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DECAY RATES FOR SOLUTIONS TO THERMOELASTIC BRESSE SYSTEMS OF TYPES I AND III

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Abstract. In this article, we study the energy decay for the thermoelastic Bresse system in the whole line with two dissipative mechanisms, given by heat conduction (Types I and III). We prove that the decay rate of the solutions are very slow. More precisely, we show that the solutions decay with the rate of $(1+t)^{-1/8}$ in the L^2-norm, whenever the initial data belongs to $L^1(\mathbb{R})\cap H^s(\mathbb{R})$ for a suitable s. The wave speeds of propagation have influence on the decay rate with respect to the regularity of the initial data. This phenomenon is known as *regularity-loss*. The main tool used to prove our results is the energy method in the Fourier space.

1. Introduction

In this article, we consider two Cauchy problems related to the Bresse model with two dissipative mechanisms corresponding to the heat conduction coupled to the system. The first of them is the Bresse system with thermoelasticity of Type I:

$$\begin{gathered}
\rho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_1 = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\rho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2x} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1x} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_{1t} - k_1\theta_{1xx} + m_1(\omega_x - l\varphi)_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_{2t} - k_2\theta_{2xx} + m_2\psi_{xt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),
\end{gathered} \tag{1.1}$$

with the initial data

$$(\varphi,\varphi_t,\psi,\psi_t,\omega,\omega_t,\theta_1,\theta_2)(x,0) = (\varphi_0,\varphi_1,\psi_0,\psi_1,\omega_0,\omega_1,\theta_{10},\theta_{20})(x).$$

The second one, is the Bresse system with thermoelasticity of Type III:

$$\begin{gathered}
\rho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_{1t} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\rho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2xt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1xt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_{1tt} - k_1\theta_{1xx} - \alpha_1\theta_{1xxt} + m_1(\omega_x - l\varphi)_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_{2tt} - k_2\theta_{2xx} - \alpha_2\theta_{2xxt} + m_2\psi_{xt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),
\end{gathered} \tag{1.2}$$

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with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_2, \theta_{1t}, \theta_{2t})(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{20}, \theta_{11}, \theta_{21})(x),$$

where α_1, α_2, $\rho_1, \rho_2, \gamma, b, k, k_0, k_1, k_2, l$ m_1 and m_2 are positive constants.

Terms $k_0(\omega_x - l\varphi)$, $k(\varphi - \psi - l\omega)$ and $b\psi_x$ denote the axial force, the shear force and the bending moment, where ω, φ and ψ are the longitudinal, vertical and shear angle displacements, respectively. Furthermore, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_0 = EA$, $k = k'GA$, $b = EI$ and $l = R^{-1}$, where ρ denotes the density, E is the elastic modulus, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section and R is the radius of curvature of the beam. Here, we assume that all the above coefficients are positive constants. In what concerns of the Thermoelastic of type III, we refer the work of Green and Naghdi [9, 11]. They re-examined the classical model of thermoelasticity and introduced the so-called model of thermoelasticity of type III, which the constitutive assumption on the heat flux vector is different from Fourier's law. They developed a model of thermoelasticity that includes temperature gradient and thermal displacement gradient among the constitutive variables and proposed a heat conduction law as

$$q(x,t) = -(\kappa\theta_x(x,t) + \kappa^* v_x(x,t)), \tag{1.3}$$

where $v_t = \theta$ and v is the thermal displacement gradient, κ and κ^* are constants. Combining (1.3) with the energy balance law

$$\rho\theta_t + \varrho\, div\, q = 0, \tag{1.4}$$

lead to the equation

$$\rho\theta_{tt} - \varrho\kappa\theta_{xx} - \varrho\kappa^*\theta_{xx} = 0,$$

which permits propagation of thermal waves at finite speed. The common feature of these theories, is that all of them lead to hyperbolic differential equations and model heat flow as thermal waves traveling at finite speed. More information about mathematical modeling can be found in [3, 11, 15].

The main purpose of this article is to investigate the asymptotic behavior of the solutions to the Cauchy problems (1.1) and (1.2) posed on $\mathbb{R}$. To the best of our knowledge, the stability of the Bresse model does not have any physical explanation when it is considered in the real line. Be that as it may, from mathematical point of view, a considerable number of stability issues concerning the Bresse model in a whole space, have received considerable attention in the last years [10, 24, 25, 28, 31]. This has been due to *the regularity-loss phenomenon* that usually appears in the pure Cauchy problems (for instance, see [5, 6, 12, 13, 14, 32] and references therein). Roughly speaking, the decay rate of the solution is of the *regularity-loss type*, when it is obtained only by assuming some additional order regularity on the initial date. Thus, based on this refinement of the initial data, we investigate the relationship between damping terms, the wave speeds of propagation and their influence on the decay rate of the vector solutions V_1 and V_2 (see (3.1)-(3.2) below) of the systems (1.1) and (1.2), respectively. Thus, our main result reads as follows.

Theorem 1.1. *Let s be a non-negative integer, suppose that $V_j^0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ for $j = 1, 2$. Then, the vector solutions V_j of thermoelastic Bresse problems* (1.1) *and* (1.2)*, respectively, satisfy the following decay estimates:*

(1) *If* $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ *and* $k = k_0$, *then*
$$\|\partial_x^k V_j(t)\|_2 \le C_1(1+t)^{-\frac{1}{8}-\frac{k}{4}}\|V_j^0\|_1 + C_2(1+t)^{-\frac{l}{4}}\|\partial_x^{k+l}V_j^0\|_2, \tag{1.5}$$
for $j = 1, 2$ *and* $t \ge 0$.

(2) *If* $\frac{\rho_1}{\rho_2} \ne \frac{k}{b}$ *or* $k \ne k_0$, *then*
$$\|\partial_x^k V_j(t)\|_2 \le C_1(1+t)^{-\frac{1}{8}-\frac{k}{4}}\|V_j^0\|_1 + C_2(1+t)^{-\frac{l}{6}}\|\partial_x^{k+l}V_j^0\|_2, \tag{1.6}$$
for $j = 1, 2$, *and* $t \ge 0$.

where $k + l \le s$, C_1, C_2 *are two positive constants.*

Our proof is based on some estimates for the Fourier image of the solution as well as a suitable linear combination of series of energy estimates. The key idea is to construct functionals to capture the dissipation of all the components of the vector solution. These functional allows to build an appropriate Lyapunov functionals equivalent to the energy, which gives the dissipation of all the components in the vector $\hat{V}_1^0(\xi, t)$ and $\hat{V}_2^0(\xi, t)$ (See (3.3) below). Finally, we rely on the Plancherel theorem and some asymptotic inequalities to show the desired decay estimates.

The decay rate $(1+t)^{-1/8}$ can be obtained only under the regularity $V_0 \in H^s(\mathbb{R})$. This regularity loss comes to analyze the Fourier image of the solution. Indeed, for $\hat{V}(\xi, t)$, we have (see (2.32), (2.66) and (3.3) below) that
$$|\hat{V}(\xi, t)|^2 \le Ce^{-\beta s(\xi)t}|\hat{V}(\xi, 0)|^2, \tag{1.7}$$
where
$$s(\xi) = \begin{cases} C_1 \frac{\xi^4}{(1+\xi^8)}, & \text{if } \frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ and } k = k_0, \\ C_2 \frac{\xi^4}{(1+\xi^2)(1+\xi^8)}, & \text{if } \frac{\rho_1}{\rho_2} \ne \frac{k}{b} \text{ or } k \ne k_0. \end{cases}$$

As we will see, the decay estimate (1.5)-(1.6) depends in a critical way on the properties of the function $s(\xi)$. Obviously, the function $s(\xi)$ behaves like ξ^4 in the low frequency region ($|\xi| \le 1$) and like ξ^{-4} near infinity whenever $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k = k_0$. Otherwise, if the wave speeds of propagation are different the function $s(\xi)$ behaves also like ξ^4 in the low frequency region and like ξ^{-6} near infinity, which means that the dissipation in the high frequency region is very weak and produces the regularity loss phenomenom. It has been known recently that this regularity loss leads to some difficulties in the nonlinear problems, see [12, 14] for more details.

There are many works on the global existence and asymptotic stability of solutions to the initial boundary value problem for the Bresse system with dissipation. In this direction, we refer the IBVP associated to (1.1) considered by Liu and Rao in [16]. They proved that the exponential decay exists only when the velocities of the wave propagation are the same. If the wave speeds are different, they showed that the energy decays polynomially to zero with the rate $t^{-\frac{1}{2}}$ and $t^{-\frac{1}{4}}$, provided that the boundary conditions are Dirichlet-Neumann-Neumann
$$\omega_x(x, t) = \varphi(x, t) = \psi_x(x, t) = \theta_1(x, t) = \theta_2(x, t) = 0, \quad \text{for } x = 0, l,$$
and Dirichlet-Dirichlet-Dirichlet type,
$$\omega(x, t) = \varphi(x, t) = \psi(x, t) = \theta_1(x, t) = \theta_2(x, t) = 0, \quad \text{for } x = 0, l.$$
An improvement of the above results was made by Fatori and Muñoz Rivera in [8]. They showed that, in general, the Thermoelastic Bresse system of Type I is not

exponentially stable, but there exists polynomial stability with rates that depend on the wave propagation and the regularity of the initial data.

As far as we know, there exist just a few results related to the stability of the pure Cauchy problem to the Bresse model. The decay rate of the solution of the IVP for Bresse system in the whole line has been first studied by Said-Houari and Soufyane in [31]. They considered the system

$$\begin{gathered}
\varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\varphi) = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\psi_{tt} - a^2\psi_{xx} - k(\varphi_x - \psi - l\omega)\gamma_1\psi_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\omega_{tt} - k_0^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) + \gamma_2\omega_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),
\end{gathered} \tag{1.8}$$

and investigated the relationship between the frictional damping terms, the wave speeds of propagation and their influence on the decay rate of the solution. In addition, they showed that the L^2-norm of the solution decays with the rate $(1+t)^{-1/4}$. Later on, the same authors in [28], proved that the vector solution V of the Bresse system damped by heat conduction:

$$\begin{gathered}
\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\varphi) = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\psi_{tt} - a^2\psi_{xx} - k(\varphi_x - \psi - l\omega) + m\theta_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\omega_{tt} - k_0^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) + \gamma\omega_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_t - k_1\theta_{xx} + m\psi_{xt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),
\end{gathered} \tag{1.9}$$

decays with the rate

$$\|\partial_x^k V(t)\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}}\|V_0\|_{L^1} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l}V_0\|_{L^2}, \tag{1.10}$$

for $a=1$, and

$$\|\partial_x^k V(t)\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}}\|V_0\|_{L^1} + C(1+t)^{-\frac{l}{4}}\|\partial_x^{k+l}V_0\|_{L^2}, \tag{1.11}$$

for $a \neq 1$, $k = 1,2,\dots,s-l$. More recently, Said-Houari and Hamadouche [24] studied the decay properties of the Bresse-Cattaneo system

$$\begin{gathered}
\varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\varphi) = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\omega) + m\theta_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\omega_{tt} - k_0^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) + \gamma\omega_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_t + q_x + m\psi_{xt} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\tau_q q_t + \beta q + \theta_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty),
\end{gathered} \tag{1.12}$$

obtaining the same decay rate as the one of the solution for the Bresse-Fourier model (1.9). This fact has been also seen in [26], where the authors investigated the Timoshenko-Cattaneo and Timoshenko-Fourier models and showed the same behavior for the solutions of both systems. Finally, concerning to the Termoelasticity type III (in one-dimensional space), Said-Houari and Hamadouche in [25] have been recently analyzed the system

$$\begin{gathered}
\varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k_0^2 l(\omega_x - l\varphi) = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\omega) + m\theta_{tx} = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\omega_{tt} - k_0^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) + \gamma\omega_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty),\\
\theta_{tt} - k_1\theta_{xx} + \beta\psi_{tx} - k_2\theta_{txx} = 0 \quad \text{in } \mathbb{R}\times(0,\infty).
\end{gathered}$$

They proved that the solution decay with the rate

$$\|\partial_x^k V(t)\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}}\|V_0\|_{L^1} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l}V_0\|_{L^2},$$

for $a = 1$, and

$$\|\partial_x^k V(t)\|_{L^2} \le C(1+t)^{-\frac{1}{12}-\frac{k}{6}}\|V_0\|_{L^1} + C(1+t)^{-\frac{l}{8}}\|\partial_x^{k+l}V_0\|_{L^2},$$

for $a \neq 1$, $k = 1, 2, \dots, s-l$.

This paper is organized as follows: In Section 2, we analyze the ODE system generated by the Fourier transform applies to the Cauchy problem, obtaining appropriate decay estimates for the Fourier image of the solution. Section 3 is dedicated to proof our main result.

2. Energy method in the Fourier space

In this section, we establish decay rates for the Fourier image of the solutions of thermoelastic Bresse systems. To obtain the estimates of the Fourier image is actually the hardest and technical part. These estimates will play to a crucial role in proving the Theorems 2.11 and 2.23, below.

2.1. **Thermoelastic Bresse system of type I.** Taking Fourier Transform in (1.1), we obtain the ODE system

$$\rho_1\hat{\varphi}_{tt} - ik\xi(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}) - k_0 l(i\xi\hat{\omega} - l\hat{\varphi}) + l\gamma\hat{\theta}_1 = 0 \quad \text{in } \mathbb{R}\times(0,\infty) \tag{2.1}$$

$$\rho_2\hat{\psi}_{tt} + b\xi^2\hat{\psi} - k(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}) + i\gamma\xi\hat{\theta}_2 = 0 \quad \text{in } \mathbb{R}\times(0,\infty) \tag{2.2}$$

$$\rho_1\hat{\omega}_{tt} - ik_0\xi(i\xi\hat{\omega} - l\hat{\varphi}) - kl(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}) + i\gamma\xi\hat{\theta}_1 = 0 \quad \text{in } \mathbb{R}\times(0,\infty) \tag{2.3}$$

$$\hat{\theta}_{1t} + k_1\xi^2\hat{\theta}_1 + m_1(i\xi\hat{\omega} - l\hat{\varphi})_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty) \tag{2.4}$$

$$\hat{\theta}_{2t} + k_2\xi^2\hat{\theta}_2 + im_2\xi\hat{\psi}_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty) \tag{2.5}$$

The energy functional associated to the above system is defined by

$$\begin{aligned}\hat{E}(\xi,t) &= \rho_1|\hat{\varphi}_t|^2 + \rho_2|\hat{\psi}_t|^2 + \rho_1|\hat{\omega}_t|^2 + \frac{\gamma}{m_1}|\hat{\theta}_1|^2 + \frac{\gamma}{m_2}|\hat{\theta}_2|^2 \\ &\quad + b|\xi|^2|\hat{\psi}|^2 + k|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + k_0|i\xi\hat{\omega} - l\hat{\varphi}|^2.\end{aligned} \tag{2.6}$$

Lemma 2.1. *Consider the energy functional $\hat{E}$ associated to the system* (2.1)-(2.5). *Then*

$$\frac{d}{dt}\hat{E}(\xi,t) = -2\gamma\xi^2\big(\frac{k_1}{m_1}|\hat{\theta}_1|^2 + \frac{k_2}{m_2}|\hat{\theta}_2|^2\big). \tag{2.7}$$

Proof. Multiplying (2.1) by $\overline{\hat{\varphi}}_t$, (2.2) by $\overline{\hat{\psi}}_t$, (2.3) by $\overline{\hat{\omega}}_t$, (2.4) by $\frac{\gamma}{m_1}\overline{\hat{\theta}_1}$, and (2.5) by $\frac{\gamma}{m_2}\overline{\hat{\theta}_2}$, adding and taking real part, (2.7) follows. □

We show that the decay rate of the solution will depend on the wave speeds of propagation. More precisely, we analyze two cases: First, we suppose that

$$\frac{\rho_1}{\rho_2} = \frac{k}{b} \quad \text{and} \quad k = k_0.$$

Otherwise, we consider the case when the wave speeds of propagation are different ($\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ or $k \neq k_0$). The proof of our main results in this section (Theorems 2.11 and 2.23 below) are based on the following lemmas:

Lemma 2.2. *The functional*

$$J_1(\xi,t)=\operatorname{Re}(i\rho_2\xi\hat{\psi}_t\overline{\hat{\theta}_2}),$$

satisfies

$$\begin{aligned}&\frac{d}{dt}J_1(\xi,t)+\frac{m_2\rho_2}{2}\xi^2|\hat{\psi}_t|^2\\&\le b|\xi|^3|\hat{\psi}||\hat{\theta}_2|+k|\xi||\hat{\theta}_2||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|+C_1(1+\xi^2)\xi^2|\hat{\theta}_2|^2,\end{aligned}\tag{2.8}$$

where C_1 is a positive constant.

Proof. Multiplying (2.5) by $-i\rho_2\xi\overline{\hat{\psi}_t}$ and taking real part, we obtain

$$\frac{d}{dt}\operatorname{Re}(-i\rho_2\xi\overline{\hat{\psi}_t}\hat{\theta}_2)+\operatorname{Re}(i\rho_2\xi\overline{\hat{\psi}_{tt}}\hat{\theta}_2)-\operatorname{Re}(ik_2\rho_2\xi^3\overline{\hat{\psi}_t}\hat{\theta}_2)+m_2\rho_2\xi^2|\hat{\psi}_t|^2=0.$$

By (2.2), we have

$$\begin{aligned}&\frac{dRe}{dt}(-i\rho_2\xi\overline{\hat{\psi}_t}\hat{\theta}_2)+m_2\rho_2\xi^2|\hat{\psi}_t|^2\\&\le k_2\rho_2|\xi|^3|\hat{\psi}_t||\hat{\theta}_2|+b|\xi|^3|\hat{\psi}||\hat{\theta}_2|+k|\xi||\hat{\theta}_2||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|+\gamma\xi^2|\hat{\theta}_2|^2\end{aligned}$$

Applying Young's inequality, we obtain (2.8). □

Lemma 2.3. *The functional*

$$T_1(\xi,t)=\operatorname{Re}\big(-\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}-\frac{\rho_1}{m_1}\hat{\varphi}_t\overline{\hat{\theta}_1}\big)$$

satisfies

$$\begin{aligned}&\frac{d}{dt}T_1(\xi,t)+\frac{k_0l}{2}|i\xi\hat{\omega}-l\hat{\varphi}|^2\\&\le\frac{\rho_1k_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_1|-\operatorname{Re}(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\&\quad+\frac{k}{m_1}|\xi||\hat{\theta}_1||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|+C_2|\hat{\theta}_1|^2,\end{aligned}\tag{2.9}$$

where C_2 is a positive constant.

Proof. Multiplying (2.1) by $-\overline{(i\xi\hat{\omega}-l\hat{\varphi})}$ and taking real part, we have

$$\begin{aligned}&\frac{d}{dt}\operatorname{Re}(-\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+\operatorname{Re}(\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}_t)\\&+\operatorname{Re}\Big(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)+k_0l|i\xi\hat{\omega}-l\hat{\varphi}|^2-\operatorname{Re}\Big(l\gamma\hat{\theta}_1\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)=0.\end{aligned}$$

Inequality (2.4) implies

$$\begin{aligned}&\frac{dRe}{dt}\Big(-\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)-\frac{\rho_1}{m_1}\operatorname{Re}(\hat{\varphi}_t\overline{\hat{\theta}}_{1t})-\frac{\rho_1k_1}{m_1}\operatorname{Re}(\xi^2\hat{\varphi}_t\overline{\hat{\theta}}_1)\\&+\operatorname{Re}(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+k_0l|i\xi\hat{\omega}-l\hat{\varphi}|^2\\&-\operatorname{Re}\Big(l\gamma\hat{\theta}_1\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)=0.\end{aligned}\tag{2.10}$$

On the other hand, multiplying (2.1) by $-\frac{\overline{\hat{\theta}}_1}{m_1}$ and taking real part, it follows that

$$\begin{aligned}&\frac{dRe}{dt}(-\frac{\rho_1}{m_1}\hat{\varphi}_t\overline{\hat{\theta}}_1)+\frac{\rho_1}{m_1}\operatorname{Re}(\hat{\varphi}_t\overline{\hat{\theta}}_{1t})+\operatorname{Re}\Big(\frac{ik}{m_1}\xi\overline{\hat{\theta}}_1(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\Big)\\&+\operatorname{Re}\Big(\frac{k_0l}{m_1}\overline{\hat{\theta}}_1(i\xi\hat{\omega}-l\hat{\varphi})\Big)-\frac{l\gamma}{m_1}|\hat{\theta}_1|^2=0.\end{aligned}\tag{2.11}$$

Adding (2.10) and (2.11),

$$
\begin{aligned}
\frac{d}{dt}T_1(\xi,t)+k_0l|i\xi\hat{\omega}-l\hat{\varphi}|^2 &\le \frac{\rho_1k_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_1|-\operatorname{Re}(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&\quad + l\gamma|\hat{\theta}_1||i\xi\hat{\omega}-l\hat{\varphi}|+\frac{k}{m_1}|\xi||\hat{\theta}_1||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|\\
&\quad + \frac{k_0l}{m_1}|\hat{\theta}_1||i\xi\hat{\omega}-l\hat{\varphi}|+\frac{l\gamma}{m_1}|\hat{\theta}_1|^2,
\end{aligned}
$$

applying Young's inequality, (2.9) follows. □

Lemma 2.4. *The functional*

$$
T_2(\xi,t)=\operatorname{Re}(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}+i\frac{\rho_1}{m_1}\xi\hat{\omega}_t\overline{\hat{\theta}_1}),
$$

satisfies

$$
\begin{aligned}
&\frac{d}{dt}T_2(\xi,t)+\frac{k_0}{2}|\xi|^2|i\xi\hat{\omega}-l\hat{\varphi}|^2\\
&\le \frac{\rho_1k_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_1|+\operatorname{Re}(ikl\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&\quad+\frac{kl}{m_1}|\xi||\hat{\theta}_1||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|+C_3|\xi|^2|\hat{\theta}_1|^2,
\end{aligned}
\tag{2.12}
$$

where C_3 is a positive constant.

Proof. Multiplying (2.3) by $i\xi\overline{(i\xi\hat{\omega}-l\hat{\varphi})}$ and taking real part, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})})-\operatorname{Re}(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}_t)+k_0|\xi|^2|i\xi\hat{\omega}-l\hat{\varphi}|^2\\
&-\operatorname{Re}(ikl\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&\quad-\operatorname{Re}\left(\gamma\xi^2\hat{\theta}_1\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\right)=0,
\end{aligned}
\tag{2.13}
$$

using (2.4), it follows that

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+\frac{\rho_1}{m_1}\operatorname{Re}(i\xi\hat{\omega}_t\overline{\hat{\theta}}_{1t})+\frac{\rho_1k_1}{m_1}\operatorname{Re}(i\xi^3\hat{\omega}_t\overline{\hat{\theta}}_1)\\
&+k_0|\xi|^2|i\xi\hat{\omega}-l\hat{\varphi}|^2-\operatorname{Re}(ikl\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&\quad-\operatorname{Re}(\gamma\xi^2\hat{\theta}_1\overline{(i\xi\hat{\omega}-l\hat{\varphi})})=0.
\end{aligned}
\tag{2.14}
$$

On the other hand, multiplying (2.3) by $\frac{i\xi}{m_1}\overline{\hat{\theta}}_1$ and taking real part,

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(\frac{i\rho_1\xi}{m_1}\hat{\omega}_t\overline{\hat{\theta}}_1)-\frac{\rho_1}{m_1}\operatorname{Re}(i\xi\hat{\omega}_t\overline{\hat{\theta}}_{1t})+\operatorname{Re}(\frac{k_0\xi^2}{m_1}(i\xi\hat{\omega}-l\hat{\varphi})\overline{\hat{\theta}}_1)\\
&-\operatorname{Re}(\frac{ikl\xi}{m_1}(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{\hat{\theta}}_1)-\frac{\gamma}{m_1}|\xi|^2|\hat{\theta}_1|^2=0.
\end{aligned}
\tag{2.15}
$$

Adding (2.14) and (2.15), applying Young's inequality, (2.12) follows. □

Lemma 2.5. *Consider the functional*

$$
J_2(\xi,t):=lT_1(\xi,t)+T_2(\xi,t).
$$

Then, there exists $\delta>0$ such that

$$
\frac{d}{dt}J_2(\xi,t)+k_0\delta|i\xi\hat{\omega}-l\hat{\varphi}|^2\le\frac{\rho_1lk_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_1|+\frac{\rho_1k_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_1|
$$

$$+\frac{2kl}{m_1}|\xi||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}||\hat{\theta}_1|+C_4(1+\xi^2)|\hat{\theta}_1|^2,$$

where C_4 is a positive constant.

Proof. Lemmas (2.3) and (2.4) imply that

$$\begin{aligned}&\frac{d}{dt}J_2(\xi,t)+\frac{k_0}{2}(l^2+\xi^2)|i\xi\hat{\omega}-l\hat{\varphi}|^2\\ &\le \frac{\rho_1 lk_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_1|+\frac{2kl}{m_1}|\xi||\hat{\theta}_1||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|+\frac{\rho_1 k_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_1|+C_4(1+\xi^2)|\hat{\theta}_1|^2.\end{aligned}$$

Note that there exists $\delta>0$ such that $2\delta\le\frac{l^2+\xi^2}{1+\xi^2}$. Thus,

$$\begin{aligned}&\frac{d}{dt}J_2(\xi,t)+k_0\delta(1+\xi^2)|i\xi\hat{\omega}-l\hat{\varphi}|^2\\ &\le \frac{\rho_1 lk_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_1|+\frac{2kl}{m_1}|\xi||\hat{\theta}_1||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|\\ &\quad+\frac{\rho_1 k_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_1|+C_4(1+\xi^2)|\hat{\theta}_1|^2.\end{aligned}\tag{2.16}$$

□

Lemma 2.6. *Consider the functional*

$$J_3(\xi,t)=\mathrm{Re}(-\rho_2\hat{\psi}_t\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}-i\frac{\rho_1 b}{k}\xi\hat{\psi}\overline{\hat{\varphi}_t}).$$

If $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ and $k=k_0$, then

$$\begin{aligned}&\frac{d}{dt}J_3(\xi,t)+\frac{k}{2}|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2\\ &\le \rho_2|\hat{\psi}_t|^2+\rho_2 l\,\mathrm{Re}(\hat{\psi}_t\overline{\hat{\omega}_t})-bl\,\mathrm{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+\frac{bl\gamma}{k}|\xi||\hat{\psi}||\hat{\theta}_1|+C_5|\xi|^2|\hat{\theta}_2|^2.\end{aligned}\tag{2.17}$$

Moreover, if $\frac{\rho_1}{\rho_2}\neq\frac{k}{b}$ or $k\neq k_0$, then

$$\begin{aligned}&\frac{d}{dt}J_3(\xi,t)+\frac{k}{2}|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2\\ &\le \rho_2|\hat{\psi}_t|^2+\rho_2 l\,\mathrm{Re}(\hat{\psi}_t\overline{\hat{\omega}_t})+(\rho_2-\frac{b\rho_1}{k})\,\mathrm{Re}(i\xi\hat{\psi}_t\overline{\hat{\varphi}_t})\\ &\quad-\frac{k_0 bl}{k}\,\mathrm{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+\frac{bl\gamma}{k}|\xi||\hat{\psi}||\hat{\theta}_1|+C_5|\xi|^2|\hat{\theta}_2|^2,\end{aligned}\tag{2.18}$$

where C_5 is a positive constant.

Proof. Multiplying (2.2) by $-\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}$ and taking real part,

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}\Big(-\rho_2\hat{\psi}_t\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big)-\mathrm{Re}(i\rho_2\xi\hat{\psi}_t\overline{\hat{\varphi}_t})-\rho_2|\hat{\psi}_t|^2\\ &-\mathrm{Re}\Big(\rho_2 l\hat{\psi}_t\overline{\hat{\omega}_t}\Big)-\mathrm{Re}(b\xi^2\hat{\psi}\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})})\\ &+k|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2-\mathrm{Re}\Big(i\gamma\xi\hat{\theta}_2\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big)=0.\end{aligned}\tag{2.19}$$

First, suppose that

$$\frac{\rho_1}{\rho_2}=\frac{k}{b}\quad\text{and}\quad k=k_0\,.\tag{2.20}$$

By (2.1) and (2.20),

$$
\begin{aligned}
&\operatorname{Re}\Big(b\xi^2\hat{\psi}\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big)\\
&= b\operatorname{Re}\Big(i\xi\hat{\psi}\overline{i\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big)\\
&= \frac{\rho_1 b}{k}\operatorname{Re}(i\xi\hat{\psi}\overline{\hat{\varphi}_{tt}}) - bl\operatorname{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})}) + \frac{bl\gamma}{k}\operatorname{Re}(i\xi\hat{\psi}\overline{\hat{\theta}_1})\\
&= \rho_2\frac{d}{dt}\operatorname{Re}(i\xi\hat{\psi}\overline{\hat{\varphi}_t}) - \rho_2\operatorname{Re}(i\xi\hat{\psi}_t\overline{\hat{\varphi}_t}) - bl\operatorname{Re}\Big(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big) + \frac{bl\gamma}{k}\operatorname{Re}(i\xi\hat{\psi}\overline{\hat{\theta}_1}).
\end{aligned}
$$

Substituting this in (2.19), we have

$$
\begin{aligned}
&\frac{d}{dt}J_3(\xi,t) + k|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2\\
&\le \rho_2|\hat{\psi}_t|^2 + \operatorname{Re}(\rho_2 l\hat{\psi}_t\overline{\hat{\omega}_t}) - bl\operatorname{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})}) + \frac{bl\gamma}{k}|\xi||\hat{\psi}||\hat{\theta}_1|\\
&\quad + \gamma|\xi||\hat{\theta}_2||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|.
\end{aligned}
$$

Applying Young's inequality, (2.17) follows. Now, suppose that

$$
\frac{\rho_1}{\rho_2} \neq \frac{k}{b} \quad \text{or} \quad k \neq k_0. \tag{2.21}
$$

Proceeding as above, (2.1) implies

$$
\begin{aligned}
\operatorname{Re}\Big(b\xi^2\hat{\psi}\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big) &= \frac{\rho_1 b}{k}\frac{d}{dt}\operatorname{Re}(i\xi\hat{\psi}\overline{\hat{\varphi}_t}) - \frac{\rho_1 b}{k}\operatorname{Re}(i\xi\hat{\psi}_t\overline{\hat{\varphi}_t})\\
&\quad - \frac{k_0}{k}bl\operatorname{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})}) + \frac{bl\gamma}{k}\operatorname{Re}(i\xi\hat{\psi}\overline{\hat{\theta}_1}).
\end{aligned}
$$

Substituting in (2.19) and applying Young's inequality, we obtain (2.18). □

Lemma 2.7. *Let $0<\varepsilon_1<\frac{\rho_2 l^2}{2\rho_1}$ and consider the functional*

$$
J_4(\xi,t) = \operatorname{Re}\big(\frac{\rho_2^2 l^2}{\rho_1}\hat{\psi}_t\overline{\hat{\psi}} - \rho_2 l\hat{\omega}_t\overline{\hat{\psi}}\big).
$$

Then

$$
\begin{aligned}
&\frac{d}{dt}J_4(\xi,t) + b\big(\frac{\rho_2 l^2}{\rho_1} - \frac{\varepsilon_1}{2}\big)\xi^2|\hat{\psi}|^2\\
&\le \frac{\rho_2^2 l^2}{\rho_1}|\hat{\psi}_t|^2 - \rho_2 l\operatorname{Re}(\overline{\hat{\psi}}_t\hat{\omega}_t) + \frac{\rho_2 k_0 l}{\rho_1}\operatorname{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&\quad + C(\varepsilon_1)(|\hat{\theta}_1|^2 + |\hat{\theta}_2|^2),
\end{aligned} \tag{2.22}
$$

where $C(\varepsilon_1)$ is a positive constant.

Proof. Multiplying (2.2) by $\overline{\hat{\psi}}$ and taking real part,

$$
\frac{d}{dt}\operatorname{Re}(\rho_2\hat{\psi}_t\overline{\hat{\psi}}) - \rho_2|\hat{\psi}_t|^2 + b\xi^2|\hat{\psi}|^2 - \operatorname{Re}(k\overline{\hat{\psi}}(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})) + \operatorname{Re}(i\gamma\xi\overline{\hat{\psi}}\hat{\theta}_2) = 0. \tag{2.23}
$$

Then (2.3) implies

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(\rho_2\hat{\psi}_t\overline{\hat{\psi}}) - \frac{d}{dt}\operatorname{Re}\big(\frac{\rho_1}{l}\overline{\hat{\psi}}\hat{\omega}_t\big) + b\xi^2|\hat{\psi}|^2\\
&= \rho_2|\hat{\psi}_t|^2 - \frac{\rho_1}{l}\operatorname{Re}(\overline{\hat{\psi}_t}\hat{\omega}_t) - \frac{k_0}{l}\operatorname{Re}(i\xi\overline{\hat{\psi}}(i\xi\hat{\omega}-l\hat{\varphi})) + \frac{\gamma}{l}\operatorname{Re}(i\xi\overline{\hat{\psi}}\hat{\theta}_1) - \operatorname{Re}(i\gamma\xi\overline{\hat{\psi}}\hat{\theta}_2).
\end{aligned}
$$

Multiplying by $\frac{\rho_2 l^2}{\rho_1}$, we have

$$\begin{aligned}\frac{d}{dt}J_4(\xi,t)+\frac{b\rho_2 l^2}{\rho_1}\xi^2|\hat{\psi}|^2 &= \frac{\rho_2^2 l^2}{\rho_1}|\hat{\psi}_t|^2-\rho_2 l\operatorname{Re}(\overline{\hat{\psi}_t}\hat{\omega}_t)-\frac{\rho_2 k_0 l}{\rho_1}\operatorname{Re}(i\xi\overline{\hat{\psi}}(i\xi\hat{\omega}-l\hat{\varphi}))\\&\quad+\frac{\gamma\rho_2 l}{\rho_1}\operatorname{Re}(i\xi\overline{\hat{\psi}}\hat{\theta}_1)-\frac{\gamma\rho_2 l^2}{\rho_1}\operatorname{Re}(i\xi\overline{\hat{\psi}}\hat{\theta}_2).\end{aligned}$$

Applying Young's inequality and using the wave speeds of propagation, we obtain (2.22). □

Lemma 2.8. *Let* $0<\varepsilon_1<\frac{\rho_2 l^2}{2\rho_1}$ *and consider* $K(\xi,t)=J_3(\xi,t)+J_4(\xi,t)$. *If* $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ *and* $k=k_0$, *then*

$$\begin{aligned}&\frac{d}{dt}K(\xi,t)+(\frac{\rho_2 l^2}{\rho_1}-\varepsilon_1)b\xi^2|\hat{\psi}|^2+\frac{k}{2}|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2\\&\leq \rho_2 s_1|\hat{\psi}_t|^2+C(\varepsilon_1)|\hat{\theta}_1|^2+C(\varepsilon_1)(1+\xi^2)|\hat{\theta}_2|^2.\end{aligned}$$

Moreover, if $\frac{\rho_1}{\rho_2}\neq\frac{k}{b}$ *or* $k\neq k_0$, *then*

$$\begin{aligned}&\frac{d}{dt}K(\xi,t)+(\frac{\rho_2 l^2}{\rho_1}-\varepsilon_1)b\xi^2|\hat{\psi}|^2+\frac{k}{2}|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2\\&\leq \rho_2 s_1|\hat{\psi}_t|^2+(\frac{\rho_2}{\rho_1}-\frac{b}{k})k_0 l\operatorname{Re}(i\xi\hat{\psi}(\overline{i\xi\hat{\omega}-l\hat{\varphi}}))+(\rho_2-\frac{b\rho_1}{k})\operatorname{Re}(i\xi\hat{\psi}_t\overline{\hat{\varphi}_t})\\&\quad+C(\varepsilon_1)|\hat{\theta}_1|^2+C(\varepsilon_1)(1+\xi^2)|\hat{\theta}_2|^2,\end{aligned}$$

where $s_1=\frac{\rho_2 l^2}{\rho_1}+1$.

The above lemma follows from Lemmas 2.6 and 2.7, applying Young's inequality. For the next lemma, it is necessary to observe that

$$\begin{gathered}(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})_t-i\xi\hat{\varphi}_t+\hat{\psi}_t+l\hat{\omega}_t=0\\(i\xi\hat{\omega}-l\hat{\varphi})_t-i\xi\hat{\omega}_t+l\hat{\varphi}_t=0\end{gathered}$$

Lemma 2.9. *Consider the functional*

$$H(\xi,t)=\rho_1\operatorname{Re}\Big((i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{\hat{\omega}_t}\Big)+\rho_1\operatorname{Re}((i\xi\hat{\omega}-l\hat{\varphi})\overline{\hat{\varphi}_t}).$$

If $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ *and* $k=k_0$, *then*

$$\begin{aligned}&\frac{d}{dt}H(\xi,t)+\rho_1 l|\hat{\varphi}_t|^2+\frac{\rho_1 l}{2}|\hat{\omega}_t|^2\\&\leq\frac{\rho_2 k}{2bl}|\hat{\psi}_t|^2+\frac{3kl}{2}|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2+\frac{3k_0 l}{2}|i\xi\hat{\omega}-l\hat{\varphi}|^2+C_6(1+\xi^2)|\hat{\theta}_1|^2.\end{aligned}\tag{2.24}$$

Moreover, if $\frac{\rho_1}{\rho_2}\neq\frac{k}{b}$ *or* $k\neq k_0$, *then*

$$\begin{aligned}&\frac{d}{dt}H(\xi,t)+\rho_1 l|\hat{\varphi}_t|^2+\frac{\rho_1 l}{2}|\hat{\omega}_t|^2\\&\leq\frac{\rho_1}{2l}|\hat{\psi}_t|^2+C_1(k,k_0)|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2\\&\quad+C_2(k,k_0)(1+\xi^2)|i\xi\hat{\omega}-l\hat{\varphi}|^2+C_6(1+\xi^2)|\hat{\theta}_1|^2,\end{aligned}\tag{2.25}$$

where $C_1(k,k_0)$ *and* C_6 *are positive constants.*

Proof. Multiplying (2.1) by $\overline{(i\xi\hat{\omega}-l\hat{\varphi})}$, (2.1) by $\rho_1\overline{\hat{\varphi}_t}$, adding these equalities and taking the real part, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(\rho_1(i\xi\hat{\omega}-l\hat{\varphi})\overline{\hat{\varphi}_t})-\rho_1\operatorname{Re}(i\xi\hat{\omega}_t\overline{\hat{\varphi}_t})+\rho_1 l|\hat{\varphi}_t|^2\\
&-k\operatorname{Re}(i\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&-k_0 l|i\xi\hat{\omega}-l\hat{\varphi}|^2+l\gamma\operatorname{Re}(\hat{\theta}_1\overline{(i\xi\hat{\omega}-l\hat{\varphi})})=0.
\end{aligned}\tag{2.26}
$$

Multiplying (2.3) by $\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}$, (2.1) by $\rho_1\overline{\hat{\omega}_t}$, adding, and taking the real part,

$$
\begin{aligned}
&\frac{d}{dt}\operatorname{Re}\Big(\rho_1(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{\hat{\omega}_t}\Big)-\rho_1\operatorname{Re}(i\xi\hat{\varphi}_t\overline{\hat{\omega}_t})+\rho_1\operatorname{Re}(\hat{\psi}_t\overline{\hat{\omega}_t})+\rho_1 l|\hat{\omega}_t|^2\\
&-kl|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2-k_0\operatorname{Re}\Big(i\xi(i\xi\hat{\omega}-l\hat{\varphi})\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big)\\
&\quad+\gamma\operatorname{Re}\Big(i\xi\hat{\theta}_1\overline{(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})}\Big)=0.
\end{aligned}\tag{2.27}
$$

Adding (2.26) and (2.27),

$$
\begin{aligned}
&\frac{d}{dt}H(\xi,t)+\rho_1 l|\hat{\varphi}_t|^2+\rho_1 l|\hat{\omega}_t|^2\\
&\le kl|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2+k_0 l|i\xi\hat{\omega}-l\hat{\varphi}|^2+\gamma|\xi||\hat{\theta}_1||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|\\
&\quad+\gamma l|\hat{\theta}_1||i\xi\hat{\omega}-l\hat{\varphi}|+|k-k_0||\xi||i\xi\hat{\omega}-l\hat{\varphi}||i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|+\rho_1|\hat{\psi}_t||\hat{\omega}_t|.
\end{aligned}
$$

Applying Young's inequality and using the wave propagation properties, (2.24) and (2.25) follow. $\square$

To prove our main result, we need to establish a functional equivalent to the energy (2.6) in a polynomial sense. In particular, this kind of functional gives some dissipative terms of the vector solution. First, let us define

$$
\mathcal{L}_1(\xi,t)=\begin{cases}
J_1+\varepsilon_2\xi^2K+\xi^2J_2+\varepsilon_3\xi^2H,\\
\quad\text{if } \frac{\rho_1}{\rho_2}=\frac{k}{b} \text{ and } k=k_0,\\
\frac{\xi^2}{(1+\xi^2+\xi^4)}\big(\lambda_1\varepsilon_3J_1+\frac{\xi^2}{(1+\xi^2+\xi^4)}(\varepsilon_3\lambda_2K+J_2+\varepsilon_3H)\big),\\
\quad\text{if } \frac{\rho_1}{\rho_2}\neq\frac{k}{b} \text{ or } k\neq k_0.
\end{cases}\tag{2.28}
$$

where $\lambda_1,\lambda_2,\varepsilon_2,\varepsilon_3$ are positive constants to be determined later.

Proposition 2.10. *There exist constants $M,M'>0$ such that, if $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ and $k=k_0$, then*

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_1(\xi,t)+M\xi^2\big\{b\xi^2|\hat{\psi}|^2+k|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2+\rho_2|\hat{\psi}_t|^2\\
&+k_0|i\xi\hat{\omega}-l\hat{\varphi}|^2+\rho_1|\hat{\varphi}_t|^2+\rho_1|\hat{\omega}_t|^2\big\}\\
&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4+\xi^6)\xi^2|\hat{\theta}_1|^2+C(\varepsilon_1,\varepsilon_2)(1+\xi^2+\xi^4)|\hat{\theta}_2|^2.
\end{aligned}\tag{2.29}
$$

Moreover, if $\frac{\rho_1}{\rho_2}\neq\frac{k}{b}$ or $k\neq k_0$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_1(\xi,t)+M'\frac{\xi^4}{(1+\xi^2+\xi^4)^2}\big\{b\xi^2|\hat{\psi}|^2+k|i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}|^2+\rho_2|\hat{\psi}_t|^2\\
&+k_0|i\xi\hat{\omega}-l\hat{\varphi}|^2+\rho_1|\hat{\varphi}_t|^2+\rho_1|\hat{\omega}_t|^2\big\}\\
&\le C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)(1+\xi^2)\xi^2(|\hat{\theta}_1|^2+|\hat{\theta}_2|^2)
\end{aligned}\tag{2.30}
$$

Proof. First, we suppose that $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k = k_0$. By using Lemmas 2.2 and 2.8, we have

$$
\begin{aligned}
&\frac{d}{dt}\{J_1(\xi,t) + \varepsilon_2\xi^2 K(\xi,t)\} + b\varepsilon_2(\frac{\rho_2 l^2}{\rho_1} - \varepsilon_1)\xi^4|\hat{\psi}|^2 + k\frac{\varepsilon_2}{2}\xi^2|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&+ \rho_2(\frac{m_2}{2} - s_1\varepsilon_2)\xi^2|\hat{\psi}_t|^2 \\
&\le b|\xi|^3|\hat{\psi}||\hat{\theta}_2| + k|\xi||\hat{\theta}_2||i\xi\hat{\varphi} + \hat{\psi} + l\hat{\omega}| + C(\varepsilon_1,\varepsilon_2)\xi^2|\hat{\theta}_1|^2 + C(\varepsilon_1,\varepsilon_2)(1+\xi^2)\xi^2|\hat{\theta}_2|^2.
\end{aligned}
$$

On the other hand, By Lemmas 2.5 and2.9, it follows that

$$
\begin{aligned}
&\frac{d}{dt}\{\xi^2 J_2(\xi,t) + \varepsilon_3\xi^2 H(\xi,t)\} + k_0\delta\xi^2|i\xi\hat{\omega} - l\hat{\varphi}|^2 + \rho_1\varepsilon_3 l\xi^2|\hat{\varphi}_t|^2 + \frac{\rho_1\varepsilon_3 l}{2}\xi^2|\hat{\omega}_t|^2 \\
&\le \frac{\rho_1 l k_1}{m_1}|\xi|^4|\hat{\varphi}_t||\hat{\theta}_1| + \frac{\rho_1 k_1}{m_1}|\xi|^5|\hat{\theta}_1||\hat{\omega}_t| + \frac{2kl}{m_1}|\xi|^3|\hat{\theta}_1||i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}| \\
&\quad + C_4(1+\xi^2)\xi^2|\hat{\theta}_1|^2 + \frac{3k_0\varepsilon_3 l}{2}\xi^2|i\xi\hat{\omega} - l\hat{\varphi}|^2 + \frac{3kl\varepsilon_3}{2}|\xi|^2|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&\quad + \frac{\rho_2\varepsilon_3 k}{2bl}\xi^2|\hat{\psi}_t|^2 + C(\varepsilon_3)(1+\xi^2)\xi^2|\hat{\theta}_1|^2.
\end{aligned}
$$

Adding and using Young's inequality, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_1(\xi,t) + b\varepsilon_2(\frac{\rho_2 l^2}{\rho_1} - 2\varepsilon_1)\xi^4|\hat{\psi}|^2 + k(\frac{\varepsilon_2}{4} - 2l\varepsilon_3)\xi^2|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&+ \rho_2(\frac{m}{2} - s_1\varepsilon_2 - \frac{k}{2bl}\varepsilon_3)\xi^2|\hat{\psi}_t|^2 + k_0(\delta - \frac{3l}{2}\varepsilon_3)\xi^2|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\
&\quad + \frac{\rho_1 l}{2}\varepsilon_3|\xi|^2|\hat{\varphi}_t|^2 + \frac{\rho_1 l}{4}\varepsilon_3|\xi|^2|\hat{\omega}_t|^2 \\
&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4+\xi^6)\xi^2|\hat{\theta}_1|^2 + C(\varepsilon_1,\varepsilon_2)(1+\xi^2+\xi^4)|\hat{\theta}_2|^2.
\end{aligned}
$$

We choose our constants as follows:

$$
\varepsilon_1 < \frac{\rho_2 l^2}{2\rho_1}, \quad \varepsilon_2 < \frac{m_2}{2s_1}, \quad \varepsilon_3 < \min\big\{\frac{2\delta}{3l}, \frac{\varepsilon_2}{8l}, \frac{2bl}{k}(\frac{m_2}{2} - s_1\varepsilon_2)\big\}.
$$

Consequently, we deduce that there exists $M > 0$, such that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{L}_1(\xi,t) + M\xi^2\big\{b\xi^2|\hat{\psi}|^2 + k|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + \rho_2|\hat{\psi}_t|^2 \\
&+ k_0|i\xi\hat{\omega} - l\hat{\varphi}|^2 + \rho_1|\hat{\varphi}_t|^2 + \rho_1|\hat{\omega}_t|^2\big\} \\
&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4+\xi^6)\xi^2|\hat{\theta}_1|^2 + C(\varepsilon_1,\varepsilon_2)(1+\xi^2+\xi^4)|\hat{\theta}_2|^2.
\end{aligned}
$$

At last, we assume that $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ or $k \neq k_0$. Consider the functional

$$
P_1 = \frac{\xi^2}{1+\xi^2+\xi^4}\lambda_1\varepsilon_3 J_1(\xi,t) + \frac{\xi^4}{(1+\xi^2+\xi^4)^2}\lambda_2\varepsilon_3 K(\xi,t).
$$

By Lemmas 2.2 and 2.8 and by using Young's inequality, it follows that

$$
\begin{aligned}
&\frac{d}{dt}P_1(\xi,t) + \big(\frac{\rho_2 l^2}{\rho_1} - 3\varepsilon_1\big)\frac{\lambda_2\varepsilon_3 b\xi^6}{(1+\xi^2+\xi^4)^2}|\hat{\psi}|^2 + \frac{\varepsilon_3\lambda_2 k\xi^4}{4(1+\xi^2+\xi^4)^2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&+ \big(\frac{\lambda_1 m_2}{2} - C(\varepsilon_4,\lambda_2)\big)\frac{\rho_2\varepsilon_3\xi^4}{(1+\xi^2+\xi^4)}|\hat{\psi}_t|^2 \\
&\le \frac{\lambda_2\varepsilon_3\varepsilon_4\xi^4|\hat{\varphi}_t|^2}{(1+\xi^2+\xi^4)^2} + C(\varepsilon_1,\lambda_2)\varepsilon_3\frac{(1+\xi^2)\xi^4|i\xi\hat{\omega} - l\hat{\varphi}|^2}{(1+\xi^2+\xi^4)^2}
\end{aligned}
$$

$$+ C(\varepsilon_1, \varepsilon_3, \lambda_1, \lambda_2)(1+\xi^2)\xi^2(|\hat{\theta}_1|^2 + |\hat{\theta}_2|).$$

In the above estimate we used the following inequalities:

$$\frac{1}{(1+\xi^2+\xi^4)} \le 1, \quad \frac{1+\xi^4}{(1+\xi^2+\xi^4)} \le 1, \quad \frac{\xi^2}{(1+\xi^2+\xi^4)} \le 1. \tag{2.31}$$

On the other hand, we consider the functional

$$P_2 = \frac{\xi^4}{(1+\xi^2+\xi^4)^2}(J_2(\xi,t) + \varepsilon_3 H(\xi,t)).$$

By (2.16) in Lemma 2.5, Lemma 2.9, Young's inequality and (2.31), we obtain

$$\begin{aligned}
&\frac{d}{dt}P_2 + k_0(\delta - C(k,k_0)\varepsilon_3)\frac{(1+\xi^2)\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\
&+ \frac{\rho_1\varepsilon_3 l\xi^4}{2(1+\xi^2+\xi^4)^2}|\hat{\varphi}_t|^2 + \frac{\rho_1\varepsilon_3 l\xi^4}{4(1+\xi^2+\xi^4)^2}|\hat{\omega}_t|^2 \\
&\le \frac{\rho_1\varepsilon_3\xi^4}{2l(1+\xi^2+\xi^4)}|\hat{\psi}_t|^2 + C(k,k_0)\varepsilon_3\frac{\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&\quad + C(\varepsilon_3)(1+\xi^2)\xi^2|\hat{\theta}_1|^2.
\end{aligned}$$

Thus, adding the above estimates of P_1 and P_2, we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathcal{L}_1(\xi,t) + (\frac{\rho_2 l^2}{\rho_1} - 3\varepsilon_1)\frac{\lambda_2\varepsilon_3 b\xi^6}{(1+\xi^2+\xi^4)^2}|\hat{\psi}|^2 \\
&+ (\frac{\lambda_2}{4} - C(k,k_0))\frac{\varepsilon_3 k\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&+ \Big(\frac{\lambda_1 m_2}{2} - C(\varepsilon_4,\lambda_2) - \frac{\rho_1}{2l\rho_2}\Big)\frac{\rho_2\varepsilon_3\xi^4}{(1+\xi^2+\xi^4)}|\hat{\psi}_t|^2 + \frac{\rho_1\varepsilon_3 l\xi^4}{4(1+\xi^2+\xi^4)^2}|\hat{\omega}_t|^2 \\
&+ (\delta - C(k,k_0,\varepsilon_1,\lambda_2)\varepsilon_3)\frac{k_0(1+\xi^2)\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\
&\quad + (\frac{l}{2} - \frac{\varepsilon_4\lambda_2}{\rho_1})\frac{\rho_1\varepsilon_3\xi^4}{(1+\xi^2+\xi^4)^2}|\hat{\varphi}_t|^2 \\
&\le C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)(1+\xi^2)\xi^2(|\hat{\theta}_1|^2 + |\hat{\theta}_2|^2)
\end{aligned}$$

We choose our constants as follows:

$$\begin{gathered}
\varepsilon_1 < \frac{\rho_2 l^2}{3\rho_1}, \quad \lambda_2 > 4C(k,k_0), \quad \varepsilon_4 < \frac{\rho_1 l}{2\lambda_2}, \\
\lambda_1 > \frac{\rho_1 + 2l\rho_2 C(\varepsilon_4,\lambda_2)}{m_2 l\rho_2}, \quad \varepsilon_3 < \frac{\delta}{C(k,k_0,\varepsilon_2,\lambda_2)}.
\end{gathered}$$

Consequently, by using (2.31), we deduce that there exists $M' > 0$, such that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{L}_1(\xi,t) + M'\frac{\xi^4}{(1+\xi^2+\xi^4)^2}\big\{b\xi^2|\hat{\psi}|^2 + k|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + \rho_2|\hat{\psi}_t|^2 \\
&+ k_0|i\xi\hat{\omega} - l\hat{\varphi}|^2 + \rho_1|\hat{\varphi}_t|^2 + \rho_1|\hat{\omega}_t|^2\big\} \\
&\le C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)(1+\xi^2)\xi^2(|\hat{\theta}_1|^2 + |\hat{\theta}_2|^2)
\end{aligned}$$

$\square$

With the Proposition 2.10 in hand and making some appropriate combinations, we build a Lyapunov functional $\mathcal{L}$, which plays a crucial role in the proof of our main result.

Theorem 2.11. *For any $t \geq 0$ and $\xi \in \mathbb{R}$, we obtain the following decay rates*
$$\hat{E}(\xi,t) \leq \begin{cases} C\hat{E}(\xi,0)e^{-\beta s_1(\xi)t}, & \text{if } \frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ and } k = k_0, \\ C'\hat{E}(\xi,0)e^{-\beta' s_2(\xi)t}, & \text{if } \frac{\rho_1}{\rho_2} \neq \frac{k}{b} \text{ or } k \neq k_0, \end{cases} \tag{2.32}$$
where C, β, C', β' are positive constants and
$$s_1(\xi) = \frac{\xi^4}{(1+\xi^2+\xi^4+\xi^6+\xi^8)}, \quad s_2(\xi) = \frac{\xi^4}{(1+\xi^2)(1+\xi^2+\xi^4)^2}.$$

Proof. Consider the Lyapunov functional
$$\mathcal{L}(\xi,t) = \begin{cases} \xi^2\mathcal{L}_1(\xi,t) + N(1+\xi^2+\xi^4+\xi^6+\xi^8)\hat{E}(\xi,t), & \text{if } \frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ and } k = k_0, \\ \mathcal{L}_1(\xi,t) + N'(1+\xi^2)\hat{E}(\xi,t), & \text{if } \frac{\rho_1}{\rho_2} \neq \frac{k}{b} \text{ or } k \neq k_0. \end{cases} \tag{2.33}$$
where N and N' are positive constants to be fixed later. First, we suppose that $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k = k_0$. By Lemma 2.1 and Proposition 2.10,
$$\begin{aligned} \frac{d}{dt}\mathcal{L}(\xi,t) &\leq -M\xi^4\big\{b\xi^2|\hat{\psi}|^2 + k|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + \rho_2|\hat{\psi}_t|^2 + k_0|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\ &\quad + \rho_1|\hat{\varphi}_t|^2 + \rho_1|\hat{\omega}_t|^2\big\} \\ &\quad - (2\gamma\eta N - C(\varepsilon_1,\varepsilon_2,\varepsilon_3))(1+\xi^2+\xi^4+\xi^6+\xi^8)\xi^2(|\hat{\theta}_1|^2 + |\hat{\theta}_2|^2), \end{aligned}$$
where $\eta = \min\{\frac{k_1}{m_1}, \frac{k_2}{m_2}\}$. On the other hand, by definition of $\mathcal{L}_1$, there exists $M_1 > 0$, such that
$$|\xi^2\mathcal{L}_1(\xi,t)| \leq M_1(1+\xi^2+\xi^4+\xi^6)\hat{E}(\xi,t).$$
It follows that
$$\begin{aligned} &(N-M_1)(1+\xi^2+\xi^4+\xi^6+\xi^8)\hat{E}(\xi,t) \\ &\leq \mathcal{L}(\xi,t) \\ &\leq (N+M_1)(1+\xi^2+\xi^4+\xi^6+\xi^8)\hat{E}(\xi,t). \end{aligned} \tag{2.34}$$
Choosing $N > \max(M_1, \frac{C(\varepsilon_1,\varepsilon_2,\varepsilon_3)}{2\gamma\eta})$ and by using
$$(1+\xi^2+\xi^4+\xi^6+\xi^8) \geq \xi^2,$$
there exists $M_2 > 0$, such that
$$\frac{d}{dt}\mathcal{L}(\xi,t) \leq -M_2\xi^4\hat{E}(\xi,t). \tag{2.35}$$
Note that (2.34) implies
$$\frac{d}{dt}\mathcal{L}(\xi,t) \leq -\beta\frac{\xi^4}{(1+\xi^2+\xi^4+\xi^6+\xi^8)}\mathcal{L}(\xi,t), \tag{2.36}$$
where $\beta = \frac{M_2}{N+M_1}$. By using Gronwall inequality, it follows that
$$\mathcal{L}(\xi,t) \leq \mathcal{L}(\xi,0)e^{-\beta s_1(\xi)t}. \tag{2.37}$$
Now, (2.34) yields
$$\hat{E}(\xi,t) \leq C\hat{E}(\xi,0)e^{-\beta s_1(\xi)t},$$

where $C = \frac{N+M_1}{N-M_1} > 0$.

At last, we assume that $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ or $k \neq k_0$. By Lemma 2.1 and Proposition 2.10,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(\xi,t) \leq & -M'\frac{\xi^4}{(1+\xi^2+\xi^4)^2}\big\{b\xi^2|\hat{\psi}|^2 + k|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + \rho_2|\hat{\psi}_t|^2 \\ & + k_0|i\xi\hat{\omega} - l\hat{\varphi}|^2 + \rho_1|\hat{\varphi}_t|^2 + \rho_1|\hat{\omega}_t|^2\big\} \\ & - (2\gamma\eta N' - C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2))(1+\xi^2)\xi^2(|\hat{\theta}_1|^2 + |\hat{\theta}_2|^2)\end{aligned}$$

On the other hand, by definition of $\mathcal{L}_2$ and by using Young's inequality, there exists $M_1' > 0$ such that

$$(N' - M_1')(1+\xi^2)\hat{E}(\xi,t) \leq \mathcal{L}(\xi,t) \leq (N' + M_1')(1+\xi^2)\hat{E}(\xi,t). \tag{2.38}$$

Choosing $N' > \max(M_1', \frac{C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)}{2\gamma\eta})$ and by using

$$(1+\xi^2) \geq \frac{\xi^2}{(1+\xi^2+\xi^4)^2},$$

there exists $M_2' > 0$, such that

$$\frac{d}{dt}\mathcal{L}(\xi,t) \leq -M_2'\frac{\xi^4}{(1+\xi^2+\xi^4)^2}\hat{E}(\xi,t). \tag{2.39}$$

From (2.38), we obtain

$$\frac{d}{dt}\mathcal{L}(\xi,t) \leq -\beta'\frac{\xi^4}{(1+\xi^2)(1+\xi^2+\xi^4)^2}\mathcal{L}(\xi,t), \tag{2.40}$$

where $\beta' = \frac{M_2'}{N'+M_1'}$. By using Gronwall inequality, we conclude that

$$\mathcal{L}(\xi,t) \leq \mathcal{L}(\xi,0)e^{-\beta' s_2(\xi)t}. \tag{2.41}$$

The last inequality together with (2.38) leads to the second inequality of the theorem, which completes the proof. □

2.2. **Thermoelastic Bresse system of type III.** In this subsection, we establish decay rates for the Fourier image of the solutions of Thermoelastic Bresse system of Type III. Taking Fourier Transform in (1.2), we obtain the ODE system

$$\rho_1\hat{\varphi}_{tt} - ik\xi(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}) - k_0 l(i\xi\hat{\omega} - l\hat{\varphi}) + l\gamma\hat{\theta}_{1t} = 0 \quad \text{in } \mathbb{R}\times(0,\infty), \tag{2.42}$$

$$\rho_2\hat{\psi}_{tt} + b\xi^2\hat{\psi} - k(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}) + i\gamma\xi\hat{\theta}_{2t} = 0 \quad \text{in } \mathbb{R}\times(0,\infty), \tag{2.43}$$

$$\rho_1\hat{\omega}_{tt} - ik_0\xi(i\xi\hat{\omega} - l\hat{\varphi}) - kl(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}) + i\gamma\xi\hat{\theta}_{1t} = 0 \quad \text{in } \mathbb{R}\times(0,\infty), \tag{2.44}$$

$$\hat{\theta}_{1tt} + k_1\xi^2\hat{\theta}_1 + \alpha_1\xi^2\hat{\theta}_{1t} + m_1(i\xi\hat{\omega} - l\hat{\varphi})_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty), \tag{2.45}$$

$$\hat{\theta}_{2tt} + k_2\xi^2\hat{\theta}_2 + \alpha_2\xi^2\hat{\theta}_{2t} + im_2\xi\hat{\psi}_t = 0 \quad \text{in } \mathbb{R}\times(0,\infty). \tag{2.46}$$

The energy functional associated with the above system is

$$\begin{aligned}\hat{\mathbb{E}}(\xi,t) = & \rho_1|\hat{\varphi}_t|^2 + \rho_2|\hat{\psi}_t|^2 + \rho_1|\hat{\omega}_t|^2 + \frac{\gamma}{m_1}|\hat{\theta}_{1t}|^2 + \frac{k_1\gamma}{m_1}\xi^2|\hat{\theta}_1|^2 + \frac{\gamma}{m_2}|\hat{\theta}_{2t}|^2 \\ & + \frac{k_2\gamma}{m_2}\xi^2|\hat{\theta}_2|^2 + b|\xi|^2|\hat{\psi}|^2 + k|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + k_0|i\xi\hat{\omega} - l\hat{\varphi}|^2\end{aligned} \tag{2.47}$$

Lemma 2.12. *Let $\hat{\mathbb{E}}$ the energy functional associated with system* (2.42)-(2.46). *Then*

$$\frac{d}{dt}\hat{\mathbb{E}}(\xi,t) = -2\gamma\xi^2(\frac{\alpha_1}{m_1}|\hat{\theta}_{1t}|^2 + \frac{\alpha_2}{m_2}|\hat{\theta}_{2t}|^2). \tag{2.48}$$

Proof. Multiplying (2.42) by $\overline{\hat{\varphi}}_t$, (2.43) by $\overline{\hat{\psi}}_t$, (2.44) by $\overline{\hat{\omega}}_t$, (2.45) by $\frac{\gamma}{m_1}\overline{\hat{\theta}_{1t}}$, (2.46) by $\frac{\gamma}{m_2}\overline{\hat{\theta}_{2t}}$, adding these equalities and taking the real part, (2.48) follows. □

To establish the main result of this subsection and based in the approach done in the previous subsection, we establish the following lemmas:

Lemma 2.13. *The functional*

$$\mathbb{J}_1(\xi,t) = \mathrm{Re}(i\rho_2\xi\hat{\psi}_t\overline{\hat{\theta}_{2t}}) + \mathrm{Re}(ik_2\rho_2\xi^3\hat{\psi}\overline{\hat{\theta}_2}),$$

satisfies

$$\begin{aligned} &\frac{d}{dt}\mathbb{J}_1(\xi,t) + \frac{m_2\rho_2}{2}\xi^2|\hat{\psi}_t|^2 \\ &\le k_2\rho_2|\xi|^3|\hat{\psi}||\hat{\theta}_{2t}| + b|\xi|^3|\hat{\psi}||\hat{\theta}_{2t}| + k|\xi||\hat{\theta}_{2t}||i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}| \\ &\quad + C_1(1+\xi^2)\xi^2|\hat{\theta}_{2t}|^2, \end{aligned} \tag{2.49}$$

where C_1 is a positive constant.

Proof. Multiplying (2.46) by $-i\rho_2\xi\overline{\hat{\psi}_t}$ and taking real part, we obtain

$$\begin{aligned} &\frac{d}{dt}\mathrm{Re}(-i\rho_2\xi\overline{\hat{\psi}_t}\hat{\theta}_{2t}) + \mathrm{Re}(i\rho_2\xi\overline{\hat{\psi}_{tt}}\hat{\theta}_{2t}) - \frac{d}{dt}\mathrm{Re}(ik_2\rho_2\xi^3\overline{\hat{\psi}}\hat{\theta}_2) \\ &\quad + \mathrm{Re}(ik_2\rho_2\xi^3\overline{\hat{\psi}}\hat{\theta}_{2t}) - \mathrm{Re}(i\alpha_2\rho_2\xi^3\overline{\hat{\psi}_t}\hat{\theta}_{2t}) + m_2\rho_2\xi^2|\hat{\psi}_t|^2 = 0. \end{aligned}$$

By (2.43), it follows that

$$\begin{aligned} \frac{d}{dt}\mathbb{J}_1(\xi,t) + m_2\rho_2\xi^2|\hat{\psi}_t|^2 &\le k_2\rho_2|\xi|^3|\hat{\psi}||\hat{\theta}_{2t}| + \alpha_2\rho_2|\xi|^3|\hat{\psi}_t||\hat{\theta}_{2t}| + b|\xi|^3|\hat{\psi}||\hat{\theta}_{2t}| \\ &\quad + k|\xi||\hat{\theta}_{2t}||i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}| + \gamma\xi^2|\hat{\theta}_{2t}|^2. \end{aligned}$$

Applying Young's inequality, (2.49) follows. □

Lemma 2.14. *The functional*

$$\mathbb{T}_1(\xi,t) = \mathrm{Re}\Big(-\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega} - l\hat{\varphi})} - \frac{\rho_1}{m_1}\hat{\varphi}_t\overline{\hat{\theta}_{1t}}\Big),$$

satisfies

$$\begin{aligned} &\frac{d}{dt}\mathbb{T}_1(\xi,t) + \frac{k_0 l}{2}|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\ &\le \frac{\alpha_1\rho_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_{1t}| - \mathrm{Re}(ik\xi(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega})\overline{(i\xi\hat{\omega} - l\hat{\varphi})}) \\ &\quad - \frac{k}{m_1}\mathrm{Re}(i\xi\overline{\hat{\theta}_{1t}}(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega})) + \frac{\rho_1 k_1}{m_1}\mathrm{Re}(\xi^2\hat{\varphi}_t\overline{\hat{\theta}}_1) + C_2|\hat{\theta}_{1t}|^2, \end{aligned} \tag{2.50}$$

where C_2 is a positive constant.

Proof. Multiplying (2.42) by $-\overline{(i\xi\hat{\omega}-l\hat{\varphi})}$ and taking real part, we have

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Re}(-\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+\operatorname{Re}(\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}_t)\\
&+\operatorname{Re}(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})+k_0l|i\xi\hat{\omega}-l\hat{\varphi}|^2-\operatorname{Re}(l\gamma\hat{\theta}_{1t}\overline{(i\xi\hat{\omega}-l\hat{\varphi})})=0,
\end{aligned}$$

by using (2.45), we have

$$\begin{aligned}
&\frac{dRe}{dt}(-\rho_1\hat{\varphi}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})})-\frac{\rho_1}{m_1}\frac{d}{dt}\operatorname{Re}(\hat{\varphi}_t\overline{\hat{\theta}}_{1t})+\frac{\rho_1}{m_1}\operatorname{Re}(\hat{\varphi}_{tt}\overline{\hat{\theta}}_{1t})\\
&-\frac{\rho_1k_1}{m_1}\operatorname{Re}(\xi^2\hat{\varphi}_t\overline{\hat{\theta}}_1)-\frac{\alpha_1\rho_1}{m_1}\operatorname{Re}(\xi^2\hat{\varphi}_t\overline{\hat{\theta}}_{1t})+\operatorname{Re}(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})})\\
&+k_0l|i\xi\hat{\omega}-l\hat{\varphi}|^2-\operatorname{Re}(l\gamma\hat{\theta}_{1t}\overline{(i\xi\hat{\omega}-l\hat{\varphi})})=0.
\end{aligned}$$

Note that (2.42) implies

$$\begin{aligned}
&\frac{d}{dt}\mathbb{T}_1(\xi,t)+\frac{k}{m_1}\operatorname{Re}\Big(i\xi\overline{\hat{\theta}}_{1t}(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\Big)+\operatorname{Re}\Big(ik\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)\\
&+k_0l|i\xi\hat{\omega}-l\hat{\varphi}|^2-\frac{\rho_1k_1}{m_1}\operatorname{Re}(\xi^2\hat{\varphi}_t\overline{\hat{\theta}}_1)-\frac{\alpha_1\rho_1}{m_1}\operatorname{Re}(\xi^2\hat{\varphi}_t\overline{\hat{\theta}}_{1t})\\
&\le\frac{l\gamma}{m_1}|\hat{\theta}_{1t}|^2+(l\gamma+\frac{k_0l}{m_1})|\hat{\theta}_{1t}||i\xi\hat{\omega}-l\hat{\varphi}|.
\end{aligned}$$

Applying Young's inequality, we obtain (2.50). $\square$

Lemma 2.15. *The functional*

$$\mathbb{T}_2(\xi,t)=\operatorname{Re}\Big(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}+i\frac{\rho_1}{m_1}\xi\hat{\omega}_t\overline{\hat{\theta}}_{1t}\Big),$$

satisfies

$$\begin{aligned}
&\frac{d}{dt}\mathbb{T}_2(\xi,t)+\frac{k_0}{2}|\xi|^2|i\xi\hat{\omega}-l\hat{\varphi}|^2\\
&\le\frac{\alpha_1\rho_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_{1t}|+\operatorname{Re}\Big(ikl\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)\\
&\quad+\frac{kl}{m_1}\operatorname{Re}(i\xi\overline{\hat{\theta}}_{1t}(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega}))-\frac{k_1\rho_1}{m_1}\operatorname{Re}(i\xi^3\hat{\omega}_t\overline{\hat{\theta}}_1)+C_3|\xi|^2|\hat{\theta}_{1t}|^2,
\end{aligned}\tag{2.51}$$

where C_3 is a positive constant.

Proof. Multiplying (2.44) by $i\xi\overline{(i\xi\hat{\omega}-l\hat{\varphi})}$ and taking real part,

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Re}\Big(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)-\operatorname{Re}\Big(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}_t\Big)+k_0|\xi|^2|i\xi\hat{\omega}-l\hat{\varphi}|^2\\
&-\operatorname{Re}\Big(ikl\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)-\operatorname{Re}\Big(\gamma\xi^2\hat{\theta}_{1t}\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)=0,
\end{aligned}$$

by using (2.45), we have

$$\begin{aligned}
&\frac{d}{dt}\operatorname{Re}\Big(i\rho_1\xi\hat{\omega}_t\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)+\frac{\rho_1}{m_1}\frac{d}{dt}\operatorname{Re}(i\xi\hat{\omega}_t\overline{\hat{\theta}}_{1t})-\frac{\rho_1}{m_1}\operatorname{Re}(i\xi\hat{\omega}_{tt}\overline{\hat{\theta}}_{1t})\\
&+\frac{\rho_1k_1}{m_1}\operatorname{Re}(i\xi^3\hat{\omega}_t\overline{\hat{\theta}}_1)+\frac{\alpha_1\rho_1}{m_1}\operatorname{Re}(i\xi^3\hat{\omega}_t\overline{\hat{\theta}}_{1t})+k_0|\xi|^2|i\xi\hat{\omega}-l\hat{\varphi}|^2\\
&-\operatorname{Re}\Big(ikl\xi(i\xi\hat{\varphi}-\hat{\psi}-l\hat{\omega})\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)-\operatorname{Re}\Big(\gamma\xi^2\hat{\theta}_{1t}\overline{(i\xi\hat{\omega}-l\hat{\varphi})}\Big)=0.
\end{aligned}$$

Note that, (2.44) implies that

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{T}_2(\xi,t) - \frac{kl}{m_1}\operatorname{Re}(i\xi\overline{\hat{\theta}}_{1t}(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega})) - \operatorname{Re}\Big(ikl\xi(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega})\overline{(i\xi\hat{\omega} - l\hat{\varphi})}\Big) \\
&+ k_0\xi^2|i\xi\hat{\omega} - l\hat{\varphi}|^2 + \frac{k_1\rho_1}{m_1}\operatorname{Re}(i\xi^3\hat{\omega}_t\overline{\hat{\theta}}_1) + \frac{\alpha_1\rho_1}{m_1}\operatorname{Re}(i\xi^3\hat{\omega}_t\overline{\hat{\theta}}_{1t}) \\
&\le \frac{\gamma\xi^2}{m_1}|\hat{\theta}_{1t}|^2 + (\gamma + \frac{k_0}{m_1})\xi^2|\hat{\theta}_{1t}||i\xi\hat{\omega} - l\hat{\varphi}|.
\end{aligned}
$$

Applying Young's inequality, we obtain (2.51). □

Lemma 2.16. *Consider the functional*

$$
\mathbb{J}_2(\xi,t) := l\mathbb{T}_1(\xi,t) + \mathbb{T}_2(\xi,t) + \frac{\rho_1k_1}{m_1}\operatorname{Re}\Big(\xi^2\overline{\hat{\theta}}_1(i\xi\hat{\omega} - l\hat{\varphi})\Big).
$$

Then, there exists $\delta > 0$ such that

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{J}_2(\xi,t) + k_0\delta|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\
&\le \frac{\alpha_1\rho_1 l}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_{1t}| + \frac{\alpha_1\rho_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_{1t}| + C_4(1+\xi^2)|\hat{\theta}_{1t}|^2
\end{aligned}
\tag{2.52}
$$

where C_3 is a positive constant.

Proof. By Lemma (2.14) and Lemma (2.15),

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{J}_2(\xi,t) + \frac{k_0}{2}(l^2+\xi^2)|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\
&\le \frac{\rho_1 l\alpha_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_{1t}| + \frac{\rho_1k_1}{m_1}\xi^2|\hat{\theta}_{1t}||i\xi\hat{\omega} - l\hat{\varphi}| + \frac{\alpha_1\rho_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_{1t}| + C_4(1+\xi^2)|\hat{\theta}_{1t}|^2
\end{aligned}
$$

Note that there exists $\delta > 0$ such that $4\delta \le \frac{l^2+\xi^2}{1+\xi^2}$. Thus,

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{J}_2(\xi,t) + 2k_0\delta(1+\xi^2)|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\
&\le \frac{\rho_1 l\alpha_1}{m_1}|\xi|^2|\hat{\varphi}_t||\hat{\theta}_{1t}| + \frac{\rho_1k_1}{m_1}(1+\xi^2)|\hat{\theta}_{1t}||i\xi\hat{\omega} - l\hat{\varphi}| \\
&\quad + \frac{\alpha_1\rho_1}{m_1}|\xi|^3|\hat{\omega}_t||\hat{\theta}_{1t}| + C_4(1+\xi^2)|\hat{\theta}_{1t}|^2.
\end{aligned}
\tag{2.53}
$$

Applying Young's inequality, (2.52) follows. □

Lemma 2.17. *Consider the functional*

$$
\mathbb{J}_3(\xi,t) = \operatorname{Re}\Big(-\rho_2\hat{\psi}_t\overline{(i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega})} - i\frac{\rho_1 b}{k}\xi\hat{\psi}\overline{\hat{\varphi}_t}\Big).
$$

If $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k = k_0$, then

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{J}_3(\xi,t) + \frac{k}{2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&\le \rho_2|\hat{\psi}_t|^2 + \rho_2 l\operatorname{Re}(\hat{\psi}_t\overline{\hat{\omega}_t}) - bl\operatorname{Re}\Big(i\xi\hat{\psi}\overline{(i\xi\hat{\omega} - l\hat{\varphi})}\Big) \\
&\quad + \frac{bl\gamma}{k}|\xi||\hat{\psi}||\hat{\theta}_{1t}| + C_5|\xi|^2|\hat{\theta}_{2t}|^2
\end{aligned}
\tag{2.54}
$$

Moreover, if $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ *or* $k \neq k_0$, *then*

$$\begin{aligned}
&\frac{d}{dt}\mathbb{J}_3(\xi,t) + \frac{k}{2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&\le \rho_2|\hat{\psi}_t|^2 + \rho_2 l \operatorname{Re}(\hat{\psi}_t\overline{\hat{\omega}_t}) \\
&\quad + (\rho_2 - \frac{b\rho_1}{k}) \operatorname{Re}(i\xi\hat{\psi}_t\overline{\hat{\varphi}_t}) - \frac{k_0 bl}{k} \operatorname{Re}\left(i\xi\hat{\psi}\overline{(i\xi\hat{\omega} - l\hat{\varphi})}\right) \\
&\quad + \frac{bl\gamma}{k}|\xi||\hat{\psi}||\hat{\theta}_{1t}| + C_5|\xi|^2|\hat{\theta}_{2t}|^2,
\end{aligned} \tag{2.55}$$

where C_5 *is a positive constant.*

Proof. Proceeding as proof of Lemma 2.6, we obtain (2.54) and (2.55). □

Lemma 2.18. *Let* $0 < \varepsilon_1 < \frac{\rho_2 l^2}{2\rho_1}$ *and consider the functional*

$$\mathbb{J}_4(\xi,t) = \operatorname{Re}(\frac{\rho_2^2 l^2}{\rho_1}\hat{\psi}_t\overline{\hat{\psi}} - \rho_2 l\hat{\omega}_t\overline{\hat{\psi}}).$$

If $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ *and* $k = k_0$, *then*

$$\begin{aligned}
&\frac{d\mathbb{J}_4(\xi,t)}{dt} + b(\frac{\rho_2 l^2}{\rho_1} - \frac{\varepsilon_1}{2})\xi^2|\hat{\psi}|^2 \\
&\le \frac{\rho_2^2 l^2}{\rho_1}|\hat{\psi}_t|^2 - \rho_2 l \operatorname{Re}(\overline{\hat{\psi}}_t\hat{\omega}_t) + bl \operatorname{Re}(i\xi\hat{\psi}\overline{(i\xi\hat{\omega} - l\hat{\varphi})}) + C(\varepsilon_1)(|\hat{\theta}_{1t}|^2 + |\hat{\theta}_{2t}|^2)
\end{aligned} \tag{2.56}$$

Moreover, If $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ *or* $k \neq k_0$, *then*

$$\begin{aligned}
&\frac{d}{dt}\mathbb{J}_4(\xi,t) + b(\frac{\rho_2 l^2}{\rho_1} - \frac{\varepsilon_1}{2})\xi^2|\hat{\psi}|^2 \\
&\le \frac{\rho_2^2 l^2}{\rho_1}|\hat{\psi}_t|^2 - \rho_2 l \operatorname{Re}(\overline{\hat{\psi}}_t\hat{\omega}_t) \\
&\quad + \frac{\rho_2 k_0 l}{\rho_1} \operatorname{Re}\left(i\xi\hat{\psi}\overline{(i\xi\hat{\omega} - l\hat{\varphi})}\right) + C(\varepsilon_1)(|\hat{\theta}_{1t}|^2 + |\hat{\theta}_{2t}|^2),
\end{aligned} \tag{2.57}$$

where $C(\varepsilon_1)$ *is a positive constant.*

Proof. Proceeding as proof of Lemma 2.7, we obtain (2.56) and (2.57). □

Lemma 2.19. *Let* $0 < \varepsilon_1 < \frac{\rho_2 l^2}{2\rho_1}$ *and consider* $\mathbb{K}(\xi,t) = \mathbb{J}_3(\xi,t) + \mathbb{J}_4(\xi,t)$, *If* $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ *and* $k = k_0$, *then*

$$\begin{aligned}
&\frac{d}{dt}\mathbb{K}(\xi,t) + b(\frac{\rho_2 l^2}{\rho_1} - \varepsilon_1)\xi^2|\hat{\psi}|^2 + \frac{k}{2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&\le \rho_2 s_1|\hat{\psi}_t|^2 + C(\varepsilon_1)|\hat{\theta}_{1t}|^2 + C(\varepsilon_1)(1+\xi^2)|\hat{\theta}_{2t}|^2
\end{aligned}$$

Moreover, if $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ *or* $k \neq k_0$, *then*

$$\begin{aligned}
&\frac{d}{dt}\mathbb{K}(\xi,t) + b(\frac{\rho_2 l^2}{\rho_1} - \varepsilon_1)\xi^2|\hat{\psi}|^2 + \frac{k}{2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&\le \rho_2 s_1|\hat{\psi}_t|^2 + k_0(\frac{\rho_2}{\rho_1} - \frac{b}{k})l \operatorname{Re}(i\xi\hat{\psi}(\overline{i\xi\hat{\omega} - l\hat{\varphi}})) + (\rho_2 - \frac{b\rho_1}{k}) \operatorname{Re}(i\xi\hat{\psi}_t\overline{\hat{\varphi}}_t) \\
&\quad + C(\varepsilon_1)|\hat{\theta}_{1t}|^2 + C(\varepsilon_1)(1+\xi^2)|\hat{\theta}_{2t}|^2,
\end{aligned}$$

where $s_1 = \frac{\rho_2 l^2}{\rho_1} + 1$.

The above lemma follows from Lemmas 2.17 and 2.18, applying Young's inequality.

Lemma 2.20. *Consider the functional*

$$\mathbb{H}(\xi,t) = \rho_1 \operatorname{Re}\Big((i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega})\overline{\hat{\omega}_t}\Big) + \rho_1 \operatorname{Re}((i\xi\hat{\omega} - l\hat{\varphi})\overline{\hat{\varphi}_t}).$$

If $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ *and* $k = k_0$, *then*

$$\begin{aligned} &\frac{d}{dt}\mathbb{H}(\xi,t) + \rho_1 l|\hat{\varphi}_t|^2 + \frac{\rho_1 l}{2}|\hat{\omega}_t|^2 \\ &\le \frac{\rho_2 k}{2bl}|\hat{\psi}_t|^2 + \frac{3kl}{2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + \frac{3k_0 l}{2}|i\xi\hat{\omega} - l\hat{\varphi}|^2 + C_6(1+\xi^2)|\hat{\theta}_{1t}|^2 \end{aligned} \tag{2.58}$$

Moreover, if $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ *or* $k \neq k_0$, *then*

$$\begin{aligned} &\frac{d}{dt}\mathbb{H}(\xi,t) + \rho_1 l|\hat{\varphi}_t|^2 + \frac{\rho_1 l}{2}|\hat{\omega}_t|^2 \\ &\le \frac{\rho_1}{2l}|\hat{\psi}_t|^2 + C_1(k,k_0)|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 + C_2(k,k_0)(1+\xi^2)|i\xi\hat{\omega} - l\hat{\varphi}|^2 \\ &\quad + C_6(1+\xi^2)|\hat{\theta}_{1t}|^2, \end{aligned} \tag{2.59}$$

where $C_1(k,k_0), C_6$ *are positive constants.*

Proof. Proceeding as proof of Lemma 2.9, we obtain (2.58) and (2.59). □

Lemma 2.21. *The functional*

$$\begin{aligned} \mathbb{S}(\xi,t) &= \frac{\gamma}{m_1}\operatorname{Re}(\xi^2\hat{\theta}_{1t}\overline{\hat{\theta}}_1) + \frac{\gamma}{m_2}\operatorname{Re}(\xi^2\hat{\theta}_{2t}\overline{\hat{\theta}}_2) + \frac{\gamma}{2}\xi^4\big(\frac{\alpha_1}{m_1}|\hat{\theta}_{1t}|^2 + \frac{\alpha_2}{m_2}|\hat{\theta}_{2t}|^2\big) \\ &\quad + \gamma \operatorname{Re}(i\xi^3\hat{\psi}\overline{\hat{\theta}}_2 + \xi^2\overline{\hat{\theta}}_1(i\xi\hat{\omega} - l\hat{\varphi})) \end{aligned}$$

satisfies

$$\begin{aligned} &\frac{d}{dt}\mathbb{S}(\xi,t) + \frac{k_1\gamma}{m_1}\xi^4|\hat{\theta}_1|^2 + \frac{k_2\gamma}{m_2}\xi^4|\hat{\theta}_2|^2 \\ &\le \gamma\xi^2|\hat{\theta}_{1t}||i\xi\hat{\omega} - l\hat{\varphi}| + \gamma|\xi|^3|\hat{\psi}||\hat{\theta}_{2t}| + \frac{\gamma}{m_1}\xi^2|\hat{\theta}_{1t}|^2 + \frac{\gamma}{m_2}\xi^2|\hat{\theta}_{2t}|^2 \end{aligned} \tag{2.60}$$

Proof. Multiplying (2.45) by $\frac{\gamma}{m_1}\xi^2\overline{\hat{\theta}}_1$ and taking real part, we obtain

$$\begin{aligned} &\frac{d}{dt}\big\{\frac{\gamma}{m_1}\operatorname{Re}(\xi^2\hat{\theta}_{1t}\overline{\hat{\theta}}_1) + \frac{\alpha_1\gamma}{2m_1}\xi^4|\hat{\theta}_{1t}|^2 + \gamma\operatorname{Re}(\xi^2\overline{\hat{\theta}}_1(i\xi\hat{\omega} - l\hat{\varphi}))\big\} \\ &+ k_1\frac{\gamma}{m_1}\xi^4|\hat{\theta}_1|^2 \\ &\le \frac{\gamma}{m_1}\xi^2|\hat{\theta}_{1t}|^2 + \gamma\xi^2|\hat{\theta}_{1t}||i\xi\hat{\omega} - l\hat{\varphi}|. \end{aligned} \tag{2.61}$$

Moreover, multiplying (2.46) by $\frac{\gamma}{m_2}\xi^2\overline{\hat{\theta}}_2$ and taking real part,

$$\begin{aligned} &\frac{d}{dt}\big\{\frac{\gamma}{m_2}\operatorname{Re}(\xi^2\hat{\theta}_{2t}\overline{\hat{\theta}}_2) + \frac{\alpha_2\gamma}{2m_2}\xi^4|\hat{\theta}_{2t}|^2 + \gamma\operatorname{Re}(i\xi^3\hat{\psi}\overline{\hat{\theta}}_2)\big\} + k_2\frac{\gamma}{m_2}\xi^4|\hat{\theta}_2|^2 \\ &\le \frac{\gamma}{m_2}\xi^2|\hat{\theta}_{2t}|^2 + \gamma|\xi|^3|\hat{\psi}||\hat{\theta}_{2t}|. \end{aligned} \tag{2.62}$$

Adding (2.61) and (2.62), we obtain (2.60) □

Now, Consider the functional

$$\mathcal{L}_2(\xi,t)=\begin{cases}\mathbb{J}_1(\xi,t)+\varepsilon_2\xi^2\mathbb{K}(\xi,t)+\xi^2\mathbb{J}_2(\xi,t)+\varepsilon_3\xi^2\mathbb{H}(\xi,t)+\mathbb{S}(\xi,t),\\ \quad\text{if } \frac{\rho_1}{\rho_2}=\frac{k}{b} \text{ and } k=k_0,\\ \frac{\xi^2}{(1+\xi^2+\xi^4)}\big(\lambda_1\varepsilon_3\mathbb{J}_1+\frac{1}{(1+\xi^2+\xi^4)}(\varepsilon_3\lambda_2\xi^2\mathbb{K}+\xi^2\mathbb{J}_2+\varepsilon_3\xi^2\mathbb{H}+\mathbb{S})\big),\\ \quad\text{if } \frac{\rho_1}{\rho_2}\neq\frac{k}{b} \text{ or } k\neq k_0.\end{cases} \tag{2.63}$$

where $\lambda_1,\lambda_2,\varepsilon_2,\varepsilon_3$ are positive constants to be determined later.

Proposition 2.22. *There exist constants $M,M'>0$ such that if $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ and $k=k_0$, then*

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_2(\xi,t)+M\xi^2\big\{b\xi^2|\hat\psi|^2+k|i\xi\hat\varphi-\hat\psi-l\hat\omega|^2+\rho_2|\hat\psi_t|^2+k_0|i\xi\hat\omega-l\hat\varphi|^2\\&+\rho_1|\hat\varphi_t|^2+\rho_1|\hat\omega_t|^2+\frac{k_1\gamma}{m_1}\xi^2|\hat\theta_1|^2+\frac{k_2\gamma}{m_2}\xi^2|\hat\theta_2|^2\big\}\\&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4+\xi^6)\xi^2|\hat\theta_{1t}|^2+C(\varepsilon_1,\varepsilon_2)(1+\xi^2+\xi^4)|\hat\theta_{2t}|^2\end{aligned} \tag{2.64}$$

Moreover, if $\frac{\rho_1}{\rho_2}\neq\frac{k}{b}$ or $k\neq k_0$, we obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_2(\xi,t)+M'\frac{\xi^4}{(1+\xi^2+\xi^4)^2}\big\{b\xi^2|\hat\psi|^2+k|i\xi\hat\varphi-\hat\psi-l\hat\omega|^2+\rho_2|\hat\psi_t|^2\\&+k_0|i\xi\hat\omega-l\hat\varphi|^2+\rho_1|\hat\varphi_t|^2+\rho_1|\hat\omega_t|^2+\frac{k_1\gamma}{m_1}\xi^2|\hat\theta_1|^2+\frac{k_2\gamma}{m_2}\xi^2|\hat\theta_2|^2\}\\&\le C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)(1+\xi^2)\xi^2(|\hat\theta_1|^2+|\hat\theta_2|^2).\end{aligned} \tag{2.65}$$

Proof. We can prove (2.64) and (2.65) following the ideas used on the proof of Proposition 2.10, thus we omit some details. First, we suppose that $\frac{\rho_1}{\rho_2}=\frac{k}{b}$ and $k=k_0$. By Lemmas 2.13, 2.19, 2.16 and 2.20, it follows that

$$\begin{aligned}&\frac{d}{dt}\{\mathcal{L}_2(\xi,t)-\mathbb{S}(\xi,t)\}+(\frac{\rho_2l^2}{\rho_1}-2\varepsilon_1)\varepsilon_2b\xi^4|\hat\psi|^2+(\frac{\varepsilon_2}{4}-\frac{3l}{2}\varepsilon_3)k\xi^2|i\xi\hat\varphi-\hat\psi-l\hat\omega|^2\\&+(\frac{m}{2}-s_1\varepsilon_2-\frac{k}{2bl}\varepsilon_3)\rho_2\xi^2|\hat\psi_t|^2+(\delta-\frac{3l}{2}\varepsilon_3)k_0\xi^2|i\xi\hat\omega-l\hat\varphi|^2\\&+\frac{\rho_1l}{2}\varepsilon_3|\xi|^2|\hat\varphi_t|^2+\frac{\rho_1l}{4}\varepsilon_3|\xi|^2|\hat\omega_t|^2\\&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4+\xi^6)\xi^2|\hat\theta_{1t}|^2+C(\varepsilon_1,\varepsilon_2)(1+\xi^2+\xi^4)|\hat\theta_{2t}|^2.\end{aligned}$$

Adding $\mathbb{S}(\xi,t)$ in the above inequality, applying Lemma 2.21 and Young's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}_2(\xi,t)+(\frac{\rho_2l^2}{\rho_1}-3\varepsilon_1)\varepsilon_2b\xi^4|\hat\psi|^2+(\frac{\varepsilon_2}{4}-\frac{3l}{2}\varepsilon_3)k\xi^2|i\xi\hat\varphi-\hat\psi-l\hat\omega|^2\\&+(\frac{m}{2}-s_1\varepsilon_2-\frac{k}{2bl}\varepsilon_3)\rho_2\xi^2|\hat\psi_t|^2+(\delta-2l\varepsilon_3)k_0\xi^2|i\xi\hat\omega-l\hat\varphi|^2\\&+\frac{\rho_1l}{2}\varepsilon_3|\xi|^2|\hat\varphi_t|^2+\frac{\rho_1l}{4}\varepsilon_3|\xi|^2|\hat\omega_t|^2+\frac{k_1\gamma}{m_1}\xi^4|\hat\theta_1|^2+\frac{k_2\gamma}{m_2}\xi^4|\hat\theta_2|^2\\&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3)(1+\xi^2+\xi^4+\xi^6)\xi^2|\hat\theta_{1t}|^2+C(\varepsilon_1,\varepsilon_2)(1+\xi^2+\xi^4)|\hat\theta_{2t}|^2.\end{aligned}$$

We choose our constants as follows:

$$\varepsilon_1 < \frac{\rho_2 l^2}{3\rho_1}, \quad \varepsilon_2 < \frac{m_2}{2s_1}, \quad \varepsilon_3 < \min\{\frac{\delta}{2l}, \frac{\varepsilon_2}{6l}, \frac{2bl}{k}(\frac{m_2}{2} - s_1\varepsilon_2)\}$$

Consequently, we deduce that there exists $M > 0$, such that (2.64) holds.

Second, we assume that $\frac{\rho_1}{\rho_2} \neq \frac{k}{b}$ and $k \neq k_0$. By Lemmas 2.13, 2.19, the estimate (2.53) in Lemma 2.16, Lemma 2.20, adding these inequalities and by using Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\{\mathcal{L}_2(\xi,t) - \frac{\xi^2 \mathbb{S}(\xi,t)}{(1+\xi^2+\xi^4)^2}\} + (\frac{\rho_2 l^2}{\rho_1} - 3\varepsilon_1)\frac{\lambda_2\varepsilon_3 b\xi^6}{(1+\xi^2+\xi^4)^2}|\hat{\psi}|^2 \\
&+ \frac{\rho_1\varepsilon_3 l\xi^4}{4(1+\xi^2+\xi^4)^2}|\hat{\omega}_t|^2 + (\frac{\lambda_1 m_2}{2} - C(\varepsilon_4,\lambda_2) - \frac{\rho_1}{\rho_2})\frac{\rho_2\varepsilon_3\xi^4}{(1+\xi^2+\xi^4)}|\hat{\psi}_t|^2 \\
&+ (\frac{\lambda_2}{4} - C(k,k_0))\frac{\varepsilon_3 k\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&+ (\delta - C(k,k_0,\varepsilon_1,\lambda_2)\varepsilon_3)\frac{k_0(1+\xi^2)\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\omega} - l\hat{\varphi}|^2 + (\frac{l}{2} - \frac{\varepsilon_4\lambda_2}{\rho_1})\frac{\rho_1\varepsilon_3\xi^4}{(1+\xi^2+\xi^4)^2}|\hat{\varphi}_t|^2 \\
&\leq C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)(1+\xi^2)\xi^2(|\hat{\theta}_{1t}|^2 + |\hat{\theta}_{2t}|^2).
\end{aligned}$$

In the last estimate, we used also the inequalities (2.31). Adding $\frac{\xi^2}{(1+\xi^2+\xi^4)^2}\mathbb{S}(\xi,t)$ in the above inequality, applying Lemma 2.21 and Young's inequality, it follows that

$$\begin{aligned}
&\frac{d}{dt}\mathcal{L}_2(\xi,t) + (\frac{\rho_2 l^2}{\rho_1} - 4\varepsilon_1)\frac{\lambda_2\varepsilon_3 b\xi^6}{(1+\xi^2+\xi^4)^2}|\hat{\psi}|^2 \\
&+ (\frac{\lambda_2}{4} - C(k,k_0))\frac{\varepsilon_3 k\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\varphi} - \hat{\psi} - l\hat{\omega}|^2 \\
&+ (\frac{\lambda_1 m_2}{2} - C(\varepsilon_4,\lambda_2) - \frac{\rho_1}{\rho_2})\frac{\rho_2\varepsilon_3\xi^4}{(1+\xi^2+\xi^4)}|\hat{\psi}_t|^2 + \frac{\rho_1 l\varepsilon_3\xi^4}{4(1+\xi^2+\xi^4)^2}|\hat{\omega}_t|^2 \\
&+ (\delta - C(k,k_0,\varepsilon_1,\lambda_2)\varepsilon_3)\frac{k_0(1+\xi^2)\xi^4}{(1+\xi^2+\xi^4)^2}|i\xi\hat{\omega} - l\hat{\varphi}|^2 + (\frac{l}{2} - \frac{\varepsilon_4\lambda_2}{\rho_1})\frac{\rho_1\varepsilon_2\xi^4}{(1+\xi^2+\xi^4)^2}|\hat{\varphi}_t|^2 \\
&+ \frac{k_1\gamma}{m_1}\frac{\xi^6}{(1+\xi^2+\xi^4)^2}|\hat{\theta}_1|^2 + \frac{k_2\gamma}{m_2}\frac{\xi^6}{(1+\xi^2+\xi^4)^2}|\hat{\theta}_2|^2 \\
&\leq C(\varepsilon_1,\varepsilon_3,\lambda_1,\lambda_2)(1+\xi^2)\xi^2(|\hat{\theta}_{1t}|^2 + |\hat{\theta}_{2t}|^2).
\end{aligned}$$

We choose our constants as follows:

$$\begin{gathered}
\varepsilon_1 < \frac{\rho_2 l^2}{4\rho_1}, \quad \lambda_2 > 4C(k,k_0), \quad \varepsilon_4 < \frac{\rho_1 l}{2\lambda_2}, \\
\lambda_1 > \frac{2(\rho_1 + C(\varepsilon_4,\lambda_2)\rho_2)}{m_2\rho_2}, \quad \varepsilon_3 < \frac{\delta}{C(k,k_0,\varepsilon_2,\lambda_2)}.
\end{gathered}$$

Consequently, by using (2.31), we deduce that there exists $M' > 0$, such that (2.65) holds. □

Theorem 2.23. *For any $t \geq 0$ and $\xi \in \mathbb{R}$, we obtain the following decay rates,*

$$\hat{\mathbb{E}}(\xi,t) \leq \begin{cases} C\hat{\mathbb{E}}(\xi,0)e^{-\beta s_1(\xi)t}, & \text{if } \frac{\rho_1}{\rho_2} = \frac{k}{b} \text{ and } k = k_0, \\ C'\hat{\mathbb{E}}(\xi,0)e^{-\beta' s_2(\xi)t}, & \text{if } \frac{\rho_1}{\rho_2} \neq \frac{k}{b} \text{ or } k \neq k_0, \end{cases} \tag{2.66}$$

where C, β, C', β' are positive constants and

$$s_1(\xi) = \frac{\xi^4}{(1+\xi^2+\xi^4+\xi^6+\xi^4)}, \quad s_2(\xi) = \frac{\xi^4}{(1+\xi^2)(1+\xi^2+\xi^4)^2}.$$

Proof. We prove (2.66), by Proposition 2.22 and using the same approach done in the proof of Theorem 2.11, Thus, we omit the details. □

3. Main result

In this section, we establish decay estimates of the solutions to systems (1.1) and (1.2). For Bresse system (1.1), thermoelasticity of Type I, we consider the vector solution

$$\begin{aligned} V_1 := \Big(&\rho_1^{1/2}\varphi_t, \rho_2^{1/2}\psi_t, \rho_1^{1/2}\omega_t, (\frac{\gamma}{m_1})^{1/2}\theta_1, (\frac{\gamma}{m_2})^{1/2}\theta_2, b^{1/2}\psi_x, \\ &k^{1/2}(\varphi_x-\psi_x-l\omega_x), k_0^{1/2}(\omega_x-l\varphi)\Big) \end{aligned} \tag{3.1}$$

and for Bresse system (1.2), thermoelasticity of type III,

$$\begin{aligned} V_2 := \Big(&\rho_1^{1/2}\varphi_t, \rho_2^{1/2}\psi_t, \rho_1^{1/2}\omega_t, (\frac{\gamma}{m_1})^{1/2}\theta_{1t}, (\frac{k_1\gamma}{m_1})^{1/2}\theta_{1x}, (\frac{\gamma}{m_2})^{1/2}\theta_{2t}, \\ &(\frac{k_2\gamma}{m_2})^{1/2}\theta_{2x}, b^{1/2}\psi_x, k^{1/2}(\varphi_x-\psi_x-l\omega_x), k_0^{1/2}(\omega_x-l\varphi)\Big) \end{aligned} \tag{3.2}$$

Note that

$$\hat{E}(\xi,t) = |\hat{V}_1(\xi,t)|^2, \quad \hat{\mathbb{E}}(\xi,t) = |\hat{V}_2(\xi,t)|^2, \tag{3.3}$$

where $\hat{E}$ and $\hat{\mathbb{E}}$ are defined in (2.6) and (2.47), respectively. We are now in a position to prove our main result.

Proof of Theorem 1.1. Applying the Plancherel identity and (3.3), we have

$$\begin{aligned} \|\partial_x^k V_1(t)\|^2_{L^2(\mathbb{R})} &= \|(i\xi)^k \hat{V}_1(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\xi|^{2k}\hat{E}(\xi,t)^2 d\xi, \\ \|\partial_x^k V_2(t)\|^2_{L^2(\mathbb{R})} &= \|(i\xi)^k \hat{V}_2(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\xi|^{2k}\hat{\mathbb{E}}(\xi,t)^2 d\xi. \end{aligned}$$

By Theorems 2.11 and 2.23, it follows that

$$\begin{aligned} \|\partial_x^k V_j(t)\|_2^2 &\le C\int_{\mathbb{R}} |\xi|^{2k} e^{-\beta s(\xi)t}\hat{V}_j(0,\xi)^2 d\xi \\ &\le C\int_{|\xi|\le 1} |\xi|^{2k} e^{-\beta s_i(\xi)t}\hat{V_j}^2(0,\xi) d\xi + C\int_{|\xi|\ge 1} |\xi|^{2k} e^{-\beta s_i(\xi)t}\hat{V_j}^2(0,\xi) d\xi \\ &= I_1 + I_2, \quad (i,j=1,2). \end{aligned}$$

It is not difficult to see that if $\frac{\rho_1}{\rho_2} = \frac{k}{b}$ and $k = k_0$, then the function $s_1(\xi)$ satisfies

$$\begin{aligned} s_1(\xi) &\ge \frac{1}{5}\xi^4 \quad \text{if } |\xi| \le 1, \\ s_1(\xi) &\ge \frac{1}{5}\xi^{-4} \quad \text{if } |\xi| \ge 1\,. \end{aligned} \tag{3.4}$$

Thus, we estimate I_1 as follows,

$$\begin{aligned} I_1 &\le C\|\hat{V_j^0}\|_{L^\infty}^2 \int_{|\xi|\le 1} |\xi|^{2k} e^{-\frac{\beta}{5}\xi^4 t} d\xi \\ &\le C_1 \|\hat{V_j^0}\|_{L^\infty}^2 (1+t)^{-\frac{1}{4}(1+2k)} \\ &\le C_1 (1+t)^{-\frac{1}{4}(1+2k)} \|V_j^0\|_{L^1}^2, \quad j=1,2. \end{aligned} \tag{3.5}$$

On the other hand, using the second inequality in (3.4), we obtain

$$\begin{aligned} I_2 &\le C \int_{|\xi|\ge 1} |\xi|^{2k} e^{-\frac{\beta}{5}\xi^{-4} t} \hat{V_j^0}^2(\xi) d\xi \\ &\le C \sup_{|\xi|\ge 1} \{|\xi|^{-2l} e^{-\frac{\beta}{5}\xi^{-4} t}\} \int_{\mathbb{R}} |\xi|^{2(k+l)} \hat{V_j^0}^2(\xi) d\xi \\ &\le C_2 (1+t)^{-\frac{l}{2}} \|\partial_x^{k+l} V_j^0\|_2^2, \quad j=1,2. \end{aligned}$$

Combining the estimates of I_1 and I_2, we obtain (1.5). On the other hand, if $\frac{\rho_1}{\rho_2} \ne \frac{k}{b}$ or $k \ne k_0$, the function $s_2(\xi)$ satisfies

$$\begin{aligned} s_2(\xi) &\ge \frac{1}{18}\xi^4 \quad \text{if } |\xi| \le 1, \\ s_2(\xi) &\ge \frac{1}{18}\xi^{-6} \quad \text{if } |\xi| \ge 1 \end{aligned} \tag{3.6}$$

Thus, we estimate I_1 as follows,

$$\begin{aligned} I_1 &\le C\|\hat{V_j^0}_0\|_{L^\infty}^2 \int_{|\xi|\le 1} |\xi|^{2k} e^{-\frac{\beta}{18}\xi^4 t} d\xi \\ &\le C_1 \|\hat{V_j^0}\|_{L^\infty}^2 (1+t)^{-\frac{1}{4}(1+2k)} \\ &\le C_1 (1+t)^{-\frac{1}{4}(1+2k)} \|V_j^0\|_{L^1}^2 \end{aligned}$$

Moreover, by using the second inequality in (3.6), it follows that

$$\begin{aligned} I_2 &\le C \int_{|\xi|\ge 1} |\xi|^{2k} e^{-\frac{\beta}{18}\xi^{-6} t} \hat{V_j^0}^2(\xi) d\xi \\ &\le C \sup_{|\xi|\ge 1} \{|\xi|^{-2l} e^{-\frac{\beta}{18}\xi^{-6} t}\} \int_{\mathbb{R}} |\xi|^{2(k+l)} \hat{V_j^0}^2(\xi) d\xi \\ &\le C_2 (1+t)^{-\frac{l}{3}} \|\partial_x^{k+l} V_j^0\|_2^2 \end{aligned}$$

Combining the estimates of I_1 and I_2, we obtain (1.6). □

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