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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A NON-AUTONOMOUS SYSTEM OF TWO-DIMENSIONAL DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the two-dimensional Bernfeld-Haddock conjecture involving non-autonomous delay differential equations. Employing the differential inequality theory, it is shown that every bounded solution tends to a constant vector as $t \to \infty$. Numerical simulations are carried out to verify our theoretical findings.

## 1. Introduction

In 1976, Bernfeld and Haddock [1] proposed the following conjecture.

Every solution of the delay differential equation

$$x'(t) = -x^{1/3}(t) + x^{1/3}(t-r), \tag{1.1}$$

where $r > 0$, tends to a constant as $t \to \infty$.

To confirm the above conjecture, variants of the above equation, which have been used as models for some population growth and the spread of epidemics, have received considerable attention (see, for example, [2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15] and the references therein). In particular, the asymptotic behavior of the autonomous equations

$$x'(t) = -F(x(t)) + G(x(t-r)), \tag{1.2}$$

and

$$\begin{aligned} x_1'(t) &= -F_1(x_1(t)) + G_1(x_2(t-r_2)), \\ x_2'(t) &= -F_2(x_2(t)) + G_2(x_1(t-r_1)), \end{aligned} \tag{1.3}$$

and non-autonomous equations

$$x'(t) = p(t)[-x^{1/3}(t) + x^{1/3}(t - r_3(t))], \tag{1.4}$$

have been studied in [4, 5, 6, 9, 10, 11, 12, 13, 14] and [2, 3, 15], respectively. Here, $r, r_1$ and $r_2$ are positive constants, $F$, $G$, $F_i$, $G_i \in C(\mathbb{R}, \mathbb{R})$, $F$ and $F_i$ are nondecreasing on $\mathbb{R}$, $p, r_3 \in C(\mathbb{R}, (0, +\infty))$, $r_3(t) > 0$, $p(t) > 0$, $i = 1, 2$.

Furthermore, it was shown in the above mentioned references that each bounded solution of above equations tends to a constant solution as $t \to \infty$. We also found that the main methods mentioned above include two kinds, one is the analysis

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method of the monotone dynamical system [9, 10, 11, 12, 13, 14], the other is the differential inequality analysis technique [2, 3, 4, 5, 6, 14]. As pointed out in [13], there were some errors in several existing works in [2, 3, 4, 5, 6, 14], and the uniqueness of the left-hand solution of the following differential equation

$$\begin{gathered} x'(t) = -F(x(t)) + F(c), \\ x(t_0) = x_0 \quad \text{for } t_0, x_0 \in \mathbb{R}, \end{gathered} \tag{1.5}$$

played a crucial role in the discussion of above references. Consequently, to improve the proof in [2, 3, 4, 5, 6], Ding adopted the following additional assumption:

(A1) If $c \neq 0$ then the solution to (1.5) on the interval $(t_0 - \delta, t_0]$ is unique, where $\delta$ is a positive constant (this soluution is called left-hand solution in [10])

This assumption is also included in [13, Appendix].

On the other hand, delays in population and ecology models are usually time-varying and usually can be generalized as the non-autonomous functional differential equation. Thus, we can generalize the equation (1.3) in two-dimensional Bernfeld-Haddock conjecture to the following non-autonomous delay differential equations:

$$\begin{gathered} x_1'(t) = \gamma_1(t)[-F_1(x_1(t)) + G_1(x_2(t - \tau_2(t)))], \\ x_2'(t) = \gamma_2(t)[-F_2(x_2(t)) + G_2(x_1(t - \tau_1(t)))], \end{gathered} \tag{1.6}$$

and $F_i, G_i \in C(\mathbb{R}, \mathbb{R}), \gamma_i, \tau_i \in C(\mathbb{R}, (0, +\infty))$, $i = 1, 2$. Moreover, it is assumed that $F_i$ is strictly increasing on $\mathbb{R}$, $F_i$ is continuous differentiable on $\mathbb{R} \setminus \{0\}$, and

$$F_i(0) = 0, \quad F_i'(x) > 0 \text{for all } x \in \mathbb{R} \setminus \{0\},\ i = 1, 2. \tag{1.7}$$

for $i = 1, 2$. It is worth noting that system (1.6) include equation (1.2) as a special case. In fact, if $\tau_1(t) = \tau_2(t) = r$ and consider the synchronized solutions of (1.6) with $x_1(t) = x_2(t) = \varphi_0(t)$ for $t \in [-r, 0]$, then system (1.6) reduces to equation (1.2). Obviously, (1.1), (1.3) and (1.4) are the special cases of (1.6). It is well known that a non-autonomous delay differential equation generally does not generate a semiflow and hence methods for differential equations with constant delays [9, 10, 11, 12, 13, 14] are not suitable for (1.6). Moreover, the irregularity of the set of equilibria seems to cause some difficulties in the study of system (1.6) now. Hence, to the best of our knowledge, there is no result on the asymptotic behavior of solutions of non-autonomous delay differential equations (1.6) before.

Motivated by the above discussions, we aim to employ a novel argument to prove that every solution of (1.6) tends to a constant vector as $t \to +\infty$.

Throughout this article, for a bounded and continuous function $g$ defined on $\mathbb{R}$, we denote

$$g^+ = \sup_{t\in\mathbb{R}} g(t) \quad \text{and} \quad g^- = \inf_{t\in\mathbb{R}} g(t).$$

It will be always assumed that

$$r = \max\{\tau_1^+, \tau_2^+\} \geq \tau^* = \min\{\tau_1^-, \tau_2^-\} > 0, \quad 0 < \gamma_i^- \leq \gamma_i^+ < +\infty, \quad i \in J = \{1, 2\}.$$

We will denote $C = C([-\tau_1^+, 0], \mathbb{R}) \times C([-\tau_2^+, 0], \mathbb{R})$ as the Banach space equipped with a supremum norm. We define the initial condition

$$x_i(t_0 + \theta) = \varphi_i(\theta), \quad \theta \in [-\tau_i^+, 0], \quad t_0 \in \mathbb{R}, \quad \varphi = (\varphi_1, \varphi_2) \in C, \quad i \in J. \tag{1.8}$$

We write $x(t; t_0, \varphi) = (x_1(t; t_0, \varphi),\ x_2(t; t_0, \varphi))$ to denote the solution of the initial value problem (1.6) and (1.8). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $x(t; t_0, \varphi)$.

The remaining of this paper is organized as follows. In Section 2, we recall some relevant results, and give a detailed proof on the boundedness and global existence of every solution for (1.6) with the initial condition (1.8). Based on the preparation in Section 2, we state and prove our main result in Section 3. In Section 4, we give some examples to illustrate the effectiveness of the obtained results by numerical simulations.

## 2. Preliminary results

Assume that $F:\mathbb{R}\to\mathbb{R}$ is continuous and strictly increasing, and

$$\begin{gathered} F(0)=0,\quad F(x) \text{ is continuous differentiable on } \mathbb{R}\setminus\{0\},\\ F'(x)>0 \text{ for all } x\in\mathbb{R}\setminus\{0\}. \end{gathered} \tag{2.1}$$

Then, $F$ satisfies (A1). From [7, Lemma 2.1, Propositions 4* and 5*], we have the following results.

**Proposition 2.1.** *Consider the differential equation*

$$u'=-F(u)+F(c+\varepsilon), \tag{2.2}$$

*where $c\neq 0$ is a given constant, $\varepsilon$ is a parameter satisfying $0\le\varepsilon\le|c|/2$, and the initial condition is*

$$u(t_0)=u_0\quad (u_0<c). \tag{2.3}$$

*Let $u=u(t;t_0,u_0)$ be the solution of the initial value problem* (2.2) *and* (2.3)*, and $\alpha>0$ be a given constant. Then there exists a positive real number $\mu$ independent of $t_0$ and $\varepsilon$ such that*

$$(c+\varepsilon)-u(t;t_0,u_0)\ge\mu>0\quad \text{for } t\in[t_0,t_0+\alpha].$$

**Proposition 2.2.** *Consider the differential equation*

$$u'=-F(u)+F(c-\varepsilon), \tag{2.4}$$

*where $c\neq 0$ is a given constant, $\varepsilon$ is a parameter satisfying $0\le\varepsilon\le\frac{|c|}{2}$. Moreover, assume the initial condition*

$$u(t_0)=u_0\quad (u_0>c). \tag{2.5}$$

*Let $u=u(t;t_0,u_0)$ be the solution of the initial value problem* (2.4) *and* (2.5)*, and $\alpha>0$ be a given constant. Then there exists a positive real number $\nu$ independent of $t_0$ and $\varepsilon$ such that*

$$u(t;t_0,u_0)-(c-\varepsilon)\ge\nu>0\quad \text{for } t\in[t_0,t_0+\alpha].$$

One can easily see that $F(x)=x^{1/3}$ satisfies (2.1) and hence Propositions 2.1 and 2.2 hold in this case.

**Lemma 2.3** (see [8, 15])**.** *Let $t_0\in\mathbb{R}$, $\beta>0$, $\bar h\in C([t_0,t_0+\beta]\times\mathbb{R},\mathbb{R})$, and $\bar h$ is non-increasing with respect to the second variable. Then the initial value problem*

$$\begin{gathered}\frac{dx}{dt}=\bar h(t,x)\\ x(t_0)=x_0\end{gathered}$$

*has a unique solution $x=x(t)$ on $[t_0,t_0+\beta]$.*

**Lemma 2.4.** *Let $\varphi\in C$. Then $x(t;t_0,\varphi)$ exists and is unique on $[t_0,\infty)$.*

*Proof.* Let $x(t) = x(t; t_0, \varphi)$. We will show that $x(t)$ exists and is unique on $[t_0, t_0 + \tau^*]$. To see this, let

$$\begin{aligned} d_1(t) &= G_1(x_2(t-\tau_2(t))) = G_1(\varphi_2(t-\tau_2(t)-t_0)), \\ d_2(t) &= G_2(x_1(t-\tau_1(t))) = G_2(\varphi_1(t-\tau_1(t)-t_0)) \end{aligned}$$

for any $t \in [t_0, t_0 + \tau^*]$. Consider the solution $x_i(t)$ of the initial value problem

$$\begin{gathered} x_i'(t) = \gamma_i(t)[-F_i(x_i(t)) + d_i(t)], \\ x_i(t_0) = \varphi_i(0) \end{gathered}$$

where $i \in J$. By Lemma 2.3, $x_i(t)$ exists and is unique on $[t_0, t_0+\tau^*]$, $i \in J$. Hence, $x(t)$ exists and is unique on $[t_0, t_0 + \tau^*]$. It follows from induction that $x(t)$ exists and is unique on $[t_0, +\infty)$. The proof is complete. $\square$

**Lemma 2.5.** *Let $\varphi \in C$, and $F_i(u) = G_i(u)$ for all $u \in \mathbb{R}, i \in J$. Then $x(t; t_0, \varphi)$ exists and is unique on $[t_0, +\infty)$. Moreover, $x(t; t_0, \varphi)$ is bounded on $[t_0, +\infty)$.*

*Proof.* By Lemma 2.4, $x(t) = x(t; t_0, \varphi)$ exists and is unique on $[t_0, +\infty)$. Furthermore, we claim that

$$\alpha < x_i(t; t_0, \varphi) < \beta \quad \text{for all} \quad t \in [t_0, +\infty), i \in J,$$

where $\alpha$ and $\beta$ are two constants such that $\alpha < \varphi_i(s) < \beta$ for all $s \in [-\tau_i^+, 0]$, $i \in J$. Suppose that the claim is not true. Then one of the following two cases must occur:

**Case I.** There exist $i^* \in J$ and $\theta_1 > t_0$ such that

$$x_{i^*}(\theta_1; t_0, \varphi) = \beta \text{ and } x_j(t; t_0, \varphi) < \beta \text{ for all } t \in [t_0 - \tau_j^+, \theta_1),\ j \in J. \tag{2.6}$$

**Case II.** There exist $i^* \in J$ and $\theta_2 > t_0$ such that

$$x_{i^*}(\theta_2; t_0, \varphi) = \alpha \text{ and } \alpha < x_j(t; t_0, \varphi) \text{ for all } t \in [t_0 - \tau_j^+, \theta_2),\ j \in J. \tag{2.7}$$

When Case I holds, in view of (1.6) and (2.6), we have

$$\begin{aligned} 0 &\le x_{i^*}'(\theta_1) \\ &= \gamma_{i^*}(\theta_1)[-F_{i^*}(x_{i^*}(\theta_1)) + F_{i^*}(x_{\bar{i}^*}(\theta_1 - \tau_{\bar{i}^*}(\theta_1))] \\ &< \gamma_{i^*}(\theta_1)[-F_{i^*}(\beta) + F_{i^*}(\beta)] \\ &= 0, \quad \bar{i}^* \in J \setminus \{i^*\}, \end{aligned}$$

which is a contradiction.

When Case II holds, similarly we have

$$\begin{aligned} 0 &\ge x_{i^*}'(\theta_2) \\ &= \gamma_{i^*}(\theta_2)[-F_{i^*}(x_{i^*}(\theta_2)) + F_{i^*}(x_{\bar{i}^*}(\theta_2 - \tau_{\bar{i}^*}(\theta_2))] \\ &> \gamma_{i^*}(\theta_2)[-F_{i^*}(\alpha) + F_{i^*}(\alpha)] \\ &= 0, \quad \bar{i}^* \in J \setminus \{i^*\}, \end{aligned}$$

which is also a contradiction. Thus we have proved the claim and completed the proof. $\square$

## 3. Main result

The purpose of this section is to show that every bounded solution of (1.6) tends to a constant as $t \to +\infty$, which is our main result in this paper.

**Theorem 3.1.** *Assume either $G_i \geq F_i$ or $G_i \leq F_i$, $i \in J$, Then every bounded solution of the initial value problem* (1.6) *and* (1.8) *tends to a constant vector as $t \to +\infty$.*

*Proof.* Note that Theorem 3.1 is equivalent to the statement: If either $G_i \geq F_i(i \in J)$ or $G_i \leq F_i(i \in J)$ holds and $\varphi \in C$ such that $x_i(t; t_0, \varphi)$ is bounded for all $t \in \mathbb{R}$ and $i \in J$, then

$$l_i = \liminf_{t \to +\infty} x_i(t; t_0, \varphi) = \limsup_{t \to +\infty} x_i(t; t_0, \varphi) = L_i, i \in J.$$

We only consider the case where $G_i \leq F_i(i \in J)$ since the case where $G_i \geq F_i(i \in J)$ can proved similarly. Let

$$\begin{gathered} x_i(t) = x_i(t; t_0, \varphi), \quad \text{for all } t \geq t_0, i \in J, \\ y_i(t) = \max_{t-r \leq s \leq t} x_i(s), \quad u_i(t) = \min_{t-r \leq s \leq t} x_i(s) \quad \text{for all } t \geq t_0 + r,\ i \in J, \\ y(t) = \max\{y_1(t), y_2(t)\}, u(t) = \min\{u_1(t), u_2(t)\}, \\ S = \{t | t \in [t_0 + r, +\infty), y(t) = x_i(t) \quad \text{for some } i \in J\}. \end{gathered}$$

Firstly, we show $D^+ y(t) \leq 0$ for all $t \geq t_0 + r$. We distinguish two cases to finish the proof.

**Case 1.** $t \in [t_0 + r, +\infty) \setminus S$. Then there exist $i_0 \in J$ and $t^* \in [t - r, t)$ such that

$$y(t) = y_{i_0}(t) = \max_{t-r \leq s \leq t} x_{i_0}(s) = x_{i_0}(t^*) > \max\{x_1(t), x_2(t)\}.$$

From the continuity of $x_i(\cdot)$ at $t$, we can choose a positive constant $\delta < r$ such that

$$x_i(s) < x_{i_0}(t^*) \quad \text{for all } s \in [t, t + \delta],\ i \in J,$$

which yields

$$x_i(s) \leq x_{i_0}(t^*) = \max_{t-r \leq s \leq t} x_{i_0}(s) = y_{i_0}(t) = y(t) \quad \text{for all } s \in [t - r, t + \delta],\ i \in J.$$

It follows that

$$\begin{aligned} y(t + h) &= \max\Big\{\max_{t+h-r \leq s \leq t+h} x_1(s), \max_{t+h-r \leq s \leq t+h} x_2(s)\Big\} \\ &\leq \max\Big\{\max_{t-r \leq s \leq t+\delta} x_1(s), \max_{t-r \leq s \leq t+\delta} x_2(s)\Big\} \\ &\leq \max_{t-r \leq s \leq t} x_{i_0}(s) = y_{i_0}(t) = y(t) \text{ for all } h \in (0, \delta), \end{aligned}$$

and hence

$$D^+ y(t) = \limsup_{h \to 0^+} \frac{y(t + h) - y(t)}{h} \leq \limsup_{h \to 0^+} \frac{y(t) - y(t)}{h} = 0.$$

**Case 2.** $t \in S$. Then there exists $i_0 \in J$ such that

$$y(t) = y_{i_0}(t) = x_{i_0}(t) = \max_{t-r \leq s \leq t} x_{i_0}(s).$$

Then (1.6) implies

$$\begin{aligned} 0 &\leq x'_{i_0}(t) \\ &= \gamma_{i_0}(t)[-F_{i_0}(x_{i_0}(t)) + G_{i_0}(x_{\bar{i}_0}(t - \tau_{\bar{i}_0}(t)))] \end{aligned}$$

$$
\begin{aligned}
&\le \gamma_{i_0}(t)[-F_{i_0}(x_{i_0}(t)) + F_{i_0}(x_{\bar{i}_0}(t-\tau_{\bar{i}_0}(t)))] \\
&\le \gamma_{i_0}(t)[-F_{i_0}(x_{i_0}(t)) + F_{i_0}(x_{i_0}(t))] \\
&= 0, \quad \text{where } \bar{i}_0 \in J \setminus \{i_0\},
\end{aligned}
$$

which gives $x'_{i_0}(t) = 0$. Let $\rho = \frac{1}{2}\tau^*$. Obviously, $\rho > 0$. First we assume that $y(s) = x_{i_0}(s)$ for all $s \in (t, t+\rho]$. Then we have

$$
\begin{aligned}
D^+y(t) &= \limsup_{h\to 0^+} \frac{y(t+h)-y(t)}{h} \\
&= \limsup_{h\to 0^+} \frac{y(t+h)-x_{i_0}(t)}{h} \\
&= \limsup_{h\to 0^+} \frac{x_{i_0}(t+h)-x_{i_0}(t)}{h} \\
&= x_{i_0}^{'}(t) \\
&= 0, \quad \text{where } 0 < h < \rho.
\end{aligned}
$$

Now assume that there exists $s_1 \in (t, t+\rho]$ such that $y(s_1) > x_{i_0}(s_1)$. Consequently, one can show that either

$$
y(s_1) = y_{i_0}(s_1) = \max_{s_1-r\le s\le s_1} x_{i_0}(s) \tag{3.1}
$$

or

$$
y(s_1) = y_{\bar{i}_0}(s_1) = \max_{s_1-r\le s\le s_1} x_{\bar{i}_0}(s) > y_{i_0}(s_1), \quad \text{where } \bar{i}_0 \in J \setminus \{i_0\}, \tag{3.2}
$$

holds.

If (3.1) holds, we can choose a constant $\tilde{t} \in [s_1 - r, s_1)$ such that

$$
y(s_1) = x_{i_0}(\tilde{t}) = \max_{s_1-r\le s\le s_1} x_{i_0}(s).
$$

This, together with the fact that $t - r < s_1 - r \le t + \rho - r < t < s_1$, implies

$$
x_{i_0}(\tilde{t}) \ge x_{i_0}(t) = y(t) = y_{i_0}(t) = \max_{t-r\le s\le t} x_{i_0}(s).
$$

We claim that

$$
x_{i_0}(\tilde{t}) = x_{i_0}(t) = y(t) = y_{i_0}(t). \tag{3.3}
$$

Otherwise, $x_{i_0}(\tilde{t}) > x_{i_0}(t)$. Then $t < \tilde{t} < s_1$ and

$$
\begin{aligned}
0 \le x'_{i_0}(\tilde{t}) &= \gamma_{i_0}(\tilde{t})[-F_{i_0}(x_{i_0}(\tilde{t})) + G_{i_0}(x_{\bar{i}_0}(\tilde{t}-\tau_{\bar{i}_0}(\tilde{t})))] \\
&\le \gamma_{i_0}(\tilde{t})[-F_{i_0}(x_{i_0}(\tilde{t})) + F_{i_0}(x_{\bar{i}_0}(\tilde{t}-\tau_{\bar{i}_0}(\tilde{t})))] \,.
\end{aligned}
$$

It follows that

$$
F_{i_0}(x_{i_0}(\tilde{t})) \le F_{i_0}(x_{\bar{i}_0}(\tilde{t}-\tau_{\bar{i}_0}(\tilde{t})))
$$

and

$$
x_{\bar{i}_0}(\tilde{t}-\tau_{\bar{i}_0}(\tilde{t})) \ge x_{i_0}(\tilde{t}) > x_{i_0}(t). \tag{3.4}
$$

Noting that $t - r \le t - \tau_{\bar{i}_0}(\tilde{t}) < \tilde{t} - \tau_{\bar{i}_0}(\tilde{t}) < \tilde{t} - \rho < t < s_1$, we have

$$
x_{i_0}(\tilde{t}) \le x_{\bar{i}_0}(\tilde{t}-\tau_{\bar{i}_0}(\tilde{t})) \le \max_{t-r\le s\le t} x_{\bar{i}_0}(s) \le y(t) = x_{i_0}(t),
$$

which contradicts with (3.4). Thus we have proved the claim. It follows that

$$
\max_{t-r\le s\le s_1} x_{i_0}(s) = x_{i_0}(t),
$$

which, together the fact that

$$t - r < s_1 - r \le t + \rho - r < t < s_1, \quad y_{\bar{i}_0}(t) \le y_{i_0}(t), \quad y_{\bar{i}_0}(s_1) \le y_{i_0}(s_1),$$

yields

$$\max_{t-r\le s\le s_1} x_{\bar{i}_0}(s) \le \max_{t-r\le s\le s_1} x_{i_0}(s) = x_{i_0}(t) = y(t), \quad y(t+h) = x_{i_0}(t)$$

for all $0 < h < s_1 - t$, and hence

$$\begin{aligned} D^+y(t) &= \limsup_{h\to 0^+} \frac{y(t+h) - y(t)}{h} \\ &= \limsup_{h\to 0^+} \frac{y(t+h) - x_{i_0}(t)}{h} \\ &= \limsup_{h\to 0^+} \frac{x_{i_0}(t) - x_{i_0}(t)}{h} = 0. \end{aligned}$$

If (3.2) holds, we can choose a constant $\bar{t} \in [s_1 - r, s_1]$ such that

$$y(s_1) = x_{\bar{i}_0}(\bar{t}) = \max_{s_1-r\le s\le s_1} x_{\bar{i}_0}(s) > y_{i_0}(s_1) \ge x_{i_0}(t). \tag{3.5}$$

Clearly, $t < \bar{t} \le s_1$ and

$$\begin{aligned} 0 \le x'_{\bar{i}_0}(\bar{t}) &= \gamma_{\bar{i}_0}(\bar{t})[-F_{\bar{i}_0}(x_{\bar{i}_0}(\bar{t})) + G_{\bar{i}_0}(x_{i_0}(\bar{t} - \tau_{i_0}(\bar{t})))] \\ &\le \gamma_{\bar{i}_0}(\bar{t})[-F_{\bar{i}_0}(x_{\bar{i}_0}(\bar{t})) + F_{\bar{i}_0}(x_{i_0}(\bar{t} - \tau_{i_0}(\bar{t})))], \end{aligned}$$

which follows that

$$F_{\bar{i}_0}(x_{\bar{i}_0}(\bar{t})) \le F_{\bar{i}_0}(x_{i_0}(\bar{t} - \tau_{i_0}(\bar{t})))]$$

and

$$x_{i_0}(\bar{t} - \tau_{i_0}(\bar{t})) \ge x_{\bar{i}_0}(\bar{t}) > x_{i_0}(t). \tag{3.6}$$

Noting that $t - r \le t - \tau_{i_0}(\bar{t}) < \bar{t} - \tau_{i_0}(\bar{t}) < \bar{t} - \rho < t < s_1$, we have

$$x_{\bar{i}_0}(\bar{t}) \le x_{i_0}(\bar{t} - \tau_{i_0}(\bar{t})) \le \max_{t-r\le s\le t} x_{i_0}(s) \le y(t) = x_{i_0}(t),$$

which contradicts with (3.6). Thus, (3.2) does not hold. It proves that $D^+y(t) \le 0$ for all $t \ge t_0 + r$.

Secondly, using similar arguments as those in the proof of $D^+y(t) \le 0$, we can obtain

$$D^-u(t) \ge 0 \quad \text{for all } t \ge t_0 + r.$$

From the above results, we see that $y$ is non-increasing and $u$ is non-decreasing on $[t_0 + r, +\infty)$. In view of the boundedness of $x$, we obtain

$$\begin{gathered} \lim_{t\to+\infty} y(t) = A \ge \lim_{t\to+\infty} u(t) = B, \\ A \ge L_i \ge l_i \ge B, \quad i \in J. \end{gathered}$$

It suffices to show that $L_i = l_i, i \in J$. Suppose that, on the contrary, either $L_1 > l_1$ or $L_2 > l_2$ holds. We next consider that $L_1 > l_1$. (The situation is analogous for $L_2 > l_2$.) Then, it is easily to see that $B < A$, and $A$ and $B$ are not zero simultaneously. Without loss of generality, we assume that $A \ne 0$ since the proof for the case of $B \ne 0$ is quite similar. For $\bar{H} \in (l_1, L_1) \subset (B, A)$, we can choose $t_0^* > t_0 + r$ and $\{\tau_m\}_{m=1}^{+\infty} \subset [t_0^* + r, +\infty)$ such that

$$x_1(\tau_m) = \bar{H}, \quad \lim_{m\to+\infty} \tau_m = +\infty, \quad x_i(t) \le A + \frac{|A|}{2} \quad \forall t \in [t_0^*, +\infty),\ i = 1, 2.$$

Then, for an arbitrary positive integer $m$, it follows from the monotonicity and definition of $y(t)$ that

$$F_1(A) \le F_1(y(\tau_m)) = F_1(A+\varepsilon_m), \quad 0 \le \varepsilon_m \le \frac{|A|}{2}, \quad \varepsilon_m = y(\tau_m) - A \to 0$$

as $m \to +\infty$). In the light of the fact that $\gamma^+ \ge \gamma^- > 0$ and

$$y(\tau_m) \ge y(t) \ge x_i(t) \quad \text{for all } t \in [\tau_m, \tau_m + 3r],\ i \in J,$$

we obtain

$$-F_1(x_1(t)) + F_1(y(\tau_m)) \ge 0 \quad \text{for all } t \in [\tau_m, \tau_m + 3r],$$

and

$$\begin{aligned} x_1'(t) &= \gamma_1(t)[-F_1(x_1(t)) + G_1(x_2(t-\tau_2(t)))] \\ &\le \gamma_1(t)[-F_1(x_1(t)) + F_1(x_2(t-\tau_2(t)))] \\ &\le \gamma_1(t)[-F_1(x_1(t)) + F_1(y(\tau_m))] \\ &\le \gamma_1^+[-F_1(x_1(t)) + F_1(A+\varepsilon_m)] \quad \text{for all } t \in [\tau_m, \tau_m + 3r]. \end{aligned} \tag{3.7}$$

Denote $v(t) = v(t;\tau_m,\varepsilon_m)$ the solutions of the initial-value problem

$$v'(t) = \gamma^+[-F_1(v(t)) + F_1(A+\varepsilon_m)], \quad v(\tau_m) = \bar{H}. \tag{3.8}$$

Note that $\bar{H} < A$. Proposition 2.1 implies that

$$A + \varepsilon_m - v(t;\tau_m,\varepsilon_m) \ge \mu > 0,\ t \in [\tau_m, \tau_m + 3r],$$

where the positive constant $\mu$ is independent of $\tau_m$ and $\varepsilon_m$. Furthermore, from (3.7) and (3.8), we have

$$x_1(t) \le v(t) < A + \varepsilon_m - \mu, \quad t \in [\tau_m, \tau_m + 3r], \tag{3.9}$$

$$y_1(s) = \max_{s-r\le t\le s} x_1(t) < A + \varepsilon_m - \mu, \quad s \in [\tau_m + r, \tau_m + 3r],$$

$$y_1(\tau_m + 2r) \le y_1(\tau_m + r) < A + \varepsilon_m - \mu. \tag{3.10}$$

For $s \in [\tau_m + 2r, \tau_m + 3r]$, from the fact that

$$y_2(s) = \max_{s-r\le t\le s} x_2(t),$$

it follows that there exists $t^* \in [s-r, s] \subseteq [\tau_m + r, \tau_m + 3r]$ such that

$$y_2(s) = x_2(t^*) = \max_{s-r\le t\le s} x_2(t)$$

and

$$\begin{aligned} 0 &\le x_2'(t^*) \\ &= \gamma_2(t^*)[-F_2(x_2(t^*)) + G_2(x_1(t^* - \tau_1(t^*)))] \\ &\le \gamma_2(t^*)[-F_2(x_2(t^*)) + F_2(x_1(t^* - \tau_1(t^*)))], \end{aligned}$$

which implies that

$$y_2(s) = \max_{s-r\le t\le s} x_2(t) = x_2(t^*) \le x_1(t^* - \tau_1(t^*)) < A + \varepsilon_m - \mu,$$

and

$$y_2(\tau_m + 2r) < A + \varepsilon_m - \mu. \tag{3.11}$$

From (3.10) and (3.11), we have

$$y(\tau_m + 2r) = \max\{y_1(\tau_m + 2r), y_2(\tau_m + 2r)\} < A + \varepsilon_m - \mu,$$

which contradicts that $\lim_{m\to+\infty} y(\tau_m + r) = \lim_{t\to+\infty} y(t) = A$. Hence, $L_1 = l_1$. This completes the proof. □

From Lemma 2.5, we have the following results for equation (1.6).

**Corollary 3.2.** *Let $F_i = G_i$ ($i \in J$). Then every solution of the initial value problem* (1.6) *and* (1.8) *tends to a constant vector as $t \to +\infty$.*

**Remark 3.3.** It is worth noting that system (1.6) includes the scalar equation

$$x'(t) = \gamma(t)[-F(x(t)) + F(x(t - \tau(t)))], \tag{3.12}$$

as a special case. In fact, if $F_i = G_i = F$, $\gamma_i = \gamma$, $\tau_i = \tau$ and consider the synchronized solutions of (1.6) with $x_1(t) = x_2(t) = \varphi(t)$ for $t \in [-\tau^+, 0]$, then system (1.6) reduces to scalar equation (3.12). This implies that Bernfeld and Haddock conjecture is only a special case of Corollary 3.2 with $F(x) = x^{1/3}$ and $\gamma(t) \equiv 1$. Moreover, the main results in the most recently papers [7, 8] are also a special case of Corollary 3.2. In particular, we obtain from Corollary 3.2 that every solution of the following equation

$$x'(t) = \gamma(t)[x^{\frac{n}{m}}(t) - x^{\frac{n}{m}}(t - \tau(t))], \gamma(t) > 0, \quad \frac{n}{m} \in (0, 1),$$

tends to a constant as $t \to +\infty$. Here, $\tau(t)$ and $\gamma(t)$ are continuous functions and are bounded above and below by positive constants, and $x_{t_0} = \varphi \in C([-\tau^+, 0], \mathbb{R})$. This answers the second open problem proposed in [7].

## 4. Numerical simulations

Consider the following functional differential equations with time-varying delays,

$$x'(t) = -x^{1/3}(t) + x^{1/3}(t - (1 + |\cos t|)), \quad x_{t_0} = \varphi \in C([-2, 0], \mathbb{R}), \tag{4.1}$$

$$x'(t) = (1 + \cos^2 t)[-x^{1/3}(t) + x^{1/3}(t - (1 + |\cos t|))], \quad x_{t_0} = \varphi \in C([-2, 0], \mathbb{R}) \tag{4.2}$$

$$\begin{gathered} x_1'(t) = (1 + \cos^4 t)[-x_1^{3/5}(t) + x_2^{3/5}(t - (1 + |\sin t|))], \\ x_2'(t) = (1 + 3\cos^2 t)[-x_2^{3/5}(t) + x_1^{3/5}(t - (1 + |\cos t|))], \\ x_{t_0} = \varphi \in C([-2, 0], \mathbb{R}) \times C([-2, 0], \mathbb{R}). \end{gathered} \tag{4.3}$$

It follows from Corollary 3.2 that for every solution of (4.1), (4.2) and (4.3) tends to a constant solution as $t \to +\infty$. Figures 1–3 support this result with the numerical solutions of the above equations with different initial values.

Since two-dimensional Bernfeld-Haddock conjecture involving non-autonomous delay differential equations has not been touched in [7, 8, 15], one can find that all results in the above references cannot be applied to (4.3). Moreover, the scalar equation in Bernfeld-Haddock conjecture has been included in two-dimensional non-autonomous delay differential equation (1.6), and the conclusions related to Bernfeld-Haddock conjecture in the references above can be summed up as a special case of the results of this paper. This implies that our results extend previously known results.

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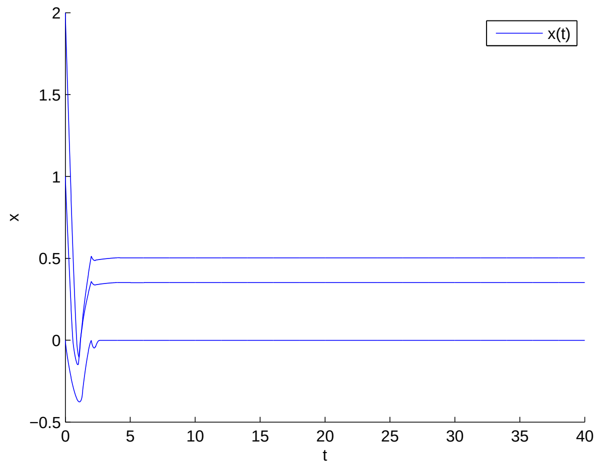


Figure 1. Numerical solutions of (4.1) for initial values $\varphi(s) = 1 + \sin s,\ 2 + \sin s,\ 6 \sin s,\ s \in [-2, 0]$.

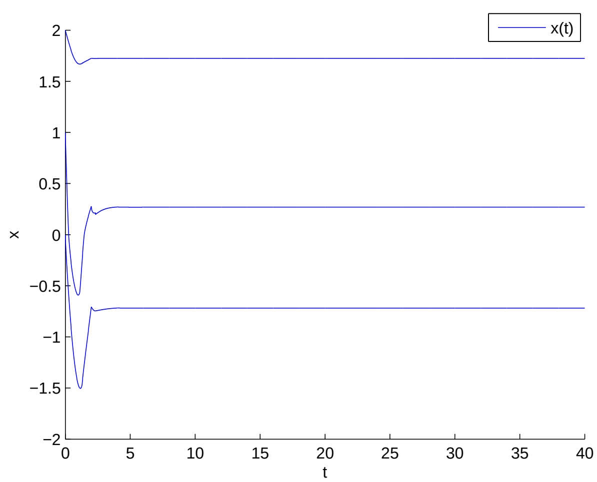


Figure 2. Numerical solutions of (4.2) for initial values $\varphi(s) = 1 + 3\sin s,\ 2 \sin s,\ 2 + 5 \sin s,\ s \in [-2, 0]$.

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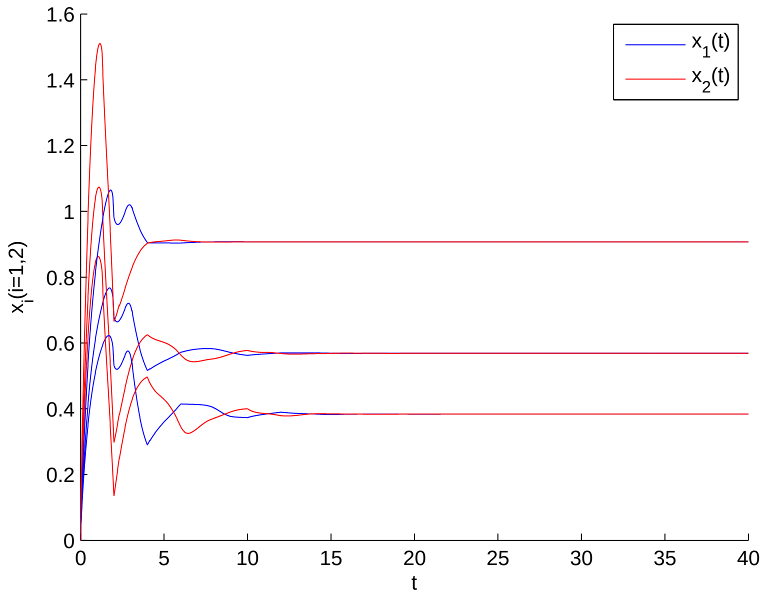


FIGURE 3. Numerical solutions of (4.3) for initial values $\varphi(s) = (-3\sin s, -3\sin s)$, $(-2\sin s, -2\sin s)$, $(-6\sin s, -6\sin s)$, $s \in [-2, 0]$.


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