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BOUNDEDLY SOLVABLE EXTENSIONS OF DELAY DIFFERENTIAL OPERATORS

BAHADIR Ö. GÜLER, BÜLENT YILMAZ, ZAMEDDIN I. ISMAILOV

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Abstract. We describe all boundedly solvable extensions of minimal operators generated by first-order linear delay differential operators in Hilbert spaces of vector-functions on finite intervals. Also, we study the structure of spectrum of these extensions. To do this we use methods from operator theory.

1. Introduction

It is known that many solvability problems arising in life sciences can be expressed as boundary value problems for linear functional (time delay, time proportional, neutral, advanced etc.) equations in corresponding functional spaces. The general theory of linear functional equations can be found in [1, 2, 3].

The solvability of the considered problems may be seen as boundedly solvability of linear differential operators in corresponding functional Banach spaces. Note that the theory of boundedly solvable extensions of a linear densely defined closed operator in Hilbert spaces was presented in the important works of Vishik in [7, 8].

Let us recall that an operator $S : D(S) \subset H \to H$ on any Hilbert space H is called boundedly solvable, if S is one-to-one and onto, and $S^{-1} \in L(H)$.

The main aim of this work is to describe of all boundedly solvable extensions of the minimal operator generated by first-order linear delay differential-operator expression in the Hilbert space of vector-functions at finite interval in terms of boundary conditions. Lastly, the structure of spectrum of these extensions will be investigated.

2. Description of solvable extensions

In the Hilbert space $L^2(H,(a,b)), a, b \in \mathbb{R}$ of H-valued vector-functions consider the linear delay differential-operator expression of first order in form

$$l(u) = (\alpha(t)u(t))' + A(t)u(t-\tau) \tag{2.1}$$

where:

(1) H is a separable Hilbert space;
(2) the function $\alpha : [a,b] \to \mathbb{R}_+$ is Lebesgue measurable;

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(3) there are positive reel numbers c and C such that for $x \in [a,b]$,
$$c \le \alpha(x) \le C;$$
(4) the operator-function $A(\cdot) : [a,b] \to L(H)$ is continuous on the uniform operator topology;
(5) $\frac{\|A(t)\|}{\alpha(t)} \in L^1(H,(a,b))$;
(6) $0 \le \tau < b-a$.

On the other hand, we shall consider the differential expression
$$m(\cdot) = d/dt \tag{2.2}$$
in the Hilbert space $L^2(H,(a,b))$ corresponding to (2.1). Using the standard way, the minimal M_0 and the maximal M operators generated by differential expression (2.2) can be defined (see [4]).

Now we define an operator $S_\tau : L^2(H,(a,b)) \to L^2(H,(a,b))$ by
$$S_\tau u(t) := \begin{cases} 0, & \text{if } a < t < a+\tau, \\ u(t-\tau), & \text{if } a+\tau < t < b. \end{cases}$$
for $u \in L^2(H,(a,b))$. It is clear that $S_\tau \in L(L^2(H,(a,b)))$ and $\|S_\tau\| = 1$.

We also define the minimal L_0 and the maximal L operators corresponding to differential-operator expression
$$l(u) = (\alpha(t)u(t))' + A(t)S_\tau u(t)$$
in $L^2(H,(a,b))$ (see [4]).

By $U(t,s)$ with $t,s \in [a,b]$ we denote the family of evolution operators corresponding to the homogeneous differential operator equation
$$\frac{\partial}{\partial t}U(t,s)f + \frac{A(t)S_\tau}{\alpha(t)}U(t,s)f = 0, \quad t,s \in [a,b],$$
$$U(s,s)f = f, \quad f \in H.$$
The operator $U(t,s)$ is linear, continuous boundedly invertible and
$$U^{-1}(t,s) = U(t,s), \quad t,s \in [a,b]\,.$$
For a detail analysis see [6].

Now we introduce the following operators:
$$\begin{gathered} Uz(t) := U(t,0)z(t), \\ Vz(t) := \frac{1}{\alpha(t)}Uz(t), \\ U,V : L^2(H,(a,b)) \to L^2(H,(a,b)). \end{gathered}$$
In this case it is easy to check that
$$\begin{aligned} l(Vz) &= (\alpha Vz)'(t) + A(t)S_\tau Vz(t) \\ &= (Uz(t))' + \frac{A(t)S_\tau}{\alpha(t)}Uz(t) \\ &= Uz'(t) + (U_t' + \frac{A(t)S_\tau}{\alpha(t)}U)z(t) \\ &= Uz'(t) = Um(z) \end{aligned}$$

Therefore,

$$U^{-1}l(Vz) = m(z).$$

Hence it is clear that if $\widetilde{L}$ is some extension of the minimal operator L_0, that is, $L_0 \subset \widetilde{L} \subset L$. Then

$$\begin{gathered} U^{-1}L_0V = M_0, \\ M_0 \subset U^{-1}\widetilde{L}V = \widetilde{M} \subset M, \\ U^{-1}LV = M. \end{gathered}$$

Now we prove the following assertions.

Theorem 2.1. $\ker L_0 = [0]$ *and* $\overline{\operatorname{Im}(L_0)} \neq L^2(H,(a,b))$.

Proof. If for any $u \in D(L_0)$

$$L_0u = 0,$$

then from the relation $U^{-1}L_0V = M_0$ it is obtained that $UM_0V^{-1}(u) = 0$. From last equation $M_0V^{-1}(u) = 0$. Since $\ker M_0 = 0$, then $V^{-1}u(t) = 0$, Consequently $u = 0$. So $\ker L_0 = 0$. To prove the relation $\overline{\operatorname{Im}(L_0)} \neq L^2(H,(a,b))$, consider the subspace $\ker(L_0^*)$ in $L^2(H,(a,b))$.

In this case it is clear that the differential equation

$$\begin{aligned} L_0^*u(t) &= (UM_0V^{-1})^*u(t) = (V^{-1})^*M_0^*U^*u(t) \\ &= -(V^{-1})^*(U^*z(t))' = 0 \end{aligned}$$

has solution of the form

$$U^*u(t) = g, \quad g \in H$$

that is,

$$u(t) = (U^*)^{-1}g, \quad g \in H,\ t \in (a,b)$$

This shows that

$$\ker(L_0^*) \neq 0$$

From this and the relation

$$\operatorname{Im}(L_0) \oplus \ker(L_0^*) = L^2(H,(a,b)),$$

we obtain that

$$\overline{\operatorname{Im}(L_0)} \neq L^2(H,(a,b)).$$

$\square$

Theorem 2.2. *For the domains of minimal L_0 and the maximal L operators*

$$\begin{gathered} D(L) = \{u \in L^2(H,(a,b)) : \alpha u \in W_2^1(H,(a,b))\} \text{ and} \\ D(L_0) = \{u \in D(L) : \lim_{t\to a^+}(\alpha u)(t) = \lim_{t\to b^-}(\alpha u)(t) = 0\} \end{gathered}$$

respectively.

Proof. First of all note that for any $u \in D(L)$, from the relations $U^{-1}L_0V = M_0$ and $ULV = M$ we obtain

$$V^{-1}u \in D(M_0), \quad V^{-1}u \in D(M_0)$$

and vice versa.

From these facts and the relations

$$D(M_0) = \overset{\circ}{W}{}_2^1(H,(a,b)),$$

$$D(M) = W_2^1(H, (a, b)),$$

the validity of assertion is obtained. □

Theorem 2.3. *Each solvable extension $\widetilde{L}$ of the minimal operator L_0 in $L^2(H, (a, b))$ is generated by the differential-operator expression* (2.1) *and the boundary condition*

$$(K + E)(\alpha u)(a) = KU(a, b)(\alpha u)(b), \tag{2.3}$$

where $K \in L(H)$, E is the identity operator in H and $(\alpha u)(a) = \lim_{t\to a^+}(\alpha u)(t)$, $(\alpha u)(b) = \lim_{t\to b^-}(\alpha u)(t)$. The operator K is determined uniquely by the extension $\widetilde{L}$, i.e $\widetilde{L} = L_K$.

On the contrary, the restriction of the maximal operator L_0 to the manifold of vector-functions satisfy the condition (2.3) *for some bounded operator $K \in L(H)$ is a boundedly solvable extension of the minimal operator L_0 in $L^2(H, (a, b))$.*

Proof. Firstly, all boundedly solvable extensions $\widetilde{M}$ of the minimal operator M_0 in $L^2(H, (a, b))$ are described in terms of boundary conditions.

Consider the so-called Cauchy extension M_c, $M_c u = u'(t)$,

$$M_c : D(M_c) \to L^2(H, (a, b)),$$
$$D(M_c) = \{u \in W_2^1(H, (a, b)) : u(0) = 0\} \subset L^2(H, (a, b)),$$

of the minimal operator M_0. It is clear that M_c is a solvable extension of M_0 and

$$M_c^{-1} f(t) = \int_a^t f(x)dx, \quad f \in L^2(H, (a, b)),$$
$$M_c^{-1} : L^2(H, (a, b)) \to L^2(H, (a, b)).$$

Now assume that $\widetilde{M}$ is a solvable extension of the minimal operator M_0 in $L^2(H, (a, b))$. In this case it is known that the domain of $\widetilde{M}$ can be written as a direct sum

$$D(\widetilde{M}) = D(M_0) \oplus (M_c^{-1} + K)V,$$

where $V = \ker M = H, K \in L(H)$ (see [8]). Therefore for each $u(t) \in D(\widetilde{M})$ the following is true

$$u(t) = u_0(t) + M_c^{-1} f + Kf, \quad u_0 \in D(M_0), \quad f \in H.$$

That is,

$$u(t) = u_0(t) + tf + Kf, \quad u_0 \in D(M_0), \quad f \in H.$$

Hence

$$u(0) = Kf, \quad u(1) = f + Kf = (K + E)f$$

and from these relations it is obtained that

$$(K + E)u(a) = Ku(b). \tag{2.4}$$

On the other hand, the uniqueness of the operator $K \in L(H)$ is clear from the work in [8]. Therefore $\widetilde{M} = M_K$. This completes of necessary part of this assertion.

On the contrary, if M_K is a operator generated by differential expression (2.2) and boundary condition (2.4), then M_K is boundedly invertible and

$$M_K^{-1} : L^2(H, (a, b)) \to L^2(H, (a, b)),$$
$$M_K^{-1} f(t) = \int_a^t f(x)dx + K \int_a^b f(x)dx, \quad f \in L^2(H, (a, b)).$$

Consequently, all solvable extensions of the minimal operator M_0 in $L^2(H,(a,b))$ is generated by the differential expression (2.2) and the boundary condition (2.4) for any linear bounded operator K.

The extension $\widetilde{L}$ of the minimal operator L_0 is solvable in $L^2(H,(a,b))$ if and only if the operator $\widetilde{M} = U^{-1}\widetilde{L}V$ is an extension of the minimal operator M_0 in $L^2(H,(a,b))$. Then $u \in D(\widetilde{L})$ if and only if

$$V^{-1}u \in D(\widetilde{M}),$$

Hence there exists $K \in L(H)$ such that

$$(K+E)V^{-1}u(a) = KV^{-1}u(b).$$

Consequently,

$$(K+E)U^{-1}(a,a)(\alpha u)(a) = KU^{-1}(b,a)(\alpha u)(b).$$

From the above equality,

$$(K+E)(\alpha u)(a) = KU(a,b)(\alpha u)(b).$$

This proves the validity of the claims in the theorem. □

Remark 2.4. Now consider in $L^2(H,(a,b))$ the differential expression

$$l(u) = (\alpha(x)u(x))' + A(t)u(t),$$

where $\alpha(x) = 0$, $x \in (c,d)$ and $a < c < d < b$ with corresponding conditions. Assume that for any $t \in [c,d]$, $A(t)$ is boundedly invertible in H and $\|A^{-1}(t)\| \in L^2(c,d)$. In this case, all boundedly solvable extensions of the minimal operator in $L^2(H,(a,b)) = L^2(H,(a,c)) \oplus L^2(H,(c,d)) \oplus L^2(H,(d,b))$ are generated by the differential-operator expression $l(\cdot)$ and the boundary conditions

$$\begin{aligned}(K_1+E)(\alpha u)(a) &= K_1U_1(a,c)(\alpha u)(c),\\ (K_2+E)(\alpha u)(d) &= K_2U_2(d,b)(\alpha u)(b),\end{aligned}$$

where $K_1, K_2 \in L(H)$; E is an identity operator in H and U_1, U_2 constitute a family of evolution operators generated by corresponding differential equations in $L^2(H,(a,c))$ and $L^2(H,(d,b))$ respectively.

3. Structure of spectrum of boundedly solvable extensions

In this section we investigated the geometric form in complex plane of boundedly solvable extensions of the minimal operators L_0 in $L^2(H,(a,b))$. First let us prove the following assertion.

Theorem 3.1. *If $\widetilde{L}$ is a boundedly solvable extension of the minimal operator L_0 and $\widetilde{M} = U^{-1}\widetilde{L}V$ is the corresponding boundedly solvable extension of the minimal operator M_0, then in order for $\lambda \in \sigma(\widetilde{L})$ the necessary and sufficient condition is $0 \in \sigma(\widetilde{M} - \lambda T_\alpha)$, where an operator $T_\alpha : L^2(H,(a,b)) \to L^2(H,(a,b))$ is a multiplication operator to $1/\alpha(t)$.*

Proof. If $\widetilde{L} = L_K$ is a boundedly solvable extension of the minimal operator L_0 and $\lambda \in \mathbb{C}$, then it is clear that

$$\begin{aligned}L_K - \lambda E &= UM_KV^{-1} - \lambda E\\ &= U(M_K - \lambda U^{-1}V)V^{-1}\end{aligned}$$

$$= U(M_K - \lambda \frac{1}{\alpha(t)} E)V^{-1}$$

The last relation explains the validity of the theorem. □

Now prove the main theorem on the spectrum structure of extensions of the minimal operator L_0.

Theorem 3.2. *The spectrum of the boundedly solvable extension L_K of the minimal operator L_0 in $L^2(H,(a,b))$ has the form*

$$\sigma(L_K) = \Big\{ \Big(\int_a^b \frac{ds}{\alpha(s)} \Big)^{-1} \big[\ln |\frac{\mu+1}{\mu}| + i \arg(\frac{\mu+1}{\mu}) + 2n\pi i \big] \\ : \mu \in \sigma(K)\backslash\{0,-1\},\ n \in \mathbb{Z} \Big\}$$

Proof. By Theorem 3.1 for the description the spectrum of boundedly solvable extension L_K in $L^2(H,(a,b))$ it is sufficient to investigate of boundedly solvability of the operator $M_K - \lambda T_\alpha$ in $L^2(H,(a,b))$ for $\lambda \in \mathbb{C}$. Now consider the spectral problem

$$M_K u = \lambda T_\alpha u + f, \lambda \in \mathbb{C}, f \in L^2(H,(a,b)).$$

From this, it is clear that

$$u'(t) = \lambda \frac{1}{\alpha(t)} u(t) + f(t),$$
$$(K+E)u(a) = Ku(b),$$
$$\lambda \in \mathbb{C}, f \in L^2(H,(a,b)), K \in L(H).$$

In this case it is evident that a general solution of above differential equation in $L^2(H,(a,b))$ has a form

$$u_\lambda(t) = e^{\lambda \int_a^t \frac{ds}{\alpha(s)}} f_0 + \int_a^t e^{\lambda \int_\tau^t \frac{ds}{\alpha(s)}} f(\tau) d\tau, f_0 \in H.$$

Therefore from the boundary condition, we obtain the expression

$$\Big(E + K\Big(1 - e^{\lambda \int_a^t \frac{ds}{\alpha(s)}}\Big)\Big) f_0 = K \int_a^b e^{\lambda \int_\tau^t \frac{ds}{\alpha(s)}} f(\tau) d\tau$$

For $\lambda_m = 2m\pi i\big(\int_a^b \frac{ds}{\alpha(s)} \big)^{-1}$ with $m \in \mathbb{Z}$, from the above relation, it follows that

$$f_0^{(m)} = K \int_a^b e^{\lambda_m \int_\tau^b \frac{ds}{\alpha(s)}} f(\tau) d\tau, \quad m \in \mathbb{Z}.$$

Consequently, in this case the inverse operator $(M_K - \lambda T_\alpha)^{-1}$ is of the form

$$(M_K - \lambda_m T_\alpha)^{-1} f(t) = K e^{\lambda_m \int_a^t \frac{ds}{\alpha(s)}} \Big(\int_a^b e^{\lambda_m \int_\tau^b \frac{ds}{\alpha(s)}} f(\tau) d\tau \Big) + \int_a^t e^{\lambda_m \int_\tau^t \frac{ds}{\alpha(s)}} f(\tau) d\tau,$$

$\lambda_m \in \mathbb{Z}$, $f \in L^2(H,(a,b))$, and it is clear that for this λ_m, $m \in \mathbb{Z}$,

$$(M_K - \lambda_m T_\alpha)^{-1} \in L(L^2(H,(a,b))).$$

On the other hand, if $\lambda \neq 2m\pi i$, $m \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, then from

$$\Big(E + K\Big(1 - e^{\lambda \int_a^b \frac{ds}{\alpha(s)}}\Big)\Big) f_0 = K \int_a^b e^{\lambda \int_\tau^b \frac{ds}{\alpha(s)}} f(\tau) d\tau,$$

we have

$$\Big(K-\frac{1}{e^{\lambda\int_a^b\frac{ds}{\alpha(s)}}-1}\Big)f_0=\Big(\frac{1}{1-e^{\lambda\int_a^b\frac{ds}{\alpha(s)}}}\Big)K\int_a^b e^{\lambda\int_\tau^b\frac{ds}{\alpha(s)}}f(\tau)d\tau,\quad f_0\in H.$$

This implies: $0\in\sigma(M_K-\lambda_m T_\alpha)$ if and only if $\mu=\frac{1}{e^{\lambda\int_a^b\frac{ds}{\alpha(s)}}-1}\in\sigma(K)$. Hence in this case we have

$$\lambda_n=\Big(\int_a^b\frac{ds}{\alpha(s)}\Big)^{-1}\big[\ln|\frac{\mu+1}{\mu}|+i\arg(\frac{\mu+1}{\mu})+2n\pi i\big],$$

where $\mu\in\sigma(K)$, $n\in\mathbb{Z}$. From this and Theorem 3.1 the validity of the claim is evident. □

Corollary 3.3. *(1) If $\sigma(K)\subset\{0,-1\}$, then for the spectrum corresponding boundedly solvable extension L_K is true $\sigma(L_K)=\emptyset$. (2) If $\sigma(K)\neq\{0,-1\}\neq\emptyset$, then $\sigma(L_K)$ is infinite.*

Example 3.4. All boundedly solvable extensions of the minimal operator L_0 in $L^2(0,1)$ generated by differential expression

$$l(u)=\big((|x-\tfrac{1}{2}|+|x-\tfrac{1}{3}|)u(x)\big)'+\int_0^x a(t)u(t-\tau)d\tau, a\in C[0,1]$$

are generated by the integro-differential expression $l(.)$ and the boundary condition

$$(k+1)\left((|x-\tfrac{1}{2}|+|x-\tfrac{1}{3}|)u(x)\right)(0)=kU(0,1)\left((|x-\tfrac{1}{2}|+|x-\tfrac{1}{3}|)u(x)\right)(1),\quad k\in\mathbb{C}$$

and $U(\cdot,\cdot)$ are the corresponding evolution operators in the Hilbert space $L^2(0,1)$. In this case, the spectrum $\sigma(L_k)$ of the extension L_k when $k\neq 0,-1$ is of the form

$$\sigma(L_k)=\Big\{\frac{1}{\ln(e\sqrt{35})}\big[\ln|\frac{k+1}{k}|+i\arg(\frac{k+1}{k})+2n\pi i\big]:n\in\mathbb{Z}\Big\}.$$

When $k=0$ or $k=-1$, the spectrum of this extension is empty by Corollary 3.3.

When $\alpha(t)=1$ for $t\in(a,b)$, Theorems 2.3 and 3.2 have been proven in [5].

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Bahadir Ö. Güler
Department of Mathematics, Karadeniz Technical University, Turkey
E-mail address: boguler@ktu.edu.tr

Bülent Yilmaz
Department of Mathematics, Marmara University, Turkey
E-mail address: bulentyilmaz@marmara.edu.tr

Zameddin Ismailov
Department of Mathematics, Karadeniz Technical University, Turkey
E-mail address: zameddin.ismailov@gmail.com