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EXISTENCE OF POSITIVE SOLUTIONS TO PERTURBED NONLINEAR DIRICHLET PROBLEMS INVOLVING CRITICAL GROWTH

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Abstract. We consider the following perturbed nonlinear elliptic problem with critical growth

$$
\begin{gathered}
-\varepsilon^2\Delta u+V(x)u=f(x)|u|^{p-2}u+\frac{\alpha}{\alpha+\beta}K(x)|u|^{\alpha-2}u|v|^{\beta}, \quad x\in\mathbb{R}^N,\\
-\varepsilon^2\Delta v+V(x)v=g(x)|v|^{p-2}v+\frac{\beta}{\alpha+\beta}K(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x\in\mathbb{R}^N,\\
u(x),\quad v(x)\to 0 \quad \text{as } |x|\to\infty.
\end{gathered}
$$

Using variational methods, we prove the existence of positive solutions.

1. Introduction

In this article, we discuss the following perturbed elliptic system involving critical growth

$$
\begin{gathered}
-\varepsilon^2\Delta u+V(x)u=f(x)|u|^{p-2}u+\frac{\alpha}{\alpha+\beta}K(x)|u|^{\alpha-2}u|v|^{\beta}, \quad x\in\mathbb{R}^N,\\
-\varepsilon^2\Delta v+V(x)v=g(x)|v|^{p-2}v+\frac{\beta}{\alpha+\beta}K(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x\in\mathbb{R}^N,\\
u(x),\quad v(x)\to 0 \quad \text{as } |x|\to\infty,
\end{gathered}
\tag{1.1}
$$

where $2<p<2^*$, $\alpha>1,\beta>1$ satisfy $\alpha+\beta=2^*$, $2^*=2N/(N-2)(N\geq 3)$ is the critical Sobolev exponent. We assume that $V(x),K(x),f(x)$ and $g(x)$ satisfy the following conditions:

- (H1) $V\in C(\mathbb{R}^N,\mathbb{R})$, $V(0)=\inf_{x\in\mathbb{R}^N}V(x)=0$ and there exists $b>0$ such that the set $\nu^b:=\{x\in\mathbb{R}^N: V(x)<b\}$ has finite Lebesgue measure;
- (H2) $K(x)\in C(\mathbb{R}^N,\mathbb{R})$, $0<\inf_{x\in\mathbb{R}^N}K(x)\leq\sup_{x\in\mathbb{R}^N}K(x)<\infty$;
- (H3) $f(x),g(x)$ are bounded and positive functions.

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Set $\alpha = \beta$, $f(x) = g(x)$ and $u = v$. Then (1.1) reduces to the semilinear scalar perturbed elliptic equation with critical growth

$$\begin{gathered} -\varepsilon^2 \Delta u + V(x)u = f(x)|u|^{p-2}u + \frac{1}{2}K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{gathered} \tag{1.2}$$

Many studies on problem (1.2) can be found in literature [3, 4, 7, 8, 9, 11, 12, 13, 20]. For example, Ding and Lin [12] established the existence of positive solutions of (1.2) as well as those solutions changed sign exactly once.

The semilinear elliptic system involving subcritical exponents on bounded domain has also been widely studied [5, 10, 21, 22, 23]. Wu [23] obtained multiplicity results of nontrivial nonnegative solutions of the elliptic system

$$\begin{gathered} -\Delta u = \lambda f(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}h(x)|u|^{\alpha-2}u|v|^{\beta}, \quad x \in \Omega, \\ -\Delta v = \mu g(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}h(x)|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega, \\ u(x) = v(x) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

For the case of a bounded domain, system (1.1) involving critical terms with $\varepsilon = 1$ was studied in [14, 15, 17, 18]. Hsu and Lin [17] considered the problem

$$\begin{gathered} -\Delta u = \lambda |u|^{q-2}u + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \quad x \in \Omega, \\ -\Delta v = \mu |v|^{q-2}v + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega, \\ u(x) = v(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $\alpha + \beta = 2^*$. Liu and Han [18] studied the system

$$\begin{gathered} -\Delta u = \lambda u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, \quad x \in \Omega, \\ -\Delta v = \mu v + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, \quad x \in \Omega, \\ u(x) = v(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

where $\lambda, \mu \geq 0$ and $\lambda + \mu > 0$, $\alpha, \beta > 1$ satisfying $\alpha + \beta = 2^*$.

Many papers were devoted to the existence results of elliptic boundary valued problems on bounded domain with the Sobolev critical exponents. However, to the known of our knowledge, no studies were conducted on the existence of semi-classical solutions to (1.1) in $\mathbb{R}^N$. In this paper, we study system (1.1) in the whole space involving the critical growth. The main difficulty of this problem is the lack of compactness of the Sobolev embedding. To overcome this difficulty, we follow the approach originally developed in [21]. Namely, we will show that the corresponding energy functional of problem (1.1) satisfies the compactness condition at the levels less than some certain constant c. Our result complements the study made in [17, 18, 23] in the sense that, in those papers, only the subcritical growth or the problem on bounded domain were considered.

Let $\lambda = \varepsilon^{-2}$. Then (1.1) can be rewritten as

$$
\begin{gathered}
-\Delta u + \lambda V(x) u = \lambda f(x) |u|^{p-2} u + \frac{\lambda \alpha}{\alpha + \beta} K(x) |u|^{\alpha - 2} u |v|^{\beta}, \quad x \in \mathbb{R}^N, \\
-\Delta v + \lambda V(x) v = \lambda g(x) |v|^{p-2} v + \frac{\lambda \beta}{\alpha + \beta} K(x) |u|^{\alpha} |v|^{\beta - 2} v, \quad x \in \mathbb{R}^N, \\
u(x), \quad v(x) \to 0 \quad \text{as } |x| \to \infty.
\end{gathered} \tag{1.6}
$$

Since (1.1) and (1.6) are equivalent, then we will focus on system (1.6).

Theorem 1.1. *Assume* (H1)–(H3) *hold. Then for any $\sigma > 0$, there is $\Lambda_\sigma > 0$ such that if $\lambda > \Lambda_\sigma$, problem* (1.6) *has at least one solution (u_λ, v_λ) that satisfies*

$$
\frac{p-2}{2p} \int_{\mathbb{R}^N} (|\nabla u_\lambda|^2 + |\nabla v_\lambda|^2 + \lambda V(x)(|u_\lambda|^2 + |v_\lambda|^2)) \le \sigma \lambda^{1 - \frac{N}{2}}. \tag{1.7}
$$

This article is organized as follows. In section 2, we show the $(PS)_c$ condition holds for I_λ with some level c. In section 3, we obtain that the functional associated to (1.2) possesses the mountain geometry structure. Section 4 is devoted to the proof of the main result.

2. Palais-Smale condition

Let $E = E_\lambda \times E_\lambda$ be the Hilbert space with norm

$$
\|(u,v)\|_E = (\int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x) u^2 + |\nabla v|^2 + \lambda V(x) v^2))^{\frac{1}{2}},
$$

for any $(u,v) \in E$. Meanwhile, the space

$$
E_\lambda = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \lambda V(x) u^2 < \infty \ , \lambda > 0\}
$$

is a Hilbert space equipped with the inner product

$$
(u,v)_{E_\lambda} = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda V(x) uv).
$$

We will show the existence of nontrivial solutions of (1.6) by searching for critical points of the functional associated to (1.6),

$$
\begin{aligned}
I_\lambda(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} & (|\nabla u|^2 + \lambda V(x) u^2 + |\nabla v|^2 + \lambda V(x) v^2) \\
& - \frac{\lambda}{p} \int_{\mathbb{R}^N} (f(x)|u|^p + g(x)|v|^p) - \frac{\lambda}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) |u|^\alpha |v|^\beta.
\end{aligned}
$$

In fact, the critical points of the functional I_λ are the weak solutions of (1.6). Recall that the weak solution (u,v) of (1.6) satisfies

$$
\begin{aligned}
& \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + \lambda V(x) u \varphi + \nabla v \nabla \psi + \lambda V(x) v \psi) \\
& = \lambda \int_{\mathbb{R}^N} (f(x) |u|^{p-2} u \varphi + g(x) |v|^{p-2} v \psi) + \frac{\lambda \alpha}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) |u|^{\alpha - 2} u |v|^\beta \varphi \\
& \quad + \frac{\lambda \beta}{\alpha + \beta} \int_{\mathbb{R}^N} K(x) |u|^\alpha |v|^{\beta - 2} v \psi,
\end{aligned}
$$

for all $(\varphi, \psi) \in E$. Based on the assumptions of Theorem 1.1, we can show that $I_\lambda \in C^1(E, \mathbb{R})$ [19].

Notation. $L^p(\mathbb{R}^N)$, $1 \le p < \infty$, denotes the Lebesgue spaces with $\|\cdot\|_p$. The dual space of a Banach space E will be denoted by E^*. $B_r := \{x \in \mathbb{R}^N : |x| \le r\}$ is the ball in $\mathbb{R}^N$. c, c_i represent various positive constants.

Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support. $o(1)$ denotes $o(1) \to 0$ as $n \to \infty$. Let $S_{\alpha,\beta}$ be the best Sobolev embedding constant defined by

$$S_{\alpha,\beta} = \inf_{u,v \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)}{\big(\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\big)^{\frac{2}{\alpha+\beta}}}. \tag{2.1}$$

We have

$$S_{\alpha,\beta} = \Big((\frac{\alpha}{\beta})^{\frac{\beta}{\alpha+\beta}} + (\frac{\beta}{\alpha})^{\frac{\alpha}{\alpha+\beta}}\Big) S,$$

where S is the best Sobolev embedding constant defined by

$$S = \inf_{u \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\big(\int_{\mathbb{R}^N} |u|^{2^*}\big)^{2/2^*}}.$$

Next, we will find the range of c where the $(PS)_c$ condition holds for the functional I_λ.

Definition 2.1. Let $I \in C^1(E, \mathbb{R})$.

(1) A sequence $\{z_n\} \subset E$ is called a $(PS)_c$ sequence in E for I if $I(z_n) = c + o(1)$ and $I'(z_n) = o(1)$ strongly in E^* as $n \to \infty$.
(2) I satisfies $(PS)_c$ condition if any $(PS)_c$ sequence $\{z_n\}$ in E for I has a convergent subsequence.

Lemma 2.2. *If the sequence $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$ sequence for I_λ, then we have $c \ge 0$ and $\{(u_n, v_n)\}$ is bounded in the space E.*

Proof. We have

$$\begin{aligned}
&I_\lambda(u_n, v_n) - \frac{1}{p} I'_\lambda(u_n, v_n)(u_n, v_n) \\
&= \frac{1}{2}\|(u_n, v_n)\|_E^2 - \frac{\lambda}{p}\int_{\mathbb{R}^N} (f(x)|u_n|^p + g(x)|v_n|^p) - \frac{\lambda}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)|u_n|^\alpha |v_n|^\beta \\
&\quad - \frac{1}{p}[\|(u_n, v_n)\|_E^2 - \lambda\int_{\mathbb{R}^N} (f(x)|u_n|^p + g(x)|v_n|^p) - \lambda\int_{\mathbb{R}^N} K(x)|u_n|^\alpha |v_n|^\beta] \\
&= (\frac{1}{2} - \frac{1}{p})\|(u_n, v_n)\|_E^2 + (\frac{1}{p} - \frac{1}{\alpha+\beta})\lambda\int_{\mathbb{R}^N} K(x)|u_n|^\alpha |v_n|^\beta.
\end{aligned}$$

From this and $2 < p < 2^*$, we obtain

$$I_\lambda(u_n, v_n) - \frac{1}{p} I'_\lambda(u_n, v_n)(u_n, v_n) \ge (\frac{1}{2} - \frac{1}{p})\|(u_n, v_n)\|_E^2.$$

Since $I_\lambda(u_n, v_n) \to c$ and $I'_\lambda(u_n, v_n) \to 0$, the conclusion follows. □

Lemma 2.3. *There exists a subsequence $\{(u_{n_j}, v_{n_j})\}$ such that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ with $r \ge r_\varepsilon$*

$$\limsup_{j\to\infty} \int_{B_j \setminus B_r} (|u_{n_j}|^d + |v_{n_j}|^d) \le \varepsilon$$

where $2 \le d < 2^*$.

Proof. By Lemma 2.2, the $(PS)_c$ sequence $\{(u_n, v_n)\}$ for I_λ is bounded in E. So, we assume $(u_n, v_n) \rightharpoonup (u, v)$ in E, $u_n \to u$, $v_n \to v$ a.e. in $\mathbb{R}^N$ and $(u_n, v_n) \to (u, v)$ in $L^d_{\rm loc}(\mathbb{R}^N) \times L^d_{\rm loc}(\mathbb{R}^N)$ for any $2 \le d < 2^*$. For each $j \in \mathbb{N}$, we have

$$\int_{B_j} (|u_n|^d + |v_n|^d) \to \int_{B_j} (|u|^d + |v|^d).$$

Thus, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_j} (|u_n|^d + |v_n|^d - |u|^d - |v|^d) < \frac{1}{j}\,,$$

for all $n \ge n_0 + 1$. Without loss of generality, we choose $n_j = n_0 + j$ such that

$$\int_{B_j} (|u_{n_j}|^d + |v_{n_j}|^d - |u|^d - |v|^d) < \frac{1}{j}\,.$$

It is easy to show that there is a r_ε satisfying

$$\int_{\mathbb{R}^N \setminus B_r} (|u|^d + |v|^d) < \varepsilon \quad \text{for all } r \ge r_\varepsilon.$$

Since

$$\begin{aligned}
&\int_{B_j \setminus B_r} (|u_{n_j}|^d + |v_{n_j}|^d) \\
&< \frac{1}{j} + \int_{\mathbb{R}^N \setminus B_r} (|u|^d + |v|^d) + \int_{B_r} (|u|^d - |u_{n_j}|^d + |v|^d - |v_{n_j}|^d).
\end{aligned}$$

In connection with $(u_n, v_n) \to (u, v)$ in $L^d_{\rm loc}(\mathbb{R}^N) \times L^d_{\rm loc}(\mathbb{R}^N)$, the lemma follows. $\square$

Let $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$ and $\eta(t) = 0$ if $t \ge 2$. Define $\tilde{u}_j(x) = \eta(2|x|/j)u(x)$ and $\tilde{v}_j(x) = \eta(2|x|/j)v(x)$, then

$$(\tilde{u}_j, \tilde{v}_j) \to (u, v) \quad \text{in } E \text{ as } j \to \infty. \tag{2.2}$$

Lemma 2.4. *One has*

$$\begin{gathered}
\lim_{j\to\infty} \Big| \int_{\mathbb{R}^N} f(x)(|u_{n_j}|^{p-2}u_{n_j} - |u_{n_j} - \tilde{u}_j|^{p-2}(u_{n_j} - \tilde{u}_j) - |\tilde{u}_j|^{p-2}\tilde{u}_j)\varphi \Big| = 0, \\
\lim_{j\to\infty} \Big| \int_{\mathbb{R}^N} g(x)(|v_{n_j}|^{p-2}v_{n_j} - |v_{n_j} - \tilde{v}_j|^{p-2}(v_{n_j} - \tilde{v}_j) - |\tilde{v}_j|^{p-2}\tilde{v}_j)\psi \Big| = 0
\end{gathered}$$

uniformly in $(\varphi, \psi) \in E$ *with* $\|(\varphi, \psi)\|_E \le 1$.

The proof of the above lemma is similar to the one [12, Lemma 3.4], so we omit it.

Lemma 2.5. *Passing to a subsequence, we have*

$$\begin{gathered}
I_\lambda(u_n - \tilde{u}_n, v_n - \tilde{v}_n) \to c - I_\lambda(u, v), \\
I'_\lambda(u_n - \tilde{u}_n, v_n - \tilde{v}_n) \to 0 \text{ in } E^*.
\end{gathered}$$

Proof. From $(u_n, v_n) \rightharpoonup (u, v)$ and $(\tilde{u}_n, \tilde{v}_n) \to (u, v)$ in E, we obtain

$$\begin{aligned}
&I_\lambda(u_n - \tilde{u}_n, v_n - \tilde{v}_n) \\
&= I_\lambda(u_n, v_n) - I_\lambda(\tilde{u}_n, \tilde{v}_n) \\
&\quad + \frac{\lambda}{p^*} \int_{\mathbb{R}^N} K(x)(|u_n|^\alpha |v_n|^\beta - |u_n - \tilde{u}_n|^\alpha |v_n - \tilde{v}_n|^\beta - |\tilde{u}_n|^\alpha |\tilde{v}_n|^\beta)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda}{p}\int_{\mathbb{R}^N} f(x)(|u_n|^p-|u_n-\tilde u_n|^p-|\tilde u_n|^p)\\
&+\frac{\lambda}{p}\int_{\mathbb{R}^N} g(x)(|v_n|^p-|v_n-\tilde v_n|^p-|\tilde v_n|^p)+o(1).
\end{aligned}
$$

Similar to the proof of Brézis-Lieb Lemma [6], it is easy to obtain

$$
\begin{gathered}
\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|u_n|^\alpha|v_n|^\beta-|u_n-\tilde u_n|^\alpha|v_n-\tilde v_n|^\beta-|\tilde u_n|^\alpha|\tilde v_n|^\beta)=0,\\
\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)(|u_n|^p-|u_n-\tilde u_n|^p-|\tilde u_n|^p)=0,\\
\lim_{n\to\infty}\int_{\mathbb{R}^N} g(x)(|v_n|^p-|v_n-\tilde v_n|^p-|\tilde v_n|^p)=0.
\end{gathered}
$$

Observing that $I_\lambda(u_n,v_n)\to c$ and $I_\lambda(\tilde u_n,\tilde v_n)\to I_\lambda(u,v)$, we have

$$
I_\lambda(u_n-\tilde u_n,v_n-\tilde v_n)\to c-I_\lambda(u,v).
$$

In addition, for any $(\varphi,\psi)\in E$, we obtain

$$
\begin{aligned}
&I'_\lambda(u_n-\tilde u_n,v_n-\tilde v_n)(\varphi,\psi)\\
&=I'_\lambda(u_n,v_n)(\varphi,\psi)-I'_\lambda(\tilde u_n,\tilde v_n)(\varphi,\psi)+\frac{\lambda\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)\Big(|u_n|^{\alpha-2}u_n|v_n|^\beta\\
&\quad-|u_n-\tilde u_n|^{\alpha-2}(u_n-\tilde u_n)|v_n-\tilde v_n|^\beta-|\tilde u_n|^{\alpha-2}\tilde u_n|\tilde v_n|^\beta\Big)\varphi\\
&\quad+\frac{\lambda\beta}{\alpha+\beta}\int_{\mathbb{R}^N} K(x)\Big(|u_n|^\alpha|v_n|^{\beta-2}v_n-|u_n-\tilde u_n|^\alpha|v_n-\tilde v_n|^{\beta-2}(v_n-\tilde v_n)\\
&\quad-|\tilde u_n|^\alpha|\tilde v_n|^{\beta-2}\tilde v_n\Big)\psi\\
&\quad+\lambda\int_{\mathbb{R}^N} f(x)\big(|u_n|^{p-2}u_n-|u_n-\tilde u_n|^{p-2}(u_n-\tilde u_n)-|\tilde u_n|^{p-2}\tilde u_n\big)\varphi\\
&\quad+\lambda\int_{\mathbb{R}^N} g(x)\big(|v_n|^{p-2}v_n-|v_n-\tilde v_n|^{p-2}(v_n-\tilde v_n)-|\tilde v_n|^{p-2}\tilde v_n\big)\psi.
\end{aligned}
$$

It is standard to check that

$$
\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|u_n|^{\alpha-2}u_n|v_n|^\beta-|u_n-\tilde u_n|^{\alpha-2}(u_n-\tilde u_n)|v_n-\tilde v_n|^\beta\\
&-|\tilde u_n|^{\alpha-2}\tilde u_n|\tilde v_n|^\beta)\varphi=0,\\
&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)(|u_n|^\alpha|v_n|^{\beta-2}v_n-|u_n-\tilde u_n|^\alpha|v_n-\tilde v_n|^{\beta-2}(v_n-\tilde v_n)\\
&-|\tilde u_n|^\alpha|\tilde v_n|^{\beta-2}\tilde v_n)\psi=0
\end{aligned}
$$

uniformly in $\|(\varphi,\psi)\|_E\le 1$. By Lemma 2.4 and $I'_\lambda(u_n,v_n)\to 0$, we complete the proof. □

Set $u_n^1=u_n-\tilde u_n$ and $v_n^1=v_n-\tilde v_n$, then $u_n-u=u_n^1+(\tilde u_n-u)$ and $v_n-v=v_n^1+(\tilde v_n-v)$. Then $(u_n,v_n)\to(u,v)$ in E if and only if $(u_n^1,v_n^1)\to(0,0)$ in E. Observe that

$$
\begin{aligned}
I_\lambda(u_n^1,v_n^1)-\frac12 I'_\lambda(u_n^1,v_n^1)(u_n^1,v_n^1)&=\big(\frac12-\frac{1}{\alpha+\beta}\big)\lambda\int_{\mathbb{R}^N} K(x)|u_n^1|^\alpha|v_n^1|^\beta\\
&\quad+(\frac12-\frac1p)\lambda\int_{\mathbb{R}^N}(f(x)|u_n^1|^p+g(x)|v_n^1|^p)
\end{aligned}
$$

$$\geq \frac{\lambda}{N} K_0 \int_{\mathbb{R}^N} |u_n^1|^\alpha |v_n^1|^\beta,$$

where $K_0 = \inf_{x\in\mathbb{R}^N} K(x) > 0$. In connection with $I_\lambda(u_n^1, v_n^1) \to c - I_\lambda(u,v)$ and $I'_\lambda(u_n^1, v_n^1) \to 0$ in E^*, we obtain

$$\int_{\mathbb{R}^N} |u_n^1|^\alpha |v_n^1|^\beta \leq \frac{N(c - I_\lambda(u,v))}{\lambda K_0} + o(1). \tag{2.3}$$

In addition, by (H2) and (H3), for any $b > 0$, there is a constant $C_b > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (K(x)|u_n^1|^\alpha |v_n^1|^\beta + f(x)|u_n^1|^p + g(x)|v_n^1|^p) \\
&\leq b(\|u_n^1\|_2^2 + \|v_n^1\|_2^2) + C_b \int_{\mathbb{R}^N} |u_n^1|^\alpha |v_n^1|^\beta.
\end{aligned}$$

Let $V_b(x) := \max\{V(x), b\}$, where b is the positive constant in the assumption (H_1). Since the set $\nu^b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure and $(u_n^1, v_n^1) \to (0,0)$ in $L^2_{\rm loc}(\mathbb{R}^N) \times L^2_{\rm loc}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V(x)(|u_n^1|^2 + |v_n^1|^2) = \int_{\mathbb{R}^N} V_b(x)(|u_n^1|^2 + |v_n^1|^2) + o(1). \tag{2.4}$$

Thus

$$\begin{aligned}
S_{\alpha,\beta}\Big(\int_{\mathbb{R}^N} |u_n^1|^\alpha |v_n^1|^\beta\Big)^{\frac{2}{\alpha+\beta}} &\leq \int_{\mathbb{R}^N} (|\nabla u_n^1|^2 + |\nabla v_n^1|^2) \\
&= \int_{\mathbb{R}^N} (|\nabla u_n^1|^2 + |\nabla v_n^1|^2 + \lambda V(x)|u_n^1|^2 + \lambda V(x)|v_n^1|^2) \\
&\quad - \int_{\mathbb{R}^N} \lambda V(x)(|u_n^1|^2 + |v_n^1|^2) \\
&= \lambda \int_{\mathbb{R}^N} (K(x)|u_n^1|^\alpha |v_n^1|^\beta + f(x)|u_n^1|^p + g(x)|v_n^1|^p) \\
&\quad - \lambda \int_{\mathbb{R}^N} V_b(x)(|u_n^1|^2 + |v_n^1|^2) + o(1) \\
&\leq \lambda C_b \int_{\mathbb{R}^N} |u_n^1|^\alpha |v_n^1|^\beta + o(1).
\end{aligned}$$

From (2.3), we have

$$\begin{aligned}
S_{\alpha,\beta} &\leq \lambda C_b \Big(\int_{\mathbb{R}^N} |u_n^1|^\alpha |v_n^1|^\beta\Big)^{1 - \frac{2}{\alpha+\beta}} + o(1) \\
&\leq \lambda C_b \Big(\frac{N(c - I_\lambda(u,v))}{\lambda K_0}\Big)^{\frac{2}{N}} + o(1) \\
&= \lambda^{1 - \frac{2}{N}} C_b \Big(\frac{N}{K_0}\Big)^{\frac{2}{N}} (c - I_\lambda(u,v))^{\frac{2}{N}} + o(1).
\end{aligned}$$

Set $\alpha_0 = S_{\alpha,\beta}^{\frac{N}{2}} C_b^{-\frac{N}{2}} N^{-1} K_0$. This implies $\alpha_0 \lambda^{1 - \frac{N}{2}} \leq c - I_\lambda(u,v) + o(1)$.

Lemma 2.6. *Let* (H1)–(H3) *be satisfied. Then, for any* $(PS)_c$ *sequence* $\{(u_n, v_n)\}$ *for* I_λ*, there exists a constant* $\alpha_0 > 0$ *(independent of* λ*) such that the functional* $I_\lambda(u,v)$ *satisfies the* $(PS)_c$ *condition for all* $c < \alpha_0 \lambda^{1 - \frac{N}{2}}$.

Proof. We can check that, for any $(PS)_c$ sequence $\{(u_n, v_n)\} \subset E$ with $(u_n, v_n) \rightharpoonup (u, v)$, either $(u_n, v_n) \to (u, v)$ or $c - I_\lambda(u, v) \geq \alpha_0 \lambda^{1-\frac{N}{2}}$.

On the contrary, suppose that $(u_n, v_n) \not\to (u, v)$; then

$$\liminf_{n\to\infty} \|(u_n, v_n)\|_E > 0$$

and $c - I_\lambda(u, v) > 0$. Based on the above conclusions, we easily get that the functional $I_\lambda(u, v)$ satisfies the $(PS)_c$ condition for all $c < \alpha_0 \lambda^{1-\frac{N}{2}}$. □

3. Mountain-pass structure

We consider $\lambda \geq 1$ and check that the functional I_λ possesses the mountain-pass structure.

Lemma 3.1. *Assume that* (H1)–(H3) *are satisfied. Then there exists* $\alpha_\lambda, \ \rho_\lambda > 0$ *such that*

$$I_\lambda(u, v) > 0 \text{ if } 0 < \|(u, v)\|_E < \rho_\lambda, \quad I_\lambda(u, v) \geq \alpha_\lambda \text{ if } \|(u, v)\|_E = \rho_\lambda.$$

Proof. Note that

$$\begin{aligned} I_\lambda(u, v) &= \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + \lambda V(x) u^2 + |\nabla v|^2 + \lambda V(x) v^2) \\ &\quad - \frac{\lambda}{p} \int_{\mathbb{R}^N} (f(x)|u|^p + g(x)|v|^p) - \frac{\lambda}{\alpha + \beta} \int_{\mathbb{R}^N} K(x)|u|^\alpha |v|^\beta. \end{aligned}$$

It is clear that, for each $s \in [2, 2^*]$, there is c_s such that if $\lambda \geq 1$,

$$\|u\|_s \leq c_s \|u\|_{E_\lambda} \quad \text{for all } u \in E_\lambda.$$

By Young inequality, we have

$$|u|^\alpha |v|^\beta \leq \frac{\alpha}{\alpha + \beta} |u|^{\alpha+\beta} + \frac{\beta}{\alpha + \beta} |v|^{\alpha+\beta}.$$

Furthermore, we obtain

$$\int_{\mathbb{R}^N} K(x)|u|^\alpha |v|^\beta \leq c_1(\|u\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*}) \leq c_1 c_{2^*} \|(u, v)\|_E^{2^*}. \tag{3.1}$$

Combining (H3) and (3.1), there is a constant c_δ such that

$$I_\lambda(u, v) \geq \frac{1}{4} \|(u, v)\|_E^2 - c_\delta \|(u, v)\|_E^{2^*} = \frac{1}{4} \|(u, v)\|_E^2 (1 - 4c_\delta \|(u, v)\|_E^{2^*-2}).$$

Set $\rho_\lambda = (\frac{1}{8c_\delta})^{\frac{1}{2^*-2}}$, it implies

$$I_\lambda(u, v) \geq \frac{1}{8} \rho_\lambda^2 =: \alpha_\lambda > 0 \quad \text{if } \|(u, v)\|_E = \rho_\lambda.$$

The proof is complete. □

Lemma 3.2. *For any finite dimensional subspace* $F \subset E$, *we have*

$$I_\lambda(u, v) \to -\infty \quad \text{as } \|(u, v)\|_E \to \infty \quad \text{for } (u, v) \in F.$$

Proof. From assumptions (H2) and (H3), it follows that

$$I_\lambda(u, v) \leq \frac{1}{2} \|(u, v)\|_E^2 - \lambda c_0 \|(u, v)\|_p^p \quad \text{for all } (u, v) \in F.$$

In connection with the fact that all norms in a finite-dimensional space are equivalent and $p > 2$, we easily get the desired conclusion. □

Lemma 3.3. *For any $\sigma > 0$, there is $\Lambda_\sigma > 0$ such that for each $\lambda \geq \Lambda_\sigma$, there exists $\bar{e}_\lambda \in E$ with $\|\bar{e}_\lambda\|_E > \rho_\lambda$, we have $I_\lambda(\bar{e}_\lambda) \leq 0$ and*

$$\max_{t\geq 0} I_\lambda(t\bar{e}_\lambda) \leq \sigma\lambda^{1-\frac{N}{2}},$$

where ρ_λ is defined in Lemma 3.1.

Proof. Define the functionals

$$\Phi_\lambda(u,v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda V(x)|u|^2 + |\nabla v|^2 + \lambda V(x)|v|^2) - \lambda c_0\int_{\mathbb{R}^N}(|u|^p + |v|^p),$$

$$\Psi_\lambda(u,v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2 + V(\lambda^{-\frac{1}{2}}x)(|u|^2 + |v|^2)) - c_0\int_{\mathbb{R}^N}(|u|^p + |v|^p).$$

We obtain that $\Phi_\lambda \in C^1(E)$ and $I_\lambda(u,v) \leq \Phi_\lambda(u,v)$ for all $(u,v) \in E$. Observe that

$$\inf\Big\{\int_{\mathbb{R}^N}|\nabla\phi|^2 : \phi \in C_0^\infty(\mathbb{R}^N,\mathbb{R}), \|\phi\|_p = 1\Big\} = 0.$$

For any $\delta > 0$, there are $\phi_\delta, \psi_\delta \in C_0^\infty(\mathbb{R}^N,\mathbb{R})$ with $\|\phi_\delta\|_p = \|\psi_\delta\|_p = 1$ such that

$$\operatorname{supp}(\phi_\delta,\psi_\delta) \subset B_{r_\delta}(0) \quad \text{and} \quad \|\nabla\phi_\delta\|_2^2, \|\nabla\psi_\delta\|_2^2 < \delta.$$

Let $e_\lambda(x) = (\phi_\delta(\sqrt{\lambda}x), \psi_\delta(\sqrt{\lambda}x))$, then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-\frac{1}{2}}r_\delta}(0)$. Furthermore,

$$\Phi_\lambda(te_\lambda) = \lambda^{1-\frac{N}{2}}\Psi_\lambda(t\phi_\delta, t\psi_\delta).$$

It is clear that

$$\begin{aligned}\max_{t\geq 0}\Psi_\lambda(t\phi_\delta,t\psi_\delta) \leq{}& \frac{p-2}{2p(pc_0)^{\frac{2}{p-2}}}\Big\{\int_{\mathbb{R}^N}(|\nabla\phi_\delta|^2 + V(\lambda^{-\frac{1}{2}}x)|\phi_\delta|^2)\Big\}^{\frac{p}{p-2}}\\ &+ \frac{p-2}{2p(pc_0)^{\frac{2}{p-2}}}\Big\{\int_{\mathbb{R}^N}(|\nabla\psi_\delta|^2 + V(\lambda^{-\frac{1}{2}}x)|\psi_\delta|^2)\Big\}^{\frac{p}{p-2}}.\end{aligned}$$

Combining $V(0) = 0$ and $\operatorname{supp}(\phi_\delta,\psi_\delta) \subset B_{r_\delta}(0)$, there is $\Lambda_\delta > 0$ such that for all $\lambda \geq \Lambda_\delta$, we have

$$\max_{t\geq 0}\Phi_\lambda(t\phi_\delta,t\psi_\delta) \leq \lambda^{1-\frac{N}{2}}\frac{(p-2)}{p(pc_0)^{\frac{2}{p-2}}}(2\delta)^{\frac{p}{p-2}}.$$

Thus, for all $\lambda \geq \Lambda_\delta$,

$$\max_{t\geq 0} I_\lambda(te_\lambda) \leq \lambda^{1-\frac{N}{2}}\frac{(p-2)}{p(pc_0)^{\frac{2}{p-2}}}(2\delta)^{\frac{p}{p-2}}. \tag{3.2}$$

For any $\sigma > 0$, we can choose $\delta > 0$ enough small such that

$$\frac{(p-2)}{p(pc_0)^{\frac{2}{p-2}}}(2\delta)^{\frac{p}{p-2}} \leq \sigma$$

and $e_\lambda(x) = (\phi_\delta(\sqrt{\lambda}x), \psi_\delta(\sqrt{\lambda}x))$. Taking $\Lambda_\delta = \Lambda_\sigma$, there is $\bar{t}_\lambda > 0$ such that $\|\bar{t}_\lambda e_\lambda\|_E > \rho_\lambda$ and $I_\lambda(te_\lambda) \leq 0$ for all $t \geq \bar{t}_\lambda$. By (3.2), $\bar{e}_\lambda = \bar{t}_\lambda e_\lambda$ satisfies the requirements. $\square$

4. Proof of Theorem 1.1

Define

$$c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = \bar{e}_\lambda\}$. In addition, for any $\sigma > 0$ with $\sigma < \alpha_0$, there is $\Lambda_\sigma > 0$ such that $\lambda \geq \Lambda_\sigma$. We can take c_λ satisfying $c_\lambda \leq \sigma \lambda^{1-\frac{N}{2}}$.

From the above results, the functional I_λ satisfies $(PS)_{c_\lambda}$ condition and has the mountain-pass structure if $c_\lambda \leq \sigma \lambda^{1-\frac{N}{2}}$. Hence, there is $(u_\lambda, v_\lambda) \in E$ such that

$$I_\lambda(u_\lambda, v_\lambda) = c_\lambda \quad \text{and} \quad I'_\lambda(u_\lambda, v_\lambda) = 0.$$

That is to say, (u_λ, v_λ) is a weak solution of (1.6). Similar to the arguments in [12], we also obtain that (u_λ, v_λ) is a positive least energy solution. Furthermore,

$$\begin{aligned} I_\lambda(u_\lambda, v_\lambda) &= I_\lambda(u_\lambda, v_\lambda) - \frac{1}{p} I'_\lambda(u_\lambda, v_\lambda)(u_\lambda, v_\lambda) \\ &\geq \big(\frac{1}{2} - \frac{1}{p}\big)\|(u_\lambda, v_\lambda)\|_E^2. \end{aligned}$$

This shows that

$$\frac{p-2}{2p}\|(u_\lambda, v_\lambda)\|_E^2 \leq I_\lambda(u_\lambda, v_\lambda) = c_\lambda \leq \sigma \lambda^{1-\frac{N}{2}}.$$

The proof is complete.

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