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GLOBAL WELL-POSEDNESS AND DECAY RESULTS FOR 3D GENERALIZED MAGNETO-HYDRODYNAMIC EQUATIONS IN CRITICAL FOURIER-BESOV-MORREY SPACES

AZZEDDINE EL BARAKA, MOHAMED TOUMLILIN

Abstract. This article concerns the Cauchy problem of the 3D generalized incompressible magneto-hydrodynamic (GMHD) equations. By using the Fourier localization argument and the Littlewood-Paley theory as in [5, 31], we obtain global well-posedness results of the GMHD equations with small initial data belonging to the critical Fourier-Besov-Morrey spaces. Moreover, we prove that the corresponding global solution decays to zero as time approaches infinity.

1. Introduction

We investigate the generalized magneto-hydrodynamic equations in the whole space $\mathbb{R}^3$,

$$\begin{gathered} u_t + u\cdot\nabla u + \mu(-\Delta)^\alpha u - b\cdot\nabla b + \nabla\pi = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^3, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ b_t + u\cdot\nabla b + \nu(-\Delta)^\alpha b - b\cdot\nabla u = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^3, \\ (u,b)|_{t=0} = (u_0,b_0), \end{gathered} \tag{1.1}$$

where $u = u(t,x) \in \mathbb{R}^3$ denotes the velocity field of the flow, $b(t,x)$ denotes the magnetic field, $\pi(t,x) : \mathbb{R}^3 \to \mathbb{R}$ represents the pressure function, $\mu \geq 0$ and $\nu \geq 0$ are real positive parameters, $\nabla\cdot u = 0$ and $\nabla\cdot b = 0$ represent the incompressible conditions, and u_0 and b_0 are for given initial velocity and initial magnetic field with $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$, respectively. The operator $(-\Delta)^\alpha$ is the Fourier multiplier with symbol $|\xi|^{2\alpha}$.

The GMHD system plays a fundamental role in applied sciences such as astrophysics, geophysics, and plasma physics. The first equation of system (1.1) reflects the conservation of momentum, the third equation of system (1.1) is the magnetic induction equation, and the second equation of system (1.1) specifies the conservation of mass.

When $\alpha = 1$, the GMHD system (1.1) becomes the usual MHD equations, which describes the macroscopic behavior of the electrically conducting incompressible fluids in a magnetic field; when $\alpha = 1$ and $b = 0$, the GMHD equations reduce

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to the Navier-Stokes equations. The study of the generalized (1.1) equations will improve our understanding of the Navier-Stokes equations and the MHD equations, which has drawn much attention during the past twenty more years. Let's take this opportunity to briefly quote some works; Duvaut and Lions [11] constructed a global Leray-Hopf weak solution and a local strong solution of the 3D incompressible MHD system, C. Cao, J. Wu [8] proved global regularity of classical solutions for the MHD equations with mixed partial dissipation and magnetic diffusion, and they also give the global existence, conditional regularity and uniqueness of a weak solution for 2D MHD equations with only magnetic diffusion. For more results in this direction, see [6, 9] and reference therein.

On the other hand, there are numerous important progresses on the fundamental issue of the blow-up criteria or regularity criteria to the system (1.1) (see [11, 7, 10, 14, 16, 15, 22, 27, 32, 36] and the references cited therein for more details).

For the GMHD system (1.1), the global-in-time weak solution for any given divergence free initial value $(u_0, b_0) \in L^2(\mathbb{R}^n)$ was proved by Wu [29], the local-in-time existence and uniqueness of smooth solution for any sufficient smooth initial data (u_0, b_0) was established by Yuan [33], and Liu, Zhao and Cui [21] obtained the global existence and stability of solutions for system (1.1) with small initial data (u_0, b_0) belonging to the pseudomeasure space $\mathcal{PM}^\alpha$, where $\mathcal{PM}^\alpha$ is defined by

$$\mathcal{PM}^\alpha := \{f \in \mathcal{S}' : \hat{f} \in L^1_{loc}(\mathbb{R}^3), \|f\|_{\mathcal{PM}^\alpha} := \operatorname{ess\,sup}_{\xi\in\mathbb{R}^3} |\xi|^\alpha |\hat{f}(\xi)| < \infty\}.$$

Recently, Wang and Wang [28] and Ye [37] obtained the global existence results for classical 3-D MHD ($\alpha = 1$) and GMHD ($\frac{1}{2} \le \alpha \le 1$), respectively.

To give a clearer introduction to our results in this paper, we first note that system (1.1) enjoys scaling properties. Clearly, if $(u(t,x), b(t,x))$ is a solution to system (1.1), then $(u^\lambda(t,x), b^\lambda(t,x))$ is also a solution of (1.1) corresponding to the initial data $(u_0^\lambda, b_0^\lambda)$, where

$$\begin{gathered} u^\lambda(t,x) := \lambda^{2\alpha-1} u\big(\lambda^{2\alpha} t, \lambda x\big), \quad b^\lambda(t,x) := \lambda^{2\alpha-1} b\big(\lambda^{2\alpha} t, \lambda x\big), \\ u_0^\lambda(x) := \lambda^{2\alpha-1} u_0(\lambda x), \quad b_0^\lambda(x) := \lambda^{2\alpha-1} b_0(\lambda x). \end{gathered}$$

In this article, we use $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}$ to denote the homogenous Fourier Besov-Morrey spaces, C will denote constants which can be different at different places, $\mathsf{U} \lesssim \mathsf{V}$ means that there exists a constant $C > 0$ such that $\mathsf{U} \le C\mathsf{V}$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \le p \le \infty$.

Motivated by the works [37, 31, 14, 2], the aim of this article is to prove the global existence and the decay property of the system (1.1) in the Fourier Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}(\mathbb{R}^3)$.

2. Preliminaries and statement of main result

The proofs of the results presented in this paper are based on a dyadic partition of unity in the Fourier variables, the so-called, homogeneous Littlewood-Paley decomposition. We recall briefly this construction below. We start with a dyadic decomposition of $\mathbb{R}^n$.

Suppose $\chi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying

$$\operatorname{supp} \chi \subset \{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\},$$

$$\operatorname{supp}\varphi \subset \{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\},$$
$$\chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n,$$
$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n\backslash\{0\},$$

and denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\mathcal{P}$ the set of all polynomials. The space of tempered distributions is denoted by S'. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cutoff operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jn}\int h(2^j y)u(x-y)\,dy,$$
$$\dot{S}_j u = \sum_{k\le j-1} \dot{\Delta}_k u = \chi(2^{-j}D)u = 2^{jn}\int \tilde{h}(2^j y)u(x-y)\,dy,$$

where $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$.

First, we recall the definition of Morrey spaces which are a complement of L^p spaces.

Definition 2.1 ([17, 25])**.** For $1 \le p < \infty$, $0 \le \lambda < n$, the Morrey space $\mathrm{M}_p^\lambda = \mathrm{M}_p^\lambda(\mathbb{R}^n)$ is defined as the set of functions $f \in L^p_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{M}_p^\lambda} = \sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\lambda/p}\|f\|_{L^p(B(x_0,r))} < \infty, \tag{2.1}$$

where $B(x_0,r)$ denotes the ball in $\mathbb{R}^n$ with center x_0 and radius r. the space M_p^λ endowed with the norm $\|f\|_{\mathrm{M}_p^\lambda}$ is a Banach space. In the case $p=1$, M_p^λ should be understood as a space of Radon measures and $\|f\|_{\mathrm{L}^1(B(x_0,r))}$ denoting the total variation of f on $B(x_0,r)$. For various reasons we find it convenient to include L^∞ among the Morrey spaces, but the indices in the notation M_p^λ will always be restricted to $1 \le p < \infty$, $0 \le \lambda < n$, notwithstanding that (2.1) makes sense for $\lambda = n$ and the resulting space is equivalent to L^∞(irrespective of the value of p). It is not difficult to see that the relation $\mathrm{M}_{p_1}^\lambda \hookrightarrow \mathrm{M}_{p_2}^\mu$ provided $\frac{n-\mu}{p_2} \ge \frac{n-\lambda}{p_1}$ and $p_2 \le p_1$, and $\mathrm{M}_p^0 = L^p$.
If $1 \le p_1, p_2, p_3 < \infty$ and $0 \le \lambda_1, \lambda_2, \lambda_3 < n$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have the Hölder type inequality

$$\|fg\|_{\mathrm{M}_{p_3}^{\lambda_3}} \le \|f\|_{\mathrm{M}_{p_1}^{\lambda_1}}\|g\|_{\mathrm{M}_{p_2}^{\lambda_2}}.$$

Also, for $1 \le p < \infty$ and $0 \le \lambda < n$,

$$\|\varphi * g\|_{\mathrm{M}_p^\lambda} \le \|\varphi\|_{L^1}\|g\|_{\mathrm{M}_p^\lambda}, \tag{2.2}$$

for all $\varphi \in L^1$ and $g \in \mathrm{M}_p^\lambda$.

Definition 2.2 (homogeneous Besov-Morrey spaces)**.** Let $s \in \mathbb{R}$, $1 \le p < +\infty$, $1 \le q \le +\infty$, and $0 \le \lambda < n$, the space $\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n) = \Big\{u \in \mathcal{Z}'(\mathbb{R}^n);\quad \|u\|_{\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} < \infty\Big\}.$$

Here

$$\|u\|_{\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} = \begin{cases} \big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\dot{\Delta}_j u\|^q_{\mathrm{M}^\lambda_p} \big\}^{1/q} & \text{for } q<\infty, \\ \sup_{j\in\mathbb{Z}} \sup 2^{js} \|\dot{\Delta}_j u\|_{\mathrm{M}^\lambda_p} & \text{for } q=\infty\,. \end{cases}$$

The space $\mathcal{Z}'(\mathbb{R}^n)$ denotes the topological dual of the space

$$\mathcal{Z}(\mathbb{R}^n) = \big\{ f \in \mathcal{S}(\mathbb{R}^n); \partial^\alpha \widehat{f}(0) = 0 \text{ for every multi-index } \alpha \big\},$$

and it can be identified to the quotient space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, where $\mathcal{P}$ denotes the set of all polynomials on $\mathbb{R}^n$. We refer to [34, chap. 8] for more details.

Definition 2.3 (homogeneous Fourier-Besov-Morrey spaces)**.** Let $s \in \mathbb{R}$, $0 \le \lambda < n$, $1 \le p < +\infty$ and $1 \le q \le +\infty$. The space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ denotes the set of all $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)} = \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{\mathrm{M}^\lambda_p} \Big\}^{1/q} < +\infty, \tag{2.3}$$

with suitable modification made when $q = \infty$.

Note that the space $\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (2.3) is a Banach space. We also notice that the Fourier-Besov-Morrey spaces are independent of the choice of φ_j, and the advantage of working in these spaces lies in they are more adapted than the classical Besov-Morrey-spaces for estimating the bilinear paraproduct using Hölder's inequality directly, instead of Bernstein's inequality. Now, we recall the definition of the mixed space-time spaces used in [5, 31].

Definition 2.4. Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q, \rho \le \infty$, $0 \le \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\begin{aligned} &\|u(t,x)\|_{\pounds^\rho(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} \\ &= \Big\{ \sum_{j\in\mathbb{Z}} 2^{jqs} \|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I,\mathrm{M}^\lambda_p)} \Big\}^{1/q}, \end{aligned}$$

and denote by $\pounds^\rho(I, \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R}\times\mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\pounds^\rho(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}$ norm.

Our first main result is the following theorem.

Theorem 2.5. *Let $1 \le p < \infty$, $1 \le q \le 2$, $0 \le \lambda < 3$, and $\frac{1}{2} < \alpha < 1 + \frac{3}{2p'} + \frac{\lambda}{2p}$. Then there exists a constant $C_0(\alpha, p, q)$ such that, for any $(u_0, b_0) \in \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ satisfying $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$ and*

$$\|(u_0, b_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le C_0 \min\{\mu, \nu\},$$

the Cauchy problem (1.1) *admits a unique global solution*

$$(u, b) \in \mathcal{C}\big([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big) \cap \pounds^1\big([0,\infty), \mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big),$$

and it satisfies

$$\begin{aligned} &\|(u,b)\|_{\pounds^\infty\left([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)\cap\pounds^1\left([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\ &\le 2\Big(1 + (\frac{16}{9})^\alpha\Big) \|(u_0, b_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,. \end{aligned}$$

Remark 2.6. Theorem 2.5 extends the result of [20] from Fourier-Herz spaces to Fourier-Besov-Morrey spaces, in fact, for $\lambda = 0$, $\mathcal{F}\dot{\mathcal{N}}^{s}_{1,0,q} = \mathrm{F}\dot{\mathrm{B}}^{s}_{1,q} = \dot{\mathcal{B}}^s_q$ where $\dot{\mathcal{B}}^s_q$ is the homogeneous Fourier-Herz spaces (see definition 2.9).

In addition, this result also holds in the Fourier-Besov spaces, in fact, for $\lambda = 0$, $\mathcal{F}\dot{\mathcal{N}}^{s}_{p,0,q} = \mathrm{F}\dot{\mathrm{B}}^{s}_{p,q}$ where $\mathrm{F}\dot{\mathrm{B}}^{s}_{p,q}$ is the homogeneous Fourier-Besov spaces.

We also remark that for general α and $b = 0$, the equation of system (1.1) becomes the Fractional Navier-Stokes equations.

Our second purpose of this paper is to prove the non-blowup at large time and the norm of global solution in $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ goes to zero at infinity.

Theorem 2.7. *Let $1 \le p, q \le 2$, $0 \le \lambda < \frac{p}{2}$, and $\frac{5}{6} + \frac{\lambda}{3p} < \alpha \le 1$. Assume that $(u, b) \in \mathcal{C}\big([0,\infty); \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big)$ is a global solution of system (1.1) given by Theorem 2.5, then*

$$\lim_{t\to\infty} \sup \Big(\|u(t)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|b(t)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \Big) = 0\,.$$

Recently, Zhuan [37] obtained the same property in the space $\chi^s = \mathrm{F}\dot{\mathrm{B}}^{s}_{1,1} = \mathcal{F}\dot{\mathcal{N}}^{s}_{1,0,1}$. Therefore, Theorem 2.7 improves and extends his result.

Remark 2.8. The Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}(\mathbb{R}^3)$ is critical for (1.1). For this, set $u_{0,\gamma}(\xi) = \gamma^{2\alpha-1}u_0(\gamma\xi)$, then $\widehat{u_{0,\gamma}}(\xi) = \gamma^{2\alpha-4}\widehat{u}_0(\gamma^{-1}\xi)$. Next, setting

$$f_j(\xi) = \varphi(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi)\widehat{u_{0,\gamma}}(\xi),$$

we can obtain

$$\begin{aligned}
&2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\|f_j\|_{\mathrm{M}^\lambda_p}\\
&= 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})} \sup_{x_0\in\mathbb{R}^3}\sup_{r>0} r^{-\lambda/p}\|\varphi(2^{-j+[\log_2\gamma]-\log_2\gamma}\xi)\widehat{u_{0,\gamma}}(\xi)\|_{L^p(B(x_0,r))}\\
&= 2^{([\log_2\gamma]-\log_2\gamma)(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}2^{(j-[\log_2\gamma])(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\|\varphi(2^{-j+[\log_2\lambda]}\eta)\widehat{u_0}(\eta)\|_{\mathrm{M}^\lambda_p}\\
&\approx 2^{(j-[\log_2\gamma])(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\|\varphi(2^{-j+[\log_2\lambda]}\eta)\widehat{u_0}(\eta)\|_{\mathrm{M}^\lambda_p}\,.
\end{aligned}$$

This implies

$$\Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})q}\|f_j\|^q_{\mathrm{M}^\lambda_p}\Big\}^{1/q} \approx \|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,,$$

and since

$$\varphi_j(\xi)\widehat{u_{0,\gamma}}(\xi) = \sum_{|k-j|\le 2}\varphi_j(\xi)f_k(\xi)$$

we easily get

$$\|u_{0,\gamma}\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \approx \|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,.$$

Similarly,

$$\|b_{0,\gamma}\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \approx \|b_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,,$$

thus $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$ is a critical space for the GMHD equations (1.1)

Now we introduce the Fourier-Besov spaces which contain some known spaces applied in studying Navier-Stokes equations. The norm of Fourier-Besov spaces $F\dot{B}^s_{p,q}$[31] is defined as follows.

Definition 2.9. For $s \in \mathbb{R}$, $1 \le p, q \le \infty$, set

$$\|u\|_{F\dot{B}^s_{p,q}} = \begin{cases} \Big(\sum_{j\in\mathbb{Z}} 2^{jqs}\|\widehat{\dot{\Delta}_j u}\|^q_{L^p}\Big)^{1/q} & \text{for } q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\widehat{\dot{\Delta}_j u}\|_{L^p} & \text{for } q = \infty\,. \end{cases}$$

One defines the homogenous Fourier-Besov spaces $F\dot{B}^s_{p,q}$ by

$$F\dot{B}^s_{p,q} = \{u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P} : \|u\|_{F\dot{B}^s_{p,q}} < \infty\}\,.$$

Particularly, for $p = 1$ Cannone and Wu introduced the Fourier-Herz spaces $\dot{\mathcal{B}}^s_q$ [5] with the norm associated

$$\|u\|_{\dot{\mathcal{B}}^s_q} = \begin{cases} \big(\sum_{j\in\mathbb{Z}} 2^{jqs}\|\widehat{\dot{\Delta}_j u}\|^q_{L^1}\big)^{1/q} & \text{for } q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\widehat{\dot{\Delta}_j u}\|_{L^1} & \text{for } q = \infty\,. \end{cases}$$

Clearly, we have $\dot{\mathcal{B}}^s_q = F\dot{B}^s_{1,q}$. The space χ^{-1} introduced by Lei and Lin [18] is

$$\chi^{-1} = \{u \in \mathcal{S}'(\mathbb{R}^3); \int_{\mathbb{R}^3} |\xi|^{-1}|\widehat{u}|d\xi < \infty\}.$$

We have $\chi^{-1} = F\dot{B}^{-1}_{1,1} = \dot{\mathcal{B}}^{-1}_1$. We finish this section with a Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 2.10 ([13]). *Let $1 \le q \le p < \infty$, $0 \le \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p} \le \frac{n-\lambda_2}{q}$, and let γ be a multiindex. If $\operatorname{supp}(\widehat{f}) \subset \{|\xi| \le A2^j\}$ then there is a constant $C > 0$ independent of f and j such that*

$$\|(i\xi)^\gamma \widehat{f}\|_{\mathrm{M}^{\lambda_2}_q} \le C2^{j|\gamma|+j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})}\|\widehat{f}\|_{\mathrm{M}^{\lambda_1}_p}\,. \tag{2.4}$$

Note that

$$\|(i\xi)^\gamma \widehat{f}\|_{\mathrm{M}^{\lambda_2}_q} \le C2^{j|\gamma|}\|1.\widehat{f}\|_{\mathrm{M}^{\lambda_2}_p} \le 2^{j|\gamma|}C2^{j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})}\|\widehat{f}\|_{\mathrm{M}^{\lambda_1}_p}\,,$$

which gives (2.4).

3. Well-posedness

First, we consider the linear nonhomogeneous dissipative equation

$$\begin{gathered} u_t + \mu(-\Delta)^\alpha u = f(t,x) \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) = u_0(x) \quad x \in \mathbb{R}^3, \end{gathered} \tag{3.1}$$

for which we recall the following result.

Lemma 3.1 ([12]). *Let $I = [0,T)$, $0 < T \le \infty$, $s \in \mathbb{R}$, $0 \le \lambda < n$, $1 \le p < \infty$, and $1 \le q \le \infty$. Assume that $u_0 \in \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}$ and $f \in \pounds^1(I, \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$. Then the solution $u(t,x)$ to the Cauchy problem (3.1) satisfies*

$$\begin{aligned} &\|u\|_{\pounds^\infty(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} + \mu\|u\|_{\pounds^1(I,\mathcal{F}\dot{\mathcal{N}}^{s+2\alpha}_{p,\lambda,q})} \\ &\le (1 + (\frac{16}{9})^\alpha)(\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} + \|f\|_{\pounds^1(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}). \end{aligned} \tag{3.2}$$

If in addition q is finite, then u belongs to $\mathcal{C}(I, \mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})$.

Proof. Inequality (3.2) was proved in [12]. Now, we shall briefly present the proof of the continuity of $u(t,x)$ in time t when $1 \le q < \infty$. By using the definition of the Fourier-Besov-Morrey spaces, we have

$$\begin{aligned} &\|u(t_1) - u(t_2)\|^q_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} \\ &\le \sum_{j\le N} (2^{js}\|\hat{u}_j(t_1) - \hat{u}_j(t_2)\|_{\mathrm{M}^\lambda_p})^q + 2\sum_{j>N} (2^{js}\|\hat{u}_j(t)\|_{L^\infty(I,\mathrm{M}^\lambda_p)})^q \end{aligned} \tag{3.3}$$

where $\hat{u}_j = \varphi_j \hat{u}$. For any small constant $\varepsilon > 0$, let N be large enough such that

$$\sum_{j>N} 2^{jsq}\|\hat{u}_j(t)\|^q_{L^\infty(I,\mathrm{M}^\lambda_p)} \le \frac{\varepsilon}{4}\,. \tag{3.4}$$

By using Taylor's formula, we obtain

$$\begin{aligned} &\sum_{j\le N} (2^{js}\|\hat{u}_j(t_1) - \hat{u}_j(t_2)\|_{\mathrm{M}^\lambda_p})^q \\ &\le |t_1 - t_2|^q \sum_{j\le N} 2^{jsq}\|\partial_t \hat{u}_j(t)\|^q_{L^1(I,\mathrm{M}^\lambda_p)} \\ &\lesssim |t_1 - t_2|^q \|\partial_t u(t)\|^q_{\mathcal{L}^1(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} \\ &\lesssim |t_1 - t_2|^q \Big(\mu^q\|(-\Delta)^\alpha u\|^q_{\mathcal{L}^1(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})} + \|f\|^q_{\mathcal{L}^1(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}\Big) \\ &\lesssim |t_1 - t_2|^q \Big(\mu^q\|u\|^q_{\mathcal{L}^1(I,\mathcal{F}\dot{\mathcal{N}}^{s+2\alpha}_{p,\lambda,q})} + \|f\|^q_{\mathcal{L}^1(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}\Big) \\ &\lesssim |t_1 - t_2|^q \Big(2 + \Big(\frac{16}{9}\Big)^\alpha\Big)^q \Big(\|u_0\|^q_{\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q}} + \|f\|^q_{\mathcal{L}^1(I,\mathcal{F}\dot{\mathcal{N}}^s_{p,\lambda,q})}\Big)\,. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4), and (3.5), we obtain the continuity of u in time t. The proof is complete. ☐

Lemma 3.2 ([12]). *Let $1 \le p < \infty$, $1 \le \rho \le \infty$, $1 \le q \le 2$,*

$$\frac{1}{2} < \alpha < \frac{2 + \frac{3}{p'} + \frac{\lambda}{p}}{4 - \frac{2}{\rho}}\,,$$

$0 \le \lambda < 3$*, and set*

$$X = \mathcal{L}^\infty(I, \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}) \cap \mathcal{L}^\rho(I, \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p}}_{p,\lambda,q}),$$

with the norm

$$\|u\|_X = \|u\|_{\mathcal{L}^\infty(I,\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} + \min\{\mu,\nu\}\|u\|_{\mathcal{L}^\rho(I,\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p}}_{p,\lambda,q})}\,.$$

There exists a constant $C = C(\alpha,p,q) > 0$ depending on α,p,q such that

$$\|\nabla.(u\otimes v)\|_{\mathcal{L}^\rho(I,\mathcal{F}\dot{\mathcal{N}}^{1-4\alpha+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p}}_{p,\lambda,q})} \le C(\min\{\mu,\nu\})^{-1}\|u\|_X\|v\|_X\,.$$

Proof of Theorem 2.5. We use the Banach fixed point theorem to ensure the existence of global mild solutions with small initial data. Note that the functions here are vector fields, whose norm is the sum of the norms of the three components.

According to Duhamel's principle, the mild solution (u,b) for system (1.1) can be represented as

$$
\begin{aligned}
u &= e^{-t\mu(-\Delta)^{\alpha}}u_0 - \int_0^t e^{-\mu(t-\tau)(-\Delta)^{\alpha}}\mathbb{P}\nabla\cdot(u\otimes u - b\otimes b)(\cdot,\tau)\,d\tau := \psi_1(u,b),\\
b &= e^{-t\nu(-\Delta)^{\alpha}}b_0 - \int_0^t e^{-\nu(t-\tau)(-\Delta)^{\alpha}}\mathbb{P}\nabla\cdot(u\otimes b - b\otimes u)(\cdot,\tau)\,d\tau := \psi_2(u,b),
\end{aligned}
\tag{3.6}
$$

where $\mathbb{P} = Id - \nabla\Delta^{-1}\nabla$ is the Leray-Hopf projector, which is a pseudo differential operator of order 0. Let

$$
X = \pounds^{\infty}\Big([0,\infty);\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}\Big) \cap \pounds^{1}\Big([0,\infty),\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1+\frac{3}{p'}+\frac{\lambda}{p}}\Big)\,.
$$

For $u,b\in X$, we define the norm of vector (u,b) as

$$
\|(u,b)\|_X = \|u\|_X + \|b\|_X.
$$

Let

$$
B_\mu(u,v) := \int_0^t e^{-\mu(t-\tau)(-\Delta)^{\alpha}}\mathbb{P}\nabla.(u\otimes v)(\tau,x)d\tau.
$$

It is clear that the system (3.6) can be rewritten as

$$
(u,b) = (\psi_1(u,b),\psi_2(u,b)) := \Phi(u,b).
$$

We note that $B_\mu(u,b)$ can be thought as the solution to the heat equation (3.1) with $u_0=0$ and force $f=\mathbb{P}\nabla.(u\otimes v)$. According to Lemma 3.1 with $s = 1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}$ and Lemma 3.2 with $\rho=1$, and the fact that $\mathbb{P}$ is an homogeneous Fourier multiplier of degree 0, we obtain

$$
\begin{aligned}
\|B_\mu(u,b)\|_X &\le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)\|\mathbb{P}\nabla.(u\otimes b)\|_{\pounds^{1}(I,\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}})}\\
&\le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)C(\min\{\mu,\nu\})^{-1}\|u\|_X\|b\|_X\,.
\end{aligned}
\tag{3.7}
$$

We also notice that $e^{-\mu t(-\Delta)^{\alpha}}u_0$ is the solution to the dissipative equation with $u_0=u_0$ and $f=0$. So, Lemma 3.1 yields

$$
\|e^{-\mu t(-\Delta)^{\alpha}}u_0\|_X \le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}\,. \tag{3.8}
$$

By using the estimates (3.7) and (3.8), we obtain

$$
\begin{aligned}
\|\psi_1(u,b)\|_X &\le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)\|u_0\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}\\
&\quad + C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)(\min\{\mu,\nu\})^{-1}\big(\|u\|_X^2+\|b\|_X^2\big)\,.
\end{aligned}
\tag{3.9}
$$

Similarly, letting

$$
B_\nu(u,v) := \int_0^t e^{-\nu(t-\tau)(-\Delta)^{\alpha}}\mathbb{P}\nabla.(u\otimes v)(\tau,x)d\tau,
$$

we obtain

$$\begin{aligned} \|\psi_2(u,b)\|_X &\le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)\|b_0\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\ &\quad + 2C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)(\min\{\mu,\nu\})^{-1}\|u\|_X\|b\|_X \,. \end{aligned} \tag{3.10}$$

Since $\|(u_0,b_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le C_0\min\{\mu,\nu\}$, define

$$E=\Big\{(u,b)|(u,b)\in X, \|(u,b)\|_X\le 2\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)C_0\min\{\mu,\nu\}\Big\},$$

where C_0 is a constant which can be chosen later. Combining (3.8), (3.9), and (3.10), it follows that for $(u,b)\in E$ we have

$$\begin{aligned} &\|\Phi(u,b)\|_X \\ &\le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)\|(u_0,b_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)(\min\{\mu,\nu\})^{-1}\|(u,b)\|_X^2 \\ &\le \big(1+\big(\frac{16}{9}\big)^{\alpha}\big)C_0\min\{\mu,\nu\} + 4C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)^3C_0^2\min\{\mu,\nu\}, \end{aligned}$$

which implies that $\Phi(u,b)\in E$ when we choose C_0 small enough such that $C_0 < \frac{1}{16C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)^2}$.

On the other hand, for any (u_1,b_1), $(u_2,b_2)\in E$, we have

$$\begin{aligned} &\|\psi_1(u_1,b_1)-\psi_1(u_2,b_2)\|_X \\ &\le \|B_\mu(u_1,u_1)-B_\mu(u_2,u_2)\|_X + \|B_\mu(b_1,b_1)-B_\mu(b_2,b_2)\|_X \\ &\le \|B_\mu(u_1,u_1-u_2)+B_\mu(u_1-u_2,u_2)\|_X + \|B_\mu(b_1-b_2,b_2)+B_\mu(b_1,b_1-b_2)\|_X \\ &\le C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)(\min\{\mu,\nu\})^{-1}((\|u_1\|_X+\|u_2\|_X)\|u_1-u_2\|_X \\ &\quad + (\|b_1\|_X+\|b_2\|_X)\|b_1-b_2\|_X) \\ &\le 4C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)^2C_0(\|u_1-u_2\|_X+\|b_1-b_2\|_X) \\ &\le \frac{1}{4}(\|u_1-u_2\|_X+\|b_1-b_2\|_X). \end{aligned}$$

Similarly,

$$\begin{aligned} &\|\psi_2(u_1,b_1)-\psi_2(u_2,b_2)\|_X \\ &\le \|B_\nu(u_2,b_2)-B_\nu(u_1,b_1)\|_X + \|B_\nu(b_2,u_2)-B_\nu(b_1,u_1)\|_X \\ &\le 4C\big(1+\big(\frac{16}{9}\big)^{\alpha}\big)^2C_0(\|u_1-u_2\|_X+\|b_1-b_2\|_X) \\ &\le \frac{1}{4}(\|u_1-u_2\|_X+\|b_1-b_2\|_X). \end{aligned}$$

Consequently,

$$\|\Phi(u_1,b_1)-\Phi(u_2,b_2)\|_X \le \frac{1}{2}(\|u_1-u_2\|_X+\|b_1-b_2\|_X).$$

From the above estimate, we obtain that Φ is a contraction mapping from E to E. By the Banach fixed point theorem, we conclude that Φ has a unique fixed point $(u,b)\in E$ which is the solution of system (1.1). The proof is complete. $\square$

4. Decay property

In this section, we first introduce the following interpolation inequality which have their own interest in the sequel.

Lemma 4.1. *Let $\alpha < \frac{5}{4} + \frac{\lambda}{2p}$, $s > \frac{5}{2} - 2\alpha + \frac{\lambda}{p}$, and $1 \le p, q \le 2$. Then we have*

$$\|(u,v)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \lesssim \|(u,v)\|_{L^2}^{1-\frac{5/2-2\alpha+\lambda/p}{s}} \|(u,v)\|_{\dot{H}^s}^{\frac{5/2-2\alpha+\lambda/p}{s}}.$$

Proof. By definition of Fourier Besov-Morrey spaces and Hölder's inequality we have

$$\begin{aligned}
&\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&= \Big\{ \sum_{j\in\mathbb{Z}} 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})q} \|\varphi_j \hat{u}\|^q_{\mathrm{M}^\lambda_p} \Big\}^{1/q} \\
&\lesssim \Big\{ \sum_{j\in\mathbb{Z}} 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})q} \Big(\sup_{x_0\in\mathbb{R}^3} \sup_{r>0} r^{-\lambda/p} \|\varphi_j \hat{u}\|_{L^2(B(x_0,r))} 2^{(\frac{3}{p}-\frac{3}{2})j} \Big)^q \Big\}^{1/q} \\
&\lesssim \Big\{ \sum_{j\le M} 2^{j(\frac{5}{2}-2\alpha+\frac{\lambda}{p})q} \|\varphi_j \hat{u}\|^q_{L^2(\mathbb{R}^3)} \Big\}^{1/q} + \Big\{ \sum_{j> M} 2^{j(\frac{5}{2}-2\alpha+\frac{\lambda}{p}-s)q} 2^{jsq} \|\varphi_j \hat{u}\|^q_{L^2(\mathbb{R}^3)} \Big\}^{1/q} \\
&\lesssim 2^{(\frac{5}{2}-2\alpha+\frac{\lambda}{p})M} \Big\{ \sum_{j\in\mathbb{Z}} \|\varphi_j \hat{u}\|^2_{L^2(\mathbb{R}^3)} \Big\}^{1/2} + 2^{(\frac{5}{2}-2\alpha+\frac{\lambda}{p}-s)M} \Big\{ \sum_{j\in\mathbb{Z}} 2^{jsq} \|\varphi_j \hat{u}\|^2_{L^2(\mathbb{R}^3)} \Big\}^{1/2}.
\end{aligned}$$

Taking M such that $2^M = \big(\|u\|_{\dot{H}^s}/\|u\|_{L^2}\big)^{1/s}$, using $F\dot{B}^s_{2,2} = \dot{B}^s_{2,2} = \dot{H}^s$ and $\dot{B}^0_{2,2} = L^2$, we obtain

$$\begin{aligned}
\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} &\lesssim \Big(\frac{\|u\|_{\dot{H}^s}}{\|u\|_{L^2}}\Big)^{\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}} \|u\|_{F\dot{B}^0_{2,2}} + \Big(\frac{\|u\|_{\dot{H}^s}}{\|u\|_{L^2}}\Big)^{\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}-s}{s}} \|u\|_{F\dot{B}^s_{2,2}} \\
&\lesssim \|u\|_{L^2}^{1-\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}} \|u\|_{\dot{H}^s}^{\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}}.
\end{aligned}$$

Similarly,

$$\|v\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \lesssim \|v\|_{L^2}^{1-\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}} \|v\|_{\dot{H}^s}^{\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}}.$$

Finally,

$$\begin{aligned}
&\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|v\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\lesssim \|u\|_{L^2}^{1-\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}} \|u\|_{\dot{H}^s}^{\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}} + \|v\|_{L^2}^{1-\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}} \|v\|_{\dot{H}^s}^{\frac{\frac{5}{2}-2\alpha+\frac{\lambda}{p}}{s}}.
\end{aligned}$$

This completes the proof. □

Lemma 4.2. *Let $\frac{1}{2} < \alpha \le 1$ and $1 \le p, q \le 2$. Then we have*

$$\|uv\|_{\dot{H}^{1-\alpha}} \le C\|u\|_{L^2} \|v\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C\|u\|_{\dot{H}^\alpha} \|v\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}. \tag{4.1}$$

Proof. The argument of this lemma is similar to the proof of Lemma 3.2. In fact, let us introduce some notations about the standard localization operators. We set

$$u_j = \dot{\Delta}_j u, \quad \dot{S}_j u = \sum_{k\le j-1} \dot{\Delta}_k u, \quad \widetilde{\dot{\Delta}}_j u = \sum_{|k-j|\le 1} \dot{\Delta}_k u, \quad \forall j \in \mathbb{Z}\,.$$

Using Bony's paraproduct decomposition and the quasi-orthogonality property for the Littlewood-Paley decomposition, for fixed j, we have

$$\begin{aligned}\dot{\Delta}_j(uv) &= \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1}u\dot{\Delta}_k v) + \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1}v\dot{\Delta}_k u) + \sum_{k\ge j-3} \dot{\Delta}_j(\dot{\Delta}_k u\widetilde{\dot{\Delta}}_k v)\\ &= I_j + II_j + III_j\,.\end{aligned}$$

For the proof of this lemma, we can write

$$\begin{aligned}\|uv\|_{F\dot{B}^{1-\alpha}_{2,2}} \le \Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)2}\|\widehat{I}_j\|^2_{L^2}\Big\}^{1/2} + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)2}\|\widehat{II}_j\|^2_{L^2}\Big\}^{1/2}\\ + \Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)2}\|\widehat{III}_j\|^2_{L^2}\Big\}^{1/2}.\end{aligned}\tag{4.2}$$

The terms I_j and II_j are symmetrical. Using Young's inequality, and Lemma 2.10 with $|\gamma| = 0$, we obtain

$$\begin{aligned}\|\widehat{I}_j\|_{L^2} &\le \sum_{|k-j|\le 4} \|\widehat{\dot{S}_{k-1}u\dot{\Delta}_k v}\|_{L^2}\\ &\le \sum_{|k-j|\le 4} \|\widehat{v}_k\|_{L^2} \sum_{l\le k-2} \|\widehat{u}_l\|_{L^1}\\ &\lesssim \sum_{|k-j|\le 4} \|\widehat{v}_k\|_{L^2} \sum_{l\le k-2} 2^{l(\frac{3}{p'}+\frac{\lambda}{p})}\|\widehat{u}_l\|_{\mathrm{M}^\lambda_p}\\ &\lesssim \sum_{|k-j|\le 4} \|\widehat{v}_k\|_{L^2} \sum_{l\le k-2} 2^{l(\frac{3}{p'}+\frac{\lambda}{p})}2^{-l(2\alpha-1)}2^{l(2\alpha-1)}\|\widehat{u}_l\|_{\mathrm{M}^\lambda_p}\\ &\lesssim \sum_{|k-j|\le 4} 2^{k(2\alpha-1)}\|\widehat{v}_k\|_{L^2}\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,.\end{aligned}$$

Consequently

$$\Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)2}\|\widehat{I}_j\|^2_{L^2}\Big\}^{1/2} \lesssim \|v\|_{\dot{H}^\alpha}\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,.\tag{4.3}$$

To estimate the term II_j, we make a minor modification to get

$$\begin{aligned}\|\widehat{II}_j\|_{L^2} &\le \sum_{|k-j|\le 4} \|\widehat{\dot{S}_{k-1}v\dot{\Delta}_k u}\|_{L^2}\\ &\le \sum_{|k-j|\le 4} \|\widehat{u}_k\|_{L^1} \sum_{l\le k-2} \|\widehat{v}_l\|_{L^2}\\ &\lesssim \sum_{|k-j|\le 4} 2^{k(\frac{3}{p'}+\frac{\lambda}{p})}\|\widehat{u}_k\|_{\mathrm{M}^\lambda_p} \sum_{l\le k-2} \|\widehat{u}_l\|_{L^2}\,.\end{aligned}$$

So we have

$$\|\widehat{II}_j\|^2_{L^2} \lesssim \sum_{|k-j|\le 4} 2^{2k(\frac{3}{p'}+\frac{\lambda}{p})}\|\widehat{u}_k\|^2_{\mathrm{M}^\lambda_p}\|v\|^2_{F\dot{B}^0_{2,2}}\,.$$

This leads to

$$\Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)2}\|\widehat{II}_j\|_{L^2}^2\Big\}^{1/2} \lesssim \|u\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}\|v\|_{L^2}\,. \tag{4.4}$$

Now, we estimate the last term, let

$$III_{jk} := \dot{\Delta}_j\Big(\sum_{|i-k|\leq 1}\dot{\Delta}_i v\dot{\Delta}_k u\Big) = \sum_{i=-1}^{1}\dot{\Delta}_j(\dot{\Delta}_k u\dot{\Delta}_{i+k}v).$$

The estimate of the so-called "remainder term" requires a different approach. First we use the Young inequality (2.2) in Morrey spaces, and Lemma 2.10 with $|\gamma|=0$, we obtain

$$\begin{aligned}
&2^{j(1-\alpha)}\|\widehat{III_{jk}}\|_{L^2}\\
&\leq \sum_{i=-1}^{1} 2^{j\alpha}2^{j(1-2\alpha)}\|\hat{u}_k\|_{L^2}\|\hat{v}_{i+k}\|_{L^1}\\
&\lesssim \sum_{i=-1}^{1} 2^{j\alpha}2^{(1-2\alpha)(j-k)}2^{(2\alpha-1)i}2^{(1-2\alpha)i}2^{(1-2\alpha)k}\|\hat{u}_k\|_{L^2}\\
&\quad\times 2^{(\frac{3}{p'}+\frac{\lambda}{p})(i+k)}\|\hat{v}_{i+k}\|_{\mathrm{M}_p^\lambda}\\
&:= \sum_{i=-1}^{1} 2^{j\alpha}2^{(1-2\alpha)(j-k)}2^{(2\alpha-1)i}2^{(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})(i+k)}\|\hat{u}_k\|_{L^2}\|\hat{v}_{i+k}\|_{\mathrm{M}_p^\lambda}\,.
\end{aligned}$$

Taking the l^2-norm on both sides in the above estimate, and using the Hölder's inequalities for series, we obtain

$$\begin{aligned}
&\Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha)2}\|\widehat{III}_j\|_{L^2}^2\Big\}^{1/2}\\
&\lesssim \Big\{\sum_{j\in\mathbb{Z}}\Big(\sum_{l\leq 3}2^{j\alpha}2^{(1-2\alpha)l}2^{(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})(j-l)}\|\hat{u}_{j-l}\|_{L^2}\|\hat{v}_{j-l}\|_{\mathrm{M}_p^\lambda}\Big)^2\Big\}^{1/2}\\
&\lesssim \sum_{j\in\mathbb{Z}}\sum_{l\leq 3}2^{j\alpha}2^{(1-2\alpha)l}2^{(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})(j-l)}\|\hat{u}_{j-l}\|_{L^2}\|\hat{v}_{j-l}\|_{\mathrm{M}_p^\lambda}\\
&\lesssim \sum_{l\leq 3}2^{(1-\alpha)l}\Big\{\sum_{j\in\mathbb{Z}}2^{2\alpha(j-l)}\|\hat{u}_{j-l}\|_{L^2}^2\Big\}^{1/2}\|v\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,2}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}\\
&\lesssim \|u\|_{\dot{H}^\alpha}\|v\|_{\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}\,,
\end{aligned}\tag{4.5}$$

where we have used the fact that $1\leq q\leq 2$ implies

$$\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}} \hookrightarrow \mathcal{F}\dot{\mathcal{N}}_{p,\lambda,2}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}$$

and $F\dot{B}^{\alpha}_{2,2} = \dot{H}^{\alpha}$. For the case $\alpha = 1$, we have

$$
\begin{aligned}
&\Big(\sum_{j\in\mathbb{Z}} \|\widehat{III_j}\|^2_{L^2}\Big)^{1/2} \\
&\le \sum_{j\in\mathbb{Z}} \Big(\sum_{k\ge j-3} \|\varphi_j(\xi) \times \sum_{i=-1}^{1} \hat{u}_k * \hat{v}_{k+i}\|_{L^2}\Big) \\
&\le \sup_{\xi} \Big(\sum_{j\in\mathbb{Z}} \varphi_j(\xi)\Big) \sum_{k\in\mathbb{Z}} \|\sum_{i=-1}^{1} \hat{u}_k * \hat{v}_{k+i}\|_{L^2} \\
&\le \sum_{i=-1}^{1} \sum_{k\in\mathbb{Z}} \|\hat{u}_k\|_{L^2} \|\hat{v}_{k+i}\|_{L^1} \\
&\le \sum_{i=-1}^{1} \sum_{k\in\mathbb{Z}} 2^{(\frac{3}{p'}+\frac{\lambda}{p})(k+i)} \|\hat{u}_k\|_{L^2} \|\hat{v}_{k+i}\|_{\mathrm{M}^{\lambda}_{p}} \\
&\le \sum_{i=-1}^{1} \sum_{k\in\mathbb{Z}} 2^{-(1-2\alpha)(k+i)} \|\hat{u}_k\|_{L^2} 2^{(k+i)(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})} \|\hat{v}_{k+i}\|_{\mathrm{M}^{\lambda}_{p}} \\
&\le C \sum_{k\in\mathbb{Z}} 2^{k} \|\hat{u}_k\|_{L^2} 2^{k(-1+\frac{3}{p'}+\frac{\lambda}{p})} \|\hat{v}_k\|_{\mathrm{M}^{\lambda}_{p}} \\
&\le C\|u\|_{\dot{H}^1} \|v\|_{\mathcal{F}\dot{\mathcal{N}}^{-1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,2}} \\
&\le C\|u\|_{\dot{H}^1} \|v\|_{\mathcal{F}\dot{\mathcal{N}}^{-1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} .
\end{aligned}
\tag{4.6}
$$

Estimates (4.2), (4.3), (4.4), (4.5) and 4.6 yield (4.1) . ☐

Proof of Theorem 2.7. The proof is largely based on the idea from the work of Gallagher-Iftimie-Planchon [14] and (see also [37, 2, 31] and [19, chap. 11]). Let $\varepsilon > 0$ be any constant small enough such that $\varepsilon \le C_0 \min\{\mu, \nu\}$, where C_0 is the constant given in Theorem 2.5 and μ, ν are the viscosity coefficient in (1.1). For $k \in \mathbb{N}$, define

$$
\mathcal{A}_k = \{\xi \in \mathbb{R}^3; |\xi| \le k \text{ and } |\hat{u}_0| + |\hat{b}_0| \le k\} .
$$

Clearly $(\mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{u}_0), \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{b}_0))$ converge to (u_0, b_0) in $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$.

Then, there is $k \in \mathbb{N}$ such that

$$
\|u_0 - \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{u}_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|b_0 - \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{b}_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le \frac{\varepsilon}{2} .
$$

Put

$$
\begin{gathered}
u_{0,k} = \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{u}_0), \quad b_{0,k} = \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{b}_0), \\
w_{0,k} = u_0 - \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{u}_0), \quad d_{0,k} = b_0 - \mathcal{F}^{-1}(\chi_{\mathcal{A}_k}\hat{b}_0) .
\end{gathered}
$$

Then $u_{0,k}, b_{0,k} \in \mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q} \cap \mathrm{L}^2$, and we have shown that

$$
\|w_{0,k}\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|d_{0,k}\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le \frac{\varepsilon}{2} . \tag{4.7}
$$

Now, we consider the system

$$
\begin{gathered}
w_t + w\cdot\nabla w + \mu(-\Delta)^\alpha w - d\cdot\nabla d + \nabla\pi_k = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^3,\\
\nabla\cdot w = 0, \quad \nabla\cdot d = 0,\\
d_t + w\cdot\nabla d + \nu(-\Delta)^\alpha d - d\cdot\nabla u = 0 \quad \text{in } (0,+\infty)\times\mathbb{R}^3,\\
(w,d)|_{t=0} = (w_{0,k}, d_{0,k}).
\end{gathered}
\tag{4.8}
$$

Since $\frac{\varepsilon}{2} \le C_0\frac{\min\{\mu,\nu\}}{2} \le C_0\min\{\mu,\nu\}$, we deduce from Theorem 2.5 that the system (4.8) has a unique global solution

$$
(w_k,d_k) \in \mathcal{C}\big([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big) \cap \mathcal{L}^1\big([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big)\,,
$$

such that

$$
\begin{aligned}
&\Big\|(w_k,d_k)\Big\|_{\mathcal{L}^\infty\big([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big)\cap\mathcal{L}^1\big([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big)}\\
&\le C\Big\|(w_{0,k},d_{0,k})\Big\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,.
\end{aligned}
$$

Moreover, for any $t\ge 0$ we have

$$
\begin{aligned}
&\|w_k(t)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|d_k(t)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\\
&+\mu\|w_k\|_{\mathcal{L}^1\big([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big)} + \nu\|d_k\|_{\mathcal{L}^1\big([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\big)}\\
&\le C\|w_{0,k}\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C\|d_{0,k}\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,.
\end{aligned}
\tag{4.9}
$$

Next, we take into consideration the difference $u_k = u - w_k$, $b_k = b - d_k$, which satisfies

$$
\begin{gathered}
(u_k,b_k) \in \mathcal{C}\Big([0,\infty);\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\Big) \cap \mathcal{L}^1\Big([0,\infty),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\Big),\\
\partial_t u_k + \mu(-\Delta)^\alpha u_k + (u\cdot\nabla)u_k + (u_k\cdot\nabla)w_k + \nabla\pi - \nabla\pi_k = (b\cdot\nabla)b_k + (b_k\cdot\nabla)d_k,\\
\partial_t b_k + \nu(-\Delta)^\alpha b_k + (u\cdot\nabla)b_k + (u_k\cdot\nabla)d_k = (b\cdot\nabla)u_k + (b_k\cdot\nabla)w_k,\\
\nabla\cdot u_k = 0, \quad \nabla\cdot b_k = 0,
\end{gathered}
$$

where π and π_k are the correspond pressures to the solutions u and w_k, respectively. Taking the inner products of the first equation with u_k and of the second equation with b_k and integrating by parts, we can show that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u_k\|_{L^2}^2 + \|b_k\|_{L^2}^2) + \mu\|(-\Delta)^{\frac{\alpha}{2}}u_k\|_{L^2}^2 + \nu\|(-\Delta)^{\frac{\alpha}{2}}b_k\|_{L^2}^2\\
&\le \Big|\int_{\mathbb{R}^3}(u_k\cdot\nabla)w_k\cdot u_k\,dx\Big| + \Big|\int_{\mathbb{R}^3}(b_k\cdot\nabla)d_k\cdot u_k\,dx\Big|\\
&\quad + \Big|\int_{\mathbb{R}^3}(u_k\cdot\nabla)d_k\cdot b_k\,dx\Big| + \Big|\int_{\mathbb{R}^3}(b_k\cdot\nabla)w_k\cdot b_k\,dx\Big|\\
&:= I_1 + I_2 + I_3 + I_4\,,
\end{aligned}
\tag{4.10}
$$

where we have used the cancelation property

$$
\int_{\mathbb{R}^3}(b\cdot\nabla)b_k\cdot u_k\,dx + \int_{\mathbb{R}^3}(b\cdot\nabla)u_k\cdot b_k\,dx = 0\,.
$$

Integrating by parts, Hölder's inequality, and Lemma 4.2 yield

$$
I_1 := \big|\big\langle\nabla.(u_k\otimes w_k), u_k\big\rangle\big|
$$

$$
\begin{aligned}
&\le \|(-\Delta)^{\frac12-\frac{\alpha}{2}}(u_k\otimes w_k)\|_{L^2}\|(-\Delta)^{\frac{\alpha}{2}}u_k\|_{L^2}\\
&\le C\|u_k\otimes w_k\|_{\dot H^{1-\alpha}}\|u_k\|_{\dot H^\alpha}\\
&\le C\|u_k\|_{L^2}\|w_k\Big\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\|u_k\|_{\dot H^\alpha}+C\|u_k\|^2_{\dot H^\alpha}\|w_k\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\\
&\le \frac{6C^2}{\mu}\|u_k\|^2_{L^2}\|w_k\Big\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\frac{\mu}{24}\|u_k\|^2_{\dot H^\alpha}+C\|u_k\|^2_{\dot H^\alpha}\|w_k\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\,.
\end{aligned}
$$

By (4.7) and (4.9) we have $\Big\|w_k\Big\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\le C\frac{\varepsilon}{2}$. We further assume ε small enough such that $C^2\varepsilon\le\frac{\mu}{12}$, thus

$$
I_1\le\frac{6C^2}{\mu}\|u_k\|^2_{L^2}\|w_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\frac{\mu}{12}\|u_k\|^2_{\dot H^\alpha}\,. \tag{4.11}
$$

To estimate I_2, we have

$$
\begin{aligned}
I_2&=\big|\big\langle\nabla.(b_k\otimes d_k),u_k\big\rangle\big|\\
&\le \|(-\Delta)^{\frac12-\frac{\alpha}{2}}(b_k\otimes d_k)\|_{L^2}\|(-\Delta)^{\frac{\alpha}{2}}u_k\|_{L^2}\\
&\le C\|b_k\|_{L^2}\|d_k\Big\|_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\|u_k\|_{\dot H^\alpha}+C\|b_k\|_{\dot H^\alpha}\|d_k\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\|u_k\|_{\dot H^\alpha}\\
&\le \frac{6C^2}{\mu}\|b_k\|^2_{L^2}\|d_k\Big\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\frac{\mu}{24}\|u_k\|^2_{\dot H^\alpha}+\frac{6C^2}{\mu}\|b_k\|^2_{\dot H^\alpha}\|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\\
&\quad+\frac{\mu}{24}\|u_k\|^2_{\dot H^\alpha}\,.
\end{aligned}
$$

By (4.9), we have $\|d_k\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\le C\frac{\varepsilon}{2}$. We take sufficiently small ε such that $C^2\varepsilon\le\frac{2\nu^{1/2}\mu^{1/2}}{\sqrt6\sqrt{12}}$, we obtain

$$
I_2\le\frac{6C^2}{\mu}\|b_k\|^2_{L^2}\|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\frac{\nu}{12}\|b_k\|^2_{\dot H^\alpha}+\frac{\mu}{12}\|u_k\|^2_{\dot H^\alpha}\,. \tag{4.12}
$$

Similarly,

$$
I_3\le\frac{6C^2}{\nu}\|u_k\|^2_{L^2}\|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\frac{\nu}{12}\|b_k\|^2_{\dot H^\alpha}+\frac{\mu}{12}\|u_k\|^2_{\dot H^\alpha}\,, \tag{4.13}
$$

$$
I_4\le\frac{6C^2}{\nu}\|b_k\|^2_{L^2}\|w_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\frac{\nu}{12}\|b_k\|^2_{\dot H^\alpha}\,. \tag{4.14}
$$

Combining (4.10), (4.11), (4.12), (4.13), and (4.14), we obtain

$$
\begin{aligned}
&\frac{d}{dt}(\|u_k\|^2_{L^2}+\|b_k\|^2_{L^2})+\mu\|u_k\|^2_{\dot H^\alpha}+\nu\|b_k\|^2_{\dot H^\alpha}\\
&\le\max\Big(\frac{12C^2}{\mu},\frac{12C^2}{\nu}\Big)\Big(\|u_k\|^2_{L^2}+\|b_k\|^2_{L^2}\Big)\Big(\|w_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\\
&\quad+\|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big)\,.
\end{aligned}
$$

Integrating in time and using Gronwall's lemma we obtain

$$\begin{aligned}
&\|u_k\|_{L^2}^2 + \|b_k\|_{L^2}^2 + \mu\int_0^t \|u_k\|_{\dot H^\alpha}^2 + \nu\int_0^t \|b_k\|_{\dot H^\alpha}^2 \\
&\le \big(\|u_{0,k}\|_{L^2}^2 + \|b_{0,k}\|_{L^2}^2\big)\exp\Big\{\max\Big(\frac{12C^2}{\mu},\frac{12C^2}{\nu}\Big) \\
&\quad\times\Big(\int_0^t \|w_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \int_0^t \|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big)\Big\}.
\end{aligned}\tag{4.15}$$

Since $q \le 2$, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^t \|w_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\le \Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-\alpha+\frac{3}{p'}+\frac{\lambda}{p})q}\Big(\int_0^t \|\varphi_j\hat w_k\|^2_{\mathrm{M}_p^\lambda}\Big)^{\frac{q}{2}}\Big\}^{2/q} \\
&\le \Big\{\sum_{j\in\mathbb{Z}} 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})\frac{q}{2}} 2^{j(1+\frac{3}{p'}+\frac{\lambda}{p})\frac{q}{2}} \|\varphi_j\hat w_k\|^{\frac{q}{2}}_{L^\infty([0,t),\mathrm{M}_p^\lambda)} \|\varphi_j\hat w_k\|^{\frac{q}{2}}_{L^1([0,t),\mathrm{M}_p^\lambda)}\Big\}^{2/q} \\
&\le \|w_k\|_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \|w_k\|_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})}.
\end{aligned}$$

Similarly,

$$\int_0^t \|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le \|d_k\|_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \|d_k\|_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})}.$$

With the aid of (4.9), we obtain

$$\begin{aligned}
&\int_0^t \|w_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \int_0^t \|d_k\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\le \|w_k\|_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \|w_k\|_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \\
&\quad + \|d_k\|_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \|d_k\|_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \\
&\le \frac{1}{2\mu}\Big(\|w_k\|^2_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} + \mu^2\|w_k\|^2_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})}\Big) \\
&\quad + \frac{1}{2\nu}\Big(\|d_k\|^2_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} + \nu^2\|d_k\|^2_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})}\Big) \\
&\le \max\big(\frac{1}{2\mu},\frac{1}{2\nu}\big)\Big(\|w_k\|_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} + \|d_k\|_{\mathcal{L}^\infty([0,t),\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} \\
&\quad + \mu\|w_k\|_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})} + \nu\|d_k\|_{\mathcal{L}^1([0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q})}\Big)^2 \\
&\le \max\big(\frac{C^2}{2\mu},\frac{C^2}{2\nu}\big)\|(w_{0,k},d_{0,k})\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&\|u_k\|_{L^2}^2+\|b_k\|_{L^2}^2+\mu\int_0^t\|u_k\|_{\dot H^\alpha}^2+\nu\int_0^t\|b_k\|_{\dot H^\alpha}^2\\
&\le \big(\|u_{0,k}\|_{L^2}^2+\|b_{0,k}\|_{L^2}^2\big)\exp\Big\{\max\Big(\frac{6C^4}{\nu\mu},\frac{6C^4}{\nu^2},\frac{6C^4}{\mu^2}\Big)\\
&\quad\times\|(w_{0,k},d_{0,k})\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big\}.
\end{aligned}\tag{4.16}
$$

Now, noting $\sigma=\frac{5-4\alpha+2\lambda/p}{2\alpha}$ and using Lemma 4.1 we obtain

$$
\begin{aligned}
&\int_0^t\big(\|u_k\|^{\frac{2}{\sigma}}_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\|b_k\|^{\frac{2}{\sigma}}_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\big)\\
&\lesssim C^{\frac{2}{\sigma}}\int_0^t\big(\|u_k\|_{L^2}^2+\|b_k\|_{L^2}^2\big)^{(\frac{1-\sigma}{\sigma})}\big(\|u_k\|_{\dot H^\alpha}+\|b_k\|_{\dot H^\alpha}\big)^2,
\end{aligned}
$$

Now (4.16) yields

$$
\begin{aligned}
&(\|u_k\|_{L^2}^2+\|b_k\|_{L^2}^2)^{(\frac{1-\sigma}{\sigma})}\\
&\lesssim\big(\|u_{0,k}\|_{L^2}^2+\|b_{0,k}\|_{L^2}^2\big)^{(\frac{1-\sigma}{\sigma})}\exp\Big\{\max\Big(\frac{6C^4}{\nu\mu},\frac{6C^4}{\nu^2},\frac{6C^4}{\mu^2}\Big)\big(\frac{1-\sigma}{\sigma}\big)\\
&\quad\times\|(w_{0,k},d_{0,k})\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big\}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_0^t\big(\|u_k\|^{\frac{2}{\sigma}}_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\|b_k\|^{\frac{2}{\sigma}}_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\big)\\
&\lesssim C^{\frac{2}{\sigma}}\big(\|u_{0,k}\|_{L^2}^2+\|b_{0,k}\|_{L^2}^2\big)^{(\frac{1-\sigma}{\sigma})}\exp\Big\{\max\Big(\frac{6C^4}{\nu\mu},\frac{6C^4}{\nu^2},\frac{6C^4}{\mu^2}\Big)\big(\frac{1-\sigma}{\sigma}\big)\\
&\quad\times\|(w_{0,k},d_{0,k})\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big\}\int_0^t\big(\|u_k\|_{\dot H^\alpha}+\|b_k\|_{\dot H^\alpha}\big)^2.
\end{aligned}
$$

Using again (4.16),

$$
\begin{aligned}
&\int_0^t\big(\|u_k\|_{\dot H^\alpha}+\|b_k\|_{\dot H^\alpha}\big)^2\\
&\lesssim(\min\{\mu,\nu\})^{-1}\big(\|u_{0,k}\|_{L^2}^2+\|b_{0,k}\|_{L^2}^2\big)\exp\Big\{\max\Big(\frac{6C^4}{\nu\mu},\frac{6C^4}{\nu^2},\frac{6C^4}{\mu^2}\Big)\\
&\quad\times\|(w_{0,k},d_{0,k})\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big\}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
&\int_0^\infty\big(\|u_k\|^{\frac{4\alpha}{5-4\alpha+2\lambda/p}}_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}+\|b_k\|^{\frac{4\alpha}{5-4\alpha+2\lambda/p}}_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\big)\\
&\lesssim C^{\frac{4\alpha}{5-4\alpha+2\lambda/p}}(\min\{\mu,\nu\})^{-1}\big(\|u_{0,k}\|_{L^2}+\|b_{0,k}\|_{L^2}\big)^{\frac{4\alpha}{5-4\alpha+2\lambda/p}}\\
&\quad\times\exp\Big\{\max\Big(\frac{6C^4}{\nu\mu},\frac{6C^4}{\nu^2},\frac{6C^4}{\mu^2}\Big)\big(\frac{2\alpha}{5-4\alpha+2\lambda/p}\big)\|(w_{0,k},d_{0,k})\|^2_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}}\Big\}.
\end{aligned}
$$

So by continuity of u_k and b_k in $\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}$, there exists a time t_0 such that

$$\|u_k(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|b_k(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le \frac{\varepsilon}{2}\,.$$

Then we have

$$\begin{aligned}
&\|u(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|b(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\le \|u_k(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|w_k(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&+ \|b_k(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|d_k(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\,.
\end{aligned}$$

Now, we consider the generalized magneto-hydrodynamic equations starting at $t = t_0$,

$$\begin{gathered}
u_t + u\cdot\nabla u + \mu(-\Delta)^\alpha u - b\cdot\nabla b + \nabla\pi = 0,\\
\nabla\cdot u = 0, \quad \nabla\cdot b = 0,\\
b_t + u\cdot\nabla b + \nu(-\Delta)^\alpha b - b\cdot\nabla u = 0,\\
u(t_0,x) = u(t_0), \quad b(t_0,x) = b(t_0).
\end{gathered}$$

By Theorem 2.5 and using the method described in the proof of (4.9), we immediately obtain

$$\begin{aligned}
&\|u(t)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + \|b(t)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \\
&+ \mu\|u\|_{\mathcal{L}^1\left([t_0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} + \nu\|b\|_{\mathcal{L}^1\left([t_0,t),\mathcal{F}\dot{\mathcal{N}}^{1+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}\right)} \\
&\le C\|u(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} + C\|b(t_0)\|_{\mathcal{F}\dot{\mathcal{N}}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}_{p,\lambda,q}} \le C\varepsilon\,,
\end{aligned}$$

for all $t \ge t_0$. This completes the proof. $\square$

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Azzeddine El Baraka
University Mohamed Ben Abdellah, FST Fes-Saiss, Laboratory AAFA, Department of Mathematics, B.P. 2202 Route Immouzer, Fes 30000, Morocco
E-mail address: azzeddine.elbaraka@usmba.ac.ma, az.elbaraka@gmail.com

Mohamed Toumlilin
University Mohamed Ben Abdellah, FST Fes-Saiss, Laboratory AAFA, Department of Mathematics, B.P. 2202 Route Immouzer, Fes 30000, Morocco
E-mail address: mohamed.toumlilin@usmba.ac.ma