $I_\lambda(u_\lambda) \ge \delta$.

Proof. Since u_λ is the critical point of $I_\lambda(u)$, from the Pohozǎev identity (3.2), we have

$$I_\lambda(u_\lambda) = \frac{s}{N}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2 - \frac{1}{2N}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_\lambda|^2. \tag{4.1}$$

From Proposition 3.10, $I_\lambda(u_\lambda) \le c_\lambda \le c_{1/2}$ for any $\lambda \in [1/2, 1]$. By the Hölder inequality and Sobolev embedding theorem,

$$\begin{aligned}
\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2 &= \frac{N}{s} I_\lambda(u_\lambda) + \frac{1}{2s}\int_{\mathbb{R}^N} \langle \nabla V(x), x\rangle |u_\lambda|^2 \\
&\le \frac{N}{s} c_{\frac12} + \frac{1}{2sS_s}\| \max\{\langle \nabla V(x), x\rangle, 0\}\|_{L^{\frac{N}{2s}}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2.
\end{aligned}$$

Condition (A25) implies that $\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2$ is bounded uniformly independent of λ. Next, we show that $\|u_\lambda\|_{L^2}$ is bounded uniformly independent of λ. From $I'_\lambda(u_\lambda)u_\lambda = 0$, we have $\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2 + \int_{\mathbb{R}^N} V(x)|u_\lambda|^2 = \lambda \int_{\mathbb{R}^N} f(u_\lambda)u_\lambda$. Then, by (A20) and (A21),

$$\begin{aligned}
V_0 \int_{\mathbb{R}^N} |u_\lambda|^2 &\le \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2 + \int_{\mathbb{R}^N} V(x)|u_\lambda|^2 \\
&\le \lambda\varepsilon \int_{\mathbb{R}^N} |u_\lambda|^2 + \lambda C(\varepsilon)\int_{\mathbb{R}^N} |u_\lambda|^{2_s^*} \\
&\le \varepsilon \int_{\mathbb{R}^N} |u_\lambda|^2 + C(\varepsilon)\Big| \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_\lambda|^2\Big|^{2_s^*/2}.
\end{aligned}$$

Therefore, $\|u_\lambda\|_{L^2}$ is bounded uniformly. Now, we prove that $I_\lambda(u_\lambda) \ge \delta > 0$. From Lemma 3.12, there exists $\delta_0 > 0$ independent of λ such that $\|u_\lambda\| \ge \delta_0$. On the other hand,

$I_\lambda(u_\lambda)$

$$\begin{aligned}&\geq \frac{s}{N}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\lambda|^2-\frac{1}{2N}\int_{\mathbb{R}^N}\max\{\langle\nabla V(x),x\rangle,0\}|u_\lambda|^2\\&\geq \frac{s}{N}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\lambda|^2-\frac{1}{2NS_s}\|\max\{\langle\nabla V(x),x\rangle,0\}\|_{L^{\frac{N}{2s}}}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_\lambda|^2.\end{aligned}$$

Condition (A25) implies that there exists $\delta>0$ independent of λ such that

$$I_\lambda(u_\lambda)\geq\delta \tag{4.2}$$

The proof is complete. □

In the following, we denote u_λ by u_{λ_j} and let $\lambda_j\to 1$ as $j\to\infty$.

Lemma 4.2. *Assume* (A20)–(A25) *hold, if* $\max\{2,2_s^*-2\}<p<2_s^*$*, then the sequence* $\{u_{\lambda_j}\}$ *is a BPS sequence for* I *satisfying* $\limsup_{j\to\infty}I(u_{\lambda_j})\leq c_1$ *and* $\|u_{\lambda_j}\|\not\to 0$.

Proof. From Lemma 3.12, we have $\|u_{\lambda_j}\|\not\to 0$. It follows from Proposition 4.1 that $\|u_{\lambda_j}\|$ is bounded uniformly, and consequently $\int_{\mathbb{R}^N}F(u_{\lambda_j})$ is bounded by (A20) and (A21). Property (iii) in Proposition 3.10 shows that $I_{\lambda_j}(u_{\lambda_j})\leq c_{\lambda_j}$ for any u_{λ_j}. Thus, from

$$I(u_{\lambda_j})=I_{\lambda_j}(u_{\lambda_j})+(\lambda_j-1)\int_{\mathbb{R}^N}F(u_{\lambda_j}), \tag{4.3}$$

we obtain $\limsup_{j\to\infty}I(u_{\lambda_j})\leq c_1$ and $I'(u_{\lambda_j})\to 0$. □

Completion of the proof of the Theorem 1.1. From Lemma 4.2, inequality (4.2), and (4.3), there exists a subsequence still denoted by $\{u_{\lambda_j}\}$ satisfying

(i) $\{u_{\lambda_j}\}$ is bounded,
(ii) $I(u_{\lambda_j})\to c\leq c_1$,
(iii) $I'(u_{\lambda_j})\to 0$,

where $c>0$. That is to say, there exists a BPS sequence $\{u_{\lambda_j}\}$ satisfying the assumptions of Lemma 3.12 for $\lambda=1$. Thus, there exists a nontrivial critical point u_0 for I satisfying $I(u_0)\leq c_1$.

Next, we show the existence of a ground state solution. Let

$$m=\inf\{I(u):u\in H^s(\mathbb{R}^N),u\neq 0,I'(u)=0\}.$$

Obviously, $m\leq I(u_0)\leq c_1=\frac{s}{N}S_s^{\frac{N}{2s}}$. Set $\{u_n\}$ be a sequence of nontrivial critical points of I satisfying $I(u_n)\to m$. Since $I(u_n)$ is bounded, similar proof as that in Proposition 4.1 for $\lambda=1$, we obtain that $\{u_n\}$ is bounded uniformly and there exists $\delta>0$ such that $I(u_n)\geq\delta>0$. Thus $m>0$. So, $\{u_n\}$ is a BPS sequence satisfying the following conditions,

(i) $\{u_n\}$ is bounded,
(ii) $I(u_n)\to m\leq c_1$,
(iii) $I'(u_n)=0$,

From Proposition 3.10, there exists $\tilde u$ such that $I'(\tilde u)=0$ and $I(\tilde u)\leq m$.

Now, we claim $\tilde u\neq 0$. Otherwise, $\tilde u=0$. Then, by Proposition 3.10, we have

$$m=\sum_{j=1}^k I^\infty(w^j)\geq m^\infty:=\inf\{I^\infty(u):u\in H^s(\mathbb{R}^N),u\neq 0,{I^\infty}'(u)=0\}$$

for $k > 0$ and $w^j (j = 1, 2, \dots, k)$ are the critical points of I^∞. On the other hand, similar argument as that in Lemma 3.12, there exists $\gamma(t)$ such that

$$\max_{t\in[0,1]} I^\infty(\gamma(t)) = m^\infty.$$

From the definition of c_1, we obtain $m \le c_1 \le \max_{t\in[0,1]} I(\gamma(t))$. By (A24), we obtain

$$m \le c_1 < m^\infty,$$

which is a contradiction. Thus, the claim is true. Then $I(\tilde{u}) \ge m$ since $I'(\tilde{u}) = 0$ and $\tilde{u} \ne 0$. So, there exists a critical point $\tilde{u} \ne 0$ such that $I(\tilde{u}) = m$. The proof is complete. □

Acknowledgements. This work is supported by the National Natural Science Foundation of China (11271364).

References

[1] G. Alberti, G. Bouchitté, P. Seppecher; *Phase transition with the line-tenstion effect.* Arch. Ration. Mech. Anal., 144 (1998), 1-46.
[2] C. O. Alves, O. H. Miyagaki; *Existence and concentration of solution for a class of fractional elliptic equation in* $\mathbb{R}^N$ *via penalization method*, Calc. Var. Partial Differential Equations. 55 (2016), 47.
[3] B. Barriosa, E. Coloradoc, R. Servadeid, F. Soriaa; *A critical fractional equation with concave-convex power nonlinearities*, Ann. I. H. Poincaré-AN., 32 (2015), 875-900.
[4] B. Barriosa, E. Colorado, A. de Pablo, U. Sánchez; *On some critical problems for the fractional Laplacian operator*, J. Differential Equations, 252 (2012), 6133-6162.
[5] H. Berestycki, P. Lions; *Nonlinear scalar field equations I. Existence of a ground state*, Arch. Ration. Mech. Anal., 82 (1990), 90-117.
[6] G. M. Bisci, V. Radulescu; *Ground state solutions of scalar field fractional Schrödinger equation*, Calc. Var. Partial Differential Equatioans, 54(2015) , 2985-3008.
[7] X. Cabré, J. Tan; *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math., 224 (2010), 2052-2093.
[8] L. A. Caffarelli, S. Salsa, L. Silvestre; *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math., 171(2) (2008), 425-461.
[9] L. Caffarelli, L. Silvestre; *An extension problem related to the fractional Laplacian*, Commun. PDEs, 32 (2007), 1245-1260.
[10] X. J. Chang, Z. Q. Wang; *Nodal and multiple solutions of nonlinear problems involving the fractional Laplacian.* J. Differential Equations, 256 (2014), 2965-2992.
[11] X. J. Chang, Z. Q. Wang; *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity*, Nonlinearity, 26 (2013), 479-494.
[12] G. Chen, Y. Zheng; *Concentration phenomena for fractional nonlinear Schrödinger equations*, Commun. Pure Appl. Anal., 13 (2014), 2359-2376.
[13] A. Cotsiolis, N. K. Tavoularis; *Best constants for Sobolev inequalities for higher order fractional derivatives*, J. Math. Anal. Appl., 295 (2004), 225-236.
[14] J. Dávila, M. del Pino, J. Wei; *Concentrating standing waves for the fractional nonlinear Schröding equation*, J. Differ. Equ., 256 (2) (2014), 858-892.
[15] M. M. Fall, F. Mahmoudi, E. Valdinoci; *Ground states and concentration phenomena for the fractional Schrödinger equations*, Nonlinearity, 28 (2015), 1937-1961.
[16] Q. Y. Guan, Z. M. Ma; *Boundary problems for fractional Laplacians*, Stoch. Dyn., 593 (2005), 385-424.
[17] X. M. He, W. M. Zou; *Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities*, Calc. Var. Partial Differential Equations, 55(4) (2016) :91.
[18] L. Jeanjean; *On the existence of bounded Palais-Smale sequence and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh, 129A (1999), 787-809.

[19] L. Jeanjean, K. Tanaka; *A Positive solution for a nonlinear Schrödinger equation on* $\mathbb{R}^N$, Indiana. Univ. Math. J., 54 (2005), 443-464.
[20] L. Jeanjean, K. Tanaka; *A remark on least energy solutions in* $\mathbb{R}^{\mathbb{N}}$, Proc. Amer. Math. Soc., 13 (2002), 2399-2408.
[21] N. Laskin; *Fractional Schrödinger equation*, Phys. Rev. E, 66 (2002), 05618.
[22] N. Laskin; *Fractional quantum mechanics*, Phys. Rev. E, 62 (2000), 3135.
[23] N. Laskin; *Fractional quantum mechanics ans Lévy path integrals*, Phys. Lett. A, 268 (4-6) (2000), 298-305.
[24] P. L. Lions; *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Analysis, 49 (1982), 315-334.
[25] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchiker's guide to the fractional Sobolev space*, Bull. Sci. Math., 136 (2012), 521-573.
[26] S. Secchi; *Ground state solutions for nonlinear fractional Schröding equations in* $\mathbb{R}^N$, J. Math. Phys., 54 (2013), 031501.
[27] J. Seok; *Spike-layer solutions to nonlinear fractional Schröding equations with almost optimal nonlinearities*, Electron. J. Differential Equations. 2015 no. 196 (2015), 1-19.
[28] X. D. Shang, J. H. Zhang; *Concentrating solutions of nonlinear fractional Schrödinger equation with potentials*, J. Differ. Equ., 258(4) (2015), 1106-1128.
[29] L. Silvestre; *Regularity of the obstacle problem for a fractional power of the Laplace ooperator*, Comm. Pure Apple. Math., 60(1) (2007), 67-112.
[30] Y. Sire, E. Valdinoci; *Fractional Laplacian phase transtion and boundary reactions: a geometric inequality and a symmetry result*, J. Funct. Anal., 256(6) (2009), 1842-1864.
[31] S. Secchi; *On fractional Schrödinger equations in* $\mathbb{R}^N$ *without the Ambrosetti-Rabinowitz condition*. Topological Methods in Nonlinear Analysis. 47(1) (2016), 19-41.
[32] J. Tan; *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var., 42 (2011), 21-41.
[33] K. M. Teng, X. M. He; *Ground state solutions for fractional Schrödinger equations with critical Sobolev exponent*, Commun. Pure Apple. Anal., 16 (2016), 991-1008.
[34] Y. H. wei, X. F. Su; *Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian*, Calc. Var., 52 (2015), 95-124.
[35] J. Zhang, W. M. Zou; *The critical case for a Berestycki-Lions theorem*, Science China Mathematics, 57 (3) (2014), 541-554.
[36] J. J. Zhang, J. M. do O, M. Squassina; *Schrödinger-Poisson systems with a general critical nonlinearity*. Commun. Contemp. Math. (published online 2016).
[37] J. J. Zhang, J. M. do O, M. Squassina; *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud. 16 (1) (2016), 15-30.
[38] J. J. Zhang, W. M. Zou; *A Berestycki-Lions Theorem revisited*, Commun. Contemp. Math. 14 (5) (2012), 1250033.
[39] J. J. Zhang, W. M. Zou; *Solutions concentrating around the saddle points of the potential for critical schrödinger equations*, Calc. Var. Partial Differential Equations. 54 (4) (2015), 4119-4142.
[40] X. Zhang, B. L. Zhang, D. repovš; *Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials*. Nonlinear Analysis, 142 (2016), 48-68.

Hua Jin
College of Science, China University of Mining and Technology, Xuzhou 221116, China
E-mail address: `huajin@cumt.edu.cn`

Wenbin Liu (corresponding author)
College of Science, China University of Mining and Technology, Xuzhou 221116, China
E-mail address: `liuwenbin-xz@163.com, phone (86-516) 83591530`