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CHARACTERIZATION OF A HOMOGENEOUS ORLICZ SPACE

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Abstract. In this article we define and characterize the homogeneous Orlicz space $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$ where $\Phi:\mathbb{R}\to[0,+\infty)$ is the N-function generated by an odd, increasing and not-necessarily differentiable homeomorphism $\phi:\mathbb{R}\to\mathbb{R}$. The properties of $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$ are treated in connection with the ϕ-Laplacian eigenvalue problem

$$-\operatorname{div}\Big(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big)=\lambda\, g(\cdot)\phi(u)\quad\text{in } \mathbb{R}^N$$

where $\lambda\in\mathbb{R}$ and $g:\mathbb{R}^N\to\mathbb{R}$ is measurable. We use a classic Lagrange rule to prove that solutions of the ϕ-Laplace operator exist and are non-negative.

1. Introduction

Let $N\geq 2$ be an integer. A broad subclass of maximization problems in an open domain $\Omega\subset\mathbb{R}^N$ involves critical Sobolev exponents. Several articles are motivated by the ideas and methods in the seminal paper by Brezis and Nirenberg [5], mainly when Ω is bounded. The case Ω unbounded is treated in [3, 21]. The reference [2] contains significant results on semilinear problems also in the unbounded case, which are largely treated via concentration-compactness methods. In that reference the authors introduce the space

$$\mathscr{D}^{1,p}(\Omega)=\{u\in L^{p^*}(\Omega):|\nabla u|\in L^p(\Omega)\}\tag{1.1}$$

where $1<p<N$ and $p^*=pN/(N-p)$ is the conjugate exponent. This space is equipped with the norm $\|u\|_{1,p}=\|u\|_{p^*}+\||\nabla u|\|_p$ where $\|\cdot\|_p$ is the norm in $L^p(\Omega)$. On the other hand, the completion of the space $\mathcal{D}(\Omega)$ of C^∞-functions with compact support in Ω with respect to the norm $\|\cdot\|_{1,p}$ is denoted by $\mathscr{D}_{\rm o}^{1,p}(\Omega)$. Equivalently,

$$\mathscr{D}_{\rm o}^{1,p}(\Omega)=\mathrm{cl}_{\mathscr{D}^{1,p}(\Omega)}\,\mathcal{D}(\Omega)$$

where $\mathrm{cl}_X(Y)$ is the closure operator of Y in X. This space is endowed with the gradient seminorm $\|u\|_{\rm o,\,p}=\||\nabla u|\|_p$. It can be easily proved that this is actually a norm on $\mathscr{D}_{\rm o}^{1,p}(\Omega)$ which is equivalent to $\|u\|_{1,p}$. It is moreover known that the two spaces thus defined are reflexive and Banach for the respective norms. Somewhat surprisingly, a fundamental characterization (see [2, Lemma 1.2]) in the (unbounded) case $\Omega=\mathbb{R}^N$ asserts that $\mathscr{D}_{\rm o}^{1,p}(\mathbb{R}^N)=\mathscr{D}^{1,p}(\mathbb{R}^N)$. This equivalence

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motivates the problem whether this space is still meaningful in a larger context or not and raises the issue about the use and place of this *extended* space in analysis, particularly in optimization and differential equations. In this paper we answer positively the former question and provide an application which well suits the latter via a fundamental formulation in Orlicz spaces, see below. An exhaustive treatment on the theory of these function spaces can be found in the classic textbook by Krasnosel'skii and Rutic'kii [17] and, more recently, in references [16, 18, 24]. The papers and monographs by Gossez [12, 13, 15] are particularly detailed and have played a paramount role in the subject as well.

Orlicz spaces constitute a natural *extension* of the notion of an L^p space: the function $t \mapsto |t|^p$ entering the definition of L^p is replaced by a more general N-function $\Phi : \mathbb{R} \to [0, +\infty)$ (sometimes called a Young function). The typical approach in the references mentioned above is mostly developed in $\mathbb{R}^N$ with the Lebesgue measure. One is naturally led to the question whether the properties and structure of classic Orlicz spaces are preserved in a much more general measure space (Ω, Σ, μ). The monograph by J. Musielak [20] studies the properties associated with the generalized Orlicz space $L^\Phi(\Omega, \Sigma, \mu)$ (such as embeddings of and compactness in generalized Orlicz classes) in the setting of modular and parameter-dependent families of Orlicz spaces.

An interesting source of research is given by the case of exponents $p(x)$, where $p : \Omega \to (1, +\infty)$ is a bounded function. The article [22] and excellent book [23] are representatives in the case of nonhomogeneous differential operators containing one or more power-type nonlinearities with variable exponents. The theory there is developed in great generality including many possible pathologies of the Young function. As a yet another significant contribution, the paper by Fu and Shan [9] gives sufficient conditions for removability of isolated singular points of elliptic equations in the Sobolev space $W^{1,p(x)}$, which was first studied by Kováčik and Rákosník.

In this manuscript we consider the homogeneous Orlicz space $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$. It corresponds to the completion of $\mathcal{D}(\mathbb{R}^N)$ with respect to a suitable norm, see Section 4. If additional hypotheses are fulfilled this space constitutes a natural source of solutions of minimization problems with constraints for a wide class of energy functionals in the generalized-Laplacian form. For example, in the article [10] the following quasilinear elliptic problem is considered,

$$-\operatorname{div}\left(\varphi(|\nabla u|)\nabla u\right) = b(|u|)u + \lambda f(x,u) \quad \text{in } \mathbb{R}^N \tag{1.2}$$

where the function $\varphi(t)t$ is non-homogeneous. The term $b(|u|)u$ denotes a critical Sobolev growth coefficient, $f(x,u)$ is a subcritical term and $\lambda > 0$ is a parameter. The authors prove that any non-negative solution of this problem can be regarded as a critical point of the variational formulation

$$\begin{aligned} &\text{minimize} \quad \int_{\mathbb{R}^N} \big(\Phi(|\nabla u|) - B(u) - \lambda F(x,u)\big)dx \\ &\text{such that} \quad u \in \mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N) \end{aligned}$$

where $B(t)$ and $F(x,t)$ are the primitives of $b(t)t$ and $f(x,t)$, respectively, and $\Phi(t) = \int_0^s \varphi(t)t\,dt$. Due to some topological restrictions on $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$ standard methods to prove convergence of minimizing sequences for this problem are useless. The techniques employed in [10] consist of a modification of the concentration-compactness principle for Mountain-pass problems.

In this article we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing, odd and not-necessarily differentiable homeomorphism and define the associated N-function

$$\Phi(t) = \int_0^t \phi(s)\, ds. \tag{1.3}$$

Motivated by the ideas discussed above, we provide a characterization of the homogeneous Orlicz space $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$ generated by Φ. This characterization asserts that the latter space is an *extension* of (1.1) in a precise sense and naturally leads to the following application. Let $g : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and λ be a real number. Under additional global restrictions on Φ and g, existence of nontrivial solutions of the ϕ-Laplacian equation

$$-\operatorname{div}\Big(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) = \lambda g(\cdot)\phi(u) \quad \text{in } \mathbb{R}^N \tag{1.4}$$

can be proved. We address this question and solve the associated optimization problem by implementing a version of Lagrange multipliers rule [6] on the source space $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$. We prove that solutions of the ϕ-Laplace operator exist and are non-negative.

2. N-functions

This is a brief overview on Orlicz spaces. Fundamental definitions and properties can be found in several monographs, articles and books. For further details we refer the reader to [17, 18, 20].

A convex, even and continuous function $\Phi : \mathbb{R} \to [0, +\infty)$ satisfying $\Phi(t) = 0$ if and only if $t = 0$ and such that

$$\frac{\Phi(t)}{t} \to 0 \text{ as } t \to 0 \quad \text{and} \quad \frac{\Phi(t)}{t} \to +\infty \text{ as } t \to +\infty$$

is called an N-function. Equivalently [13], Φ can be represented in the integral form (1.3), where $\phi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing, odd function which is right-continuous for $t \geq 0$ and which satisfies $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) \to +\infty$ as $t \to +\infty$. The N-function Φ satisfies a global Δ_2-condition (see [1, pp. 266]) if there exists $\mathcal{C} > 0$ such that

$$\Phi(2t) \leq \mathcal{C}\Phi(t)$$

for all $t \geq 0$.

Lemma 2.1 ([1]). *The N-function Φ satisfies a global Δ_2-condition if and only if*

$$q_\Phi := \sup_{s>0} \frac{s\phi(s)}{\Phi(s)} < +\infty. \tag{2.1}$$

2.1. Conjugates.

The reciprocal function $\psi(s)$ of ϕ is defined for $s \geq 0$ by

$$\psi(s) = \sup\{t : \phi(t) \leq s\}.$$

Both functions ϕ and ψ have the same properties. Hence the integral

$$\overline{\Phi}(t) = \int_0^t \psi(s)\, ds$$

is an N-function, called the conjugate (or complementary) N-function of Φ. The pair $\Phi, \overline{\Phi}$ is called a pair of complementary N-functions. If ϕ is continuous and increases monotonically then the reciprocal ψ is the ordinary inverse of ϕ.

Lemma 2.2 ([10, Lemma 2.5]). *The complementary N-function $\overline{\Phi}$ satisfies a global Δ_2-condition if and only if*

$$p_\Phi := \inf_{s>0} \frac{s\phi(s)}{\Phi(s)} > 1. \tag{2.2}$$

The Sobolev conjugate N-function Φ_* of Φ is defined as

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}}\, ds$$

where Φ^{-1} denotes the inverse function of $\Phi|_{[0,+\infty)}$. It is known [24] that the Sobolev conjugate exists if and only if

$$\int_0^1 \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}}\, ds < +\infty \quad \text{and} \quad \lim_{t\to+\infty} \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}}\, ds = +\infty. \tag{2.3}$$

Moreover, it is known [11] that if conditions (2.3) are fulfilled then

$$\lim_{t\to+\infty} \frac{\Phi(t)}{\Phi_*(kt)} = 0 \tag{2.4}$$

for all $k > 0$.

Proposition 2.3 ([10]). *If conditions* (2.3) *are met and $q_\Phi < N$ then the following estimates hold:*

(a) $\min\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\Phi(t) \le \Phi(\rho t) \le \max\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\Phi(t)$;
(b) $\min\{r^{p_\Phi^*}, r^{q_\Phi^*}\}\Phi_*(t) \le \Phi_*(rt) \le \max\{r^{p_\Phi^*}, r^{q_\Phi^*}\}\Phi_*(t)$;
(c) $\min\{r^{p_\Phi^*/(p_\Phi^*-1)}, r^{q_\Phi^*/(q_\Phi^*-1)}\}\overline{\Phi_*}(t) \le \overline{\Phi_*}(rt)$
$\le \max\{r^{p_\Phi^*/(p_\Phi^*-1)}, r^{q_\Phi^*/(q_\Phi^*-1)}\}\overline{\Phi_*}(t)$

for $r, t \ge 0$ and where $p_\Phi^* = p_\Phi N/(N - p_\Phi)$ and $q_\Phi^* = q_\Phi N/(N - q_\Phi)$ are the conjugate exponents.

Note that Proposition 2.3 ensures that both the Sobolev conjugate N-function Φ_* and its complementary $\overline{\Phi_*}$ satisfy a global Δ_2-condition provided $q_\Phi < N$.

Lemma 2.4. *Let $1 < r < N$ be such that*

$$0 < A = \liminf_{s\to 0^+} \frac{\phi(s)}{s^{r-1}} \le \mathsf{B} = \limsup_{s\to 0^+} \frac{\phi(s)}{s^{r-1}} < +\infty. \tag{2.5}$$

Then for $\varepsilon > 0$ sufficiently small there exists $s_0 = s_0(\varepsilon) > 0$ such that for all $0 < s < s_0$,

(a) $\frac{(\mathsf{A}-\varepsilon)}{r} s^r \le \Phi(s) \le \frac{(\mathsf{B}+\varepsilon)}{r} s^r$,
(b) $\left(\frac{s\, r^*}{\overline{\mathsf{A}}}\right)^{1/r^*} \le \Phi_*(s) \le \left(\frac{s\, r^*}{\overline{\mathsf{B}}}\right)^{1/r^*}$

where $\overline{\mathsf{B}} = r^{1/r}/(\mathsf{B}+\varepsilon)^{1/r}$, $\overline{\mathsf{A}} = r^{1/r}/(\mathsf{A}-\varepsilon)^{1/r}$ and $r^* = (N-r)/Nr$ is the Sobolev conjugate exponent.

Proof. If $\varepsilon > 0$ is small then there exists $s_0 = s_0(\varepsilon) > 0$ such that if $0 < s < s_0$ then by definition

$$\mathsf{A} - \varepsilon \le \frac{\phi(s)}{s^{r-1}} \le \mathsf{B} + \varepsilon.$$

Denote $t = \Phi(s)$ and $t_0 = \Phi(s_0)$. The monotonicity of Φ and simple integration yield

$$\frac{(\mathsf{A}-\varepsilon)}{r}\big(\Phi^{-1}(t)\big)^r \le t \le \frac{(\mathsf{B}+\varepsilon)}{r}\big(\Phi^{-1}(t)\big)^r$$

provided $0 < t < t_0$. Hence $\overline{\mathsf{B}}\, t^{1/r} \le \Phi^{-1}(t) \le \overline{\mathsf{A}}\, t^{1/r}$ for all $0 < t < t_0$. If $s < t < t_0$ we integrate (from s to t) the latter inequalities with respect to a new variable. This gives

$$\frac{\overline{\mathsf{B}}}{r^*}(t^{r^*} - s^{r^*}) \le \Phi_*^{-1}(t) - \Phi_*^{-1}(s) \le \frac{\overline{\mathsf{A}}}{r^*}(t^{r^*} - s^{r^*}).$$

Letting $s \to 0^+$ we get

$$\frac{\overline{\mathsf{B}}}{r^*} t^{r^*} \le \Phi_*^{-1}(t) \le \frac{\overline{\mathsf{A}}}{r^*} t^{r^*}$$

provided $0 < t < t_0$. Finally, the change of variables $s = \Phi_*^{-1}(t)$ and $s_0 = \Phi_*^{-1}(t_0)$ and the inequality above yield the estimate in (b) provided $0 < s < s_0$. $\square$

3. Function spaces

3.1. **Orlicz classes.** Let $\Phi, \overline{\Phi}$ be a pair of complementary N-functions and let Ω denote an open domain in $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_\Phi(\Omega)$ is the set of (equivalence classes of) real-valued measurable functions u such that $\Phi(u) \in L^1(\Omega)$. In general, $\mathcal{L}_\Phi(\Omega)$ is not a vector space [13]. However, the linear hull $L_\Phi(\Omega)$ of $\mathcal{L}_\Phi(\Omega)$ equipped with the Luxemburg norm

$$\|u\|_{\Phi,\Omega} = \inf\Big\{k > 0 : \int_\Omega \Phi\Big(\frac{u}{k}\Big) \le 1\Big\}$$

is a normed linear space, called the Orlicz space generated by the N-function Φ. It is known [17] that the vector space thus defined is complete.

The closure in $L_\Phi(\Omega)$ of the space of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\Phi(\Omega)$. This space is separable and Banach with the inherited norm. The following lemma gives a useful characterization of a particular type of sequences in E_Φ in the unbounded case $\Omega = \mathbb{R}^N$.

Lemma 3.1. *Let $z \in E_\Phi(\mathbb{R}^N)$ and fix an integer $k > 1$. Define the function*

$$z_k(x) = \begin{cases} z(x) & \text{if } |x| > k \\ 0 & \text{if } |x| \le k. \end{cases}$$

Then $\|z_k\|_{\Phi,\mathbb{R}^N} \to 0$ as $k \to +\infty$.

Proof. If $\varepsilon > 0$ is sufficiently small then $z/\varepsilon \in E_\Phi(\mathbb{R}^N) \subseteq \mathcal{L}_\Phi(\mathbb{R}^N)$. The latter implies $\Phi(z/\varepsilon) \in L^1(\mathbb{R}^N)$ and then there exists a positive integer k_0 such that if $k \ge k_0$ then

$$\int_{\mathbb{R}^N} \Phi\Big(\frac{z_k}{\varepsilon}\Big)\, dx = \int_{\mathbb{R}^N \setminus B_k(0)} \Phi\Big(\frac{z}{\varepsilon}\Big)\, dx \le 1$$

where $B_k(0)$ denotes the ball of radius k and center at zero in $\mathbb{R}^N$. The definition of the Luxemburg norm hence yields $\|z_k\|_{\Phi,\mathbb{R}^N} \le \varepsilon$ provided $k \ge k_0$. $\square$

In general, $E_\Phi(\Omega) \subseteq \mathcal{L}_\Phi(\Omega) \subseteq L_\Phi(\Omega)$ but if Φ satisfies a global Δ_2-condition then $E_\Phi(\Omega) = L_\Phi(\Omega)$ and *vice-versa.* In this case, a known result [1, Theorem 8.20] ensures that $L_\Phi(\Omega)$ and $L_{\overline{\Phi}}(\Omega)$ are reflexive and separable provided $\overline{\Phi}$ satisfies a global Δ_2-condition as well. Since this result remains valid after replacing Φ by its Sobolev conjugate Φ_* (provided the latter exists), Proposition 2.3 guarantees the validity of the following result.

Corollary 3.2. *If (2.3) are satisfied and $q_\Phi < N$ then the Orlicz space $L_{\Phi_*}(\Omega)$ is reflexive.*

It is well known [1, 13] that one can identify the dual space of $E_\Phi(\Omega)$ with $L_{\overline{\Phi}}(\Omega)$ and the dual space of $E_{\overline{\Phi}}(\Omega)$ with $L_\Phi(\Omega)$. Moreover, if $u \in L_\Phi(\Omega)$ and $v \in L_{\overline{\Phi}}(\Omega)$ then the inequality

$$\int_\Omega |uv|\,dx \le 2\|u\|_{\Phi,\Omega}\,\|v\|_{\overline{\Phi},\Omega} \tag{3.1}$$

holds. This estimate is an extension of Hölder's inequality to Orlicz spaces.

An Orlicz-Sobolev space. The Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ is the vector space of functions in $L_\Phi(\Omega)$ with first distributional derivatives in $L_\Phi(\Omega)$. This space is Banach with the norm

$$|||u|||_\Omega = \|u\|_{\Phi,\Omega} + \sum_{i=1}^N \|\partial_{x_i} u\|_{\Phi,\Omega} \tag{3.2}$$

where ∂_{x_i} denotes the partial derivative $\partial/\partial x_i$. Usually, $W^1L_\Phi(\Omega)$ is identified with a subspace of the product $L_\Phi(\Omega)^{N+1} = \Pi L_\Phi(\Omega)$. The space $W^1L_\Phi(\Omega)$ is not separable in general.

3.2. **Approximation properties.** In what follows we consider $\Omega = \mathbb{R}^N$ in which case further characterizations are possible. The Luxemburg norm $\|\cdot\|_{\Phi,\mathbb{R}^N}$ will be simply denoted by $\|\cdot\|_\Phi$. The symbol $\mathcal{D}(\mathbb{R}^N)$ denotes the space of C^∞-functions with compact support in $\mathbb{R}^N$. We choose a mollifier $\rho \in \mathcal{D}(\mathbb{R}^N)$; i.e. ρ is a real-valued function such that

(a) $\rho(x) \ge 0$, if $x \in \mathbb{R}^N$;
(b) $\rho(x) = 0$, if $|x| \ge 1$;
(c) $\int_{\mathbb{R}^N} \rho(x)\,dx = 1$.

If ε is positive, it is clear that the function $\rho_\varepsilon(x) = \varepsilon^{-N}\rho(x/\varepsilon)$ is non-negative, belongs to $\mathcal{D}(\mathbb{R}^N)$ and satisfies $\rho_\varepsilon(x) = 0$ provided $|x| \ge \varepsilon$. In addition,

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,dx = 1. \tag{3.3}$$

If $u \in L_\Phi(\mathbb{R}^N)$ we define the regularized function u_ε of u by the convolution

$$u_\varepsilon(x) = (\rho_\varepsilon * u)(x) = \int_{\mathbb{R}^N} u(x-y)\rho_\varepsilon(y)\,dy.$$

It is easy to see that if u has compact support in $\mathbb{R}^N$ then u_ε belongs to $\mathcal{D}(\mathbb{R}^N)$.

Proposition 3.3. *If $u \in L_\Phi(\mathbb{R}^N)$ then $u_\varepsilon \in L_\Phi(\mathbb{R}^N)$ and $\|u_\varepsilon\|_\Phi \le \|u\|_\Phi$.*

Proof. Let $\lambda = \|u\|_\Phi$. Jensen's inequality [13, pp. 18] yields

$$\int_{\mathbb{R}^N} \Phi\Big(\frac{u_\varepsilon(x)}{\lambda}\Big)\,dx \le \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \Phi\Big(\frac{u(x-y)}{\lambda}\Big)\rho_\varepsilon(y)\,dy\Big)dx. \tag{3.4}$$

Define the function $F(x,y) = \Phi(u(x-y)/\lambda)\rho_\varepsilon(y)$. It is clear from the definition of λ that

$$\int_{\mathbb{R}^N} F(x,y)\,dx = \rho_\varepsilon(y)\int_{\mathbb{R}^N} \Phi\Big(\frac{u(x-y)}{\lambda}\Big)dx \le \rho_\varepsilon(y). \tag{3.5}$$

Integration of this inequality with respect to y and condition (3.3) imply $F \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. Hence Fubini's theorem and (3.4) yield

$$\int_{\mathbb{R}^N} \Phi\Big(\frac{u_\varepsilon(x)}{\lambda}\Big)\,dx \le \int_{\mathbb{R}^N} \Big(\int_{\mathbb{R}^N} \Phi\Big(\frac{u(x-y)}{\lambda}\Big)\,dx\Big)\rho_\varepsilon(y)\,dy \le 1$$

and then $u_\varepsilon \in L_\Phi(\mathbb{R}^N)$. By definition of the Luxemburg norm, $\|u_\varepsilon\|_\Phi \le \lambda = \|u\|_\Phi$. ☐

Lemma 3.4 ([14]). *If $u \in E_\Phi(\mathbb{R}^N)$ then $\|u_\varepsilon - u\|_\Phi \to 0$ as $\varepsilon \to 0$.*

4. The homogeneous Orlicz space $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$

In what follows we assume that $\phi : \mathbb{R} \to \mathbb{R}$ is an odd, non-decreasing and not-necessarily differentiable homeomorphism which generates the N-function (1.3). We suppose that condition (2.1) is fulfilled; i.e. Φ satisfies a global Δ_2-condition. We will assume that (2.3) are met as well, so that the Sobolev conjugate Φ_* is defined. The set $B_R(x_0) \subseteq \mathbb{R}^N$ will denote the ball of radius R with center at $x_0 \in \mathbb{R}^N$. As mentioned previously, the operator ∂_{x_i} will denote the partial derivative $\partial/\partial x_i$, $i = 1, \dots, N$. We start out by defining the space

$$\mathscr{D}^{1,\Phi}(\mathbb{R}^N) = \left\{u \in L_{\Phi_*}(\mathbb{R}^N) : |\nabla u| \in L_\Phi(\mathbb{R}^N)\right\}.$$

Proposition 4.1. *The space $\mathscr{D}^{1,\Phi}(\mathbb{R}^N)$ equipped with the norm*

$$\|u\|_{1,\Phi} = \|u\|_{\Phi_*} + \||\nabla u|\|_\Phi. \tag{4.1}$$

is complete.

Proof. Let $\{u_n\}$ be a Cauchy sequence in $\mathscr{D}^{1,\Phi}(\mathbb{R}^N)$; that is,

$$\|u_n - u_m\|_{\Phi_*} \to 0 \quad \text{and} \quad \||\nabla u_n - \nabla u_m|\|_\Phi \to 0 \tag{4.2}$$

as $n, m \to +\infty$. Since $L_{\Phi_*}(\mathbb{R}^N)$ is a Banach space we can find $u \in L_{\Phi_*}(\mathbb{R}^N)$ such that $u_n \to u$ in $L_{\Phi_*}(\mathbb{R}^N)$. The second condition in (4.2) implies that $\{\partial_{x_i} u_n\}$ is a Cauchy sequence in $L_\Phi(\mathbb{R}^N)$. Then for each index $i = 1, \dots, N$ there exists $\omega_i \in L_\Phi(\mathbb{R}^N)$ such that $\partial_{x_i} u_n \to \omega_i$ in $L_\Phi(\mathbb{R}^N)$. Since $\partial_{x_i} u_n$ is the weak derivative of u_n we have $\partial_{x_i} u_n \in L_\Phi(\mathbb{R}^N)$. Then

$$-\int_{\mathbb{R}^N} u_n\, \partial_{x_i}\psi\, dx = \int_{\mathbb{R}^N} \partial_{x_i} u_n\, \psi\, dx$$

for all $\psi \in \mathcal{D}(\mathbb{R}^N)$. Hölder's inequality (3.1) and uniqueness of limits yield

$$-\int_{\mathbb{R}^N} u\, \partial_{x_i}\psi\, dx = \int_{\mathbb{R}^N} \omega_i\, \psi\, dx.$$

Thus, we get $\partial_{x_i} u = \omega_i \in L_\Phi(\mathbb{R}^N)$ and $\|u_n - u\|_{1,\Phi} \to 0$ as $n \to +\infty$. ☐

Definition 4.2. The homogeneous Orlicz space $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ is the completion of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm (4.1). Equivalently,

$$\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N) = \mathrm{cl}_{\mathscr{D}^{1,\Phi}(\mathbb{R}^N)}\, \mathcal{D}(\mathbb{R}^N)$$

where $\mathrm{cl}_{\mathscr{D}^{1,\Phi}(\mathbb{R}^N)}$ denotes the closure operator.

The space $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ is endowed with the seminorm

$$\|u\|_{\mathrm{o},\Phi} = \||\nabla u|\|_\Phi. \tag{4.3}$$

Lemma 4.3. *On $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ the seminorm (4.3) defines a norm which is equivalent to (4.1).*

Proof. By [8, Theorem 3.4], if $u \in \mathcal{D}(\mathbb{R}^N)$ then

$$\|u\|_{\Phi_*} \le \mathscr{C}(N)\, \||\nabla u|\|_\Phi = \mathscr{C}(N)\, \|u\|_{\mathrm{o},\Phi} \tag{4.4}$$

where $\mathscr{C}(N)$ is a positive constant. This inequality extends to all of $\mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N)$ by density. □

We remark that since $\mathcal{D}(\mathbb{R}^N) \subseteq \mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N)$, the inclusions

$$W^1 L_\Phi(\mathbb{R}^N) \subseteq \mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N) \subseteq \mathscr{D}^{1,\Phi}(\mathbb{R}^N) \tag{4.5}$$

hold. Example 4.7 below proves that there exist N-functions Φ for which the inclusion $W^1 L_\Phi(\mathbb{R}^N) \subseteq \mathscr{D}^{1,\Phi}(\mathbb{R}^N)$ is strict.

The following theorem is the main result in this article.

Theorem 4.4. *Assume that there exists $1 < r < N$ such that estimates (2.5) are fulfilled. If $q_\Phi < N$ then the reversed inclusion $\mathscr{D}^{1,\Phi}(\mathbb{R}^N) \subseteq \mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N)$ holds as well. That is,*

$$\mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N) = \{u \in L_{\Phi_*}(\mathbb{R}^N) : |\nabla u| \in L_\Phi(\mathbb{R}^N)\}.$$

Proof. Take $u \in \mathscr{D}^{1,\Phi}(\mathbb{R}^N)$ and define $\omega \in \mathcal{D}(\mathbb{R}^N)$ by

$$\omega(x) = \begin{cases} 0 & \text{if } |x| \ge 2, \\ 1 & \text{if } |x| \le 1. \end{cases}$$

Next, form the functions

$$\omega_k(x) = \omega\left(\frac{x}{k}\right) \quad \text{and} \quad u_k(x) = u(x)\, \omega_k(x), \quad k \in \mathbb{N}.$$

For each fixed $k \in \mathbb{N}$ we consider the sequence of regularized functions $v_n^k = \rho_{1/n} * u_k$, $n \in \mathbb{N}$, where $\rho_{1/n}(x) = (1/n)^{-N} \rho(nx)$ and ρ is the mollifier satisfying (a), (b) and (c) in §3.2. Note that as u_k has compact support the convolution $v_n^k \in \mathcal{D}(\mathbb{R}^N)$. Moreover, since $\partial_{x_i} v_n^k = \rho_{1/n} * \partial_{x_i} u_k \in E_\Phi(\mathbb{R}^N)$, Lemma 3.4 implies

$$\|\partial_{x_i} v_n^k - \partial_{x_i} u_k\|_\Phi \to 0 \quad \text{as } n \to +\infty.$$

Then, for $k \in \mathbb{N}$, we have

$$\lim_{n\to+\infty} \||\nabla v_n^k - \nabla u_k|\|_\Phi = 0.$$

For every natural number k, Cantor's diagonalization method produces an integer $n_k \in \mathbb{N}$ (which depends only on k) such that if we set $v_k = v_{n_k}^k = \rho_{1/n_k} * u_k$, then

$$\||\nabla v_k - \nabla u_k|\|_\Phi \le \frac{1}{k}, \quad k \in \mathbb{N}.$$

The triangle inequality thus implies

$$\||\nabla v_k - \nabla u|\|_\Phi \le \frac{1}{k} + \||\nabla u_k - \nabla u|\|_\Phi.$$

We must prove that

$$\lim_{k\to+\infty} \||\nabla u_k - \nabla u|\|_\Phi = 0. \tag{4.6}$$

We note that the product rule yields $\partial_{x_i} u_k = u\, \partial_{x_i} \omega_k + \omega_k\, \partial_{x_i} u$ and hence

$$\||\nabla u_k - \nabla u|\|_\Phi \le \|(1-\omega_k)|\nabla u|\|_\Phi + \||u\, \nabla \omega_k|\|_\Phi.$$

Since Φ is increasing,

$$\int_{\mathbb{R}^N} \Phi\Big((1-\omega_k)\frac{|\nabla u|}{\lambda}\Big)\,dx \le \int_{\mathbb{R}^N\setminus\overline{B_k(0)}} \Phi\Big(\frac{|\nabla u|}{\lambda}\Big)\,dx$$

where the parameter $\lambda > 0$ is arbitrary. Note that $L_\Phi(\mathbb{R}^N) = \mathcal{L}_\Phi(\mathbb{R}^N)$ since Φ satisfies a Δ_2-condition. Therefore $\Phi(|\nabla u|/\lambda) \in L^1(\mathbb{R}^N)$. The definition of the Luxemburg norm thus implies

$$\|(1-\omega_k)|\nabla u|\|_\Phi \to 0 \quad \text{as} \quad k \to +\infty.$$

To prove (4.6) we need $\||u\nabla\omega_k|\|_\Phi \to 0$ as $k \to +\infty$. This is the case. Indeed, if $\varepsilon > 0$ is sufficiently small, there exists $s_0 = s_0(\varepsilon)$ such that items (a) and (b) from Lemma 2.4 will be satisfied for all $0 < s < s_0$. Also, note that (2.4) implies

$$\mathsf{C} := \sup_{s\ge s_0} \frac{\Phi(s)}{\Phi_*(s)} < +\infty. \tag{4.7}$$

We define the sets $\Omega_1 = \{x \in \mathbb{R}^N : |u(x)| < s_0\}$ and $\Omega_2 = \{x \in \mathbb{R}^N : |u(x)| \ge s_0\}$ and take the closed annulus $A_k = \overline{B_{2k}(0)}\setminus B_k(0) \subseteq \mathbb{R}^N$. Choose λ positive and denote by $M = \sup_{\mathbb{R}^N} \partial_{x_i}\omega$. We take k sufficiently large such that $k > M/\lambda$. The monotonicity of Φ and (4.7) yield

$$\begin{aligned}
&\int_{A_k} \Phi\Big(\frac{1}{\lambda}|\partial_{x_i} w_k||u|\Big)\,dx\\
&= \int_{A_k\cap\Omega_1} \Phi\Big(\frac{1}{\lambda k}|\partial_{x_i} w||u|\Big)\,dx + \int_{A_k\cap\Omega_2} \Phi\Big(\frac{1}{\lambda k}|\partial_{x_i} w|\,|u|\Big)\,dx\\
&\le \int_{A_k\cap\Omega_1} \Phi\left(\frac{M}{\lambda k}|u|\right)\,dx + \int_{A_k\cap\Omega_2} \Phi(|u|)\,dx\\
&\le \int_{A_k\cap\Omega_1} \Phi\Big(\frac{M}{\lambda k}|u|\Big)\,dx + \mathsf{C}\int_{A_k} \Phi_*(|u|)\,dx.
\end{aligned} \tag{4.8}$$

Since $u \in L_{\Phi_*}(\mathbb{R}^N)$ it is evident that $\int_{A_k} \Phi_*(|u|)\,dx \to 0$ as $k \to +\infty$.

Note that the choice of k above implies that $M|u|/\lambda k < s_0$ on Ω_1. Item (a) in Lemma 2.4 yields the following estimate for the integral on the right-hand side in (4.8),

$$\int_{A_k\cap\Omega_1} \Phi\Big(\frac{M}{\lambda k}|u|\Big)\,dx \le (\mathsf{B}+\varepsilon)\frac{M^r}{r\lambda^r k^r}\int_{A_k\cap\Omega_1} |u|^r\,dx. \tag{4.9}$$

Since Φ_* satisfies a global Δ_2-condition, $\Phi_*(|u|) \in L_1(A_k\cap\Omega_1)$. Item (b) in Lemma 2.4 yields

$$\mathscr{A}(r,\varepsilon)|u|^{\frac{Nr}{N-r}} \le \Phi_*(|u|)$$

where $\mathscr{A}(r,\varepsilon)$ is positive. Therefore $|u|^r \in L^{\frac{N}{N-r}}(A_k\cap\Omega_1)$ and then Hölder's inequality, with $p = N/(N-r)$ and $q = N/r$, implies

$$\begin{aligned}
\int_{A_k\cap\Omega_1} |u|^r\,dx &\le \big(\operatorname{meas}(A_k\cap\Omega_1)\big)^{r/N}\Big(\int_{A_k\cap\Omega_1} |u|^{\frac{Nr}{N-r}}\,dx\Big)^{\frac{N-r}{N}}\\
&\le \big(\operatorname{meas}(\overline{B_{2k}(0)})\big)^{r/N}\Big(\int_{A_k\cap\Omega_1} |u|^{\frac{Nr}{N-r}}\,dx\Big)^{\frac{N-r}{N}}
\end{aligned}$$

where $\operatorname{meas}(\overline{B_{2k}(0)}) = \pi^{N/2}(2k)^N/\Gamma(N/2+1)$ is the volume of the closed ball $\overline{B_{2k}(0)}$ and Γ is Euler's gamma function. Thus, we obtain

$$\int_{A_k\cap\Omega_1} |u|^r\,dx \le \mathscr{B}k^r\Big(\int_{A_k\cap\Omega_1} |u|^{\frac{Nr}{N-r}}\,dx\Big)^{\frac{N-r}{N}}$$

where $\mathscr{B}=\mathscr{B}(r,N)$ is a positive constant. Therefore, estimate (4.9) yields

$$\int_{A_k\cap\Omega_1} \Phi\Big(\frac{M}{\lambda k}|u|\Big)dx \le \mathscr{B}\cdot(\mathsf{B}+\varepsilon)\frac{M^r}{r\lambda^r}\Big(\int_{A_k\cap\Omega_1} |u|^{\frac{Nr}{N-r}}\,dx\Big)^{\frac{N-r}{N}}.$$

Since the integral on the right tends to 0 as $k\to+\infty$, from (4.8) we obtain

$$\int_{A_k} \Phi\Big(\frac{1}{\lambda}|\partial_{x_i}w_k||u|\Big)\,dx \to 0 \quad \text{as } k\to+\infty.$$

The definition of the Luxemburg norm thus ensures $\||u\nabla\omega_k|\|_\Phi \to 0$ as $k\to+\infty$ and hence (4.6) holds.

To conclude the proof we must show that $\|v_k-u\|_{\Phi_*}\to 0$ as $k\to+\infty$. Notice that $v_k-u\in\mathscr{D}^{1,\Phi}(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ and hence inequality (4.4) does not apply in this case. We proceed as follows, instead. The triangle inequality and Proposition 3.3 yield

$$\begin{aligned}
\|v_k-u\|_{\Phi_*} &= \|\rho_{1/n_k} * u_k - u\|_{\Phi_*}\\
&\le \|\rho_{1/n_k} * (\omega_k\,u-u)\|_{\Phi_*} + \|\rho_{1/n_k} * u - u\|_{\Phi_*}\\
&\le \|\omega_k\,u-u\|_{\Phi_*} + \|\rho_{1/n_k} * u - u\|_{\Phi_*}.
\end{aligned}$$

Since Φ_* satisfies a global Δ_2-condition we have $\omega_k u-u\in\mathscr{D}^{1,\Phi}(\mathbb{R}^N)\subseteq L_{\Phi_*}(\mathbb{R}^N)=E_{\Phi_*}(\mathbb{R}^N)$. Lemma 3.1 (with $z_k=\omega_k u-u$) produces $\|\omega_k\,u-u\|_{\Phi_*}\to 0$ as $k\to+\infty$. Lemma 3.4 in turn implies that $\|\rho_{1/n_k} * u-u\|_{\Phi_*}\to 0$ as $k\to+\infty$ and hence the inequality above ensures that $v_k\to u$ in $L_{\Phi_*}(\mathbb{R}^N)$. Along with (4.6), the latter implies $\|v_k-u\|_{1,\Phi}\to 0$ as $k\to+\infty$. The proof of the theorem is complete. $\square$

Example 4.5. We define

$$\phi_1(s) = \frac{|s|^{p-2}s}{\log(1+|s|)},$$

where $p>2$. In this case,

$$\Phi_1(s) = \int_0^s \phi_1(t)\,dt = \frac{|s|^p}{p\log(1+|s|)} + \frac{1}{p}\int_0^{|s|} \frac{t^p}{(1+t)(\ln(1+t))^2}\,dt.$$

If we take $\alpha=p-1$ and $\beta=1$ in [7, Example III], then we obtain

$$p_{\Phi_1} = \inf_{s>0}\frac{s\phi_1(s)}{\Phi_1(s)} = p-1 \quad\text{and}\quad q_{\Phi_1} = \sup_{s>0}\frac{s\phi_1(s)}{\Phi_1(s)} = p.$$

By Lemma 2.1, Φ_1 satisfies a Δ_2-condition. Since $p>2$ estimate (2.2) is also fulfilled (i.e. the complementary N-function $\overline{\Phi_1}$ satisfies a Δ_2-condition). On the other hand, the choice $r=p-1$ and L'Hôpital's rule yield

$$\liminf_{s\to0^+}\frac{\phi_1(s)}{s^{r-1}} = \limsup_{s\to0^+}\frac{\phi_1(s)}{s^{r-1}} = \lim_{s\to0^+}\frac{\phi_1(s)}{s^{r-1}} = \lim_{s\to0^+}\frac{s}{\log(1+s)} = 1.$$

Conditions (2.5) are met in this case and hence Theorem 4.4 implies $\mathscr{D}^{1,\Phi_1}(\mathbb{R}^N)=\mathscr{D}_{\mathrm{o}}^{1,\Phi_1}(\mathbb{R}^N)$.

Example 4.6. Consider the function $\phi_2(s) = |s|^{p-2}s\, \log(1+\mu+|s|)$ where $p>1$ and $\mu>0$ is a parameter. A simple calculation shows that

$$\Phi_2(s) = \int_0^s \phi_2(t)\, dt = \frac{|s|^p}{p}\log(1+\mu+|s|) - \frac{1}{p}\int_0^{|s|} \frac{t^p}{1+\mu+t}\, dt.$$

For values $s>0$ we consider the differentiable function

$$g_\mu(s) = \frac{\int_0^s \frac{t^p}{1+\mu+t} dt}{s^p \log(1+\mu+s)}.$$

A simple application of L'Hôpital's rule proves that $g_\mu(s) \to 0$ as $s \to 0$ and also $g_\mu(s) \to 0$ as $s \to +\infty$. Since

$$s^p \log(1+\mu+s) = p\int_0^s t^{p-1}\log(1+\mu+t)dt + \int_0^s \frac{t^p}{1+\mu+t}dt$$

it is evident that $0 < g_\mu(s) < 1$ if $s>0$. It follows that

$$\frac{s\phi_2(s)}{\Phi_2(s)} = \frac{p}{1-g_\mu(s)} \geq \lim_{s\to 0^+} \frac{s\phi_2(s)}{\Phi_2(s)} = p$$

for all $s>0$. Therefore

$$p_{\Phi_2} = \inf_{s>0} \frac{s\phi_2(s)}{\Phi_2(s)} = \lim_{s\to 0^+}\frac{s\phi_2(s)}{\Phi_2(s)} = p. \tag{4.10}$$

On the other hand, the implicit function theorem allows to determine a local maximum of g_μ at $s=s^*>0$ from the equation

$$s^{p+1}\log(1+\mu+s) = \Big(\int_0^s \frac{t^p}{1+\mu+t}dt\Big)\Big(p(1+\mu+s)\log(1+\mu+s)+s\Big).$$

The condition $g_\mu(s) \to 0$ as $s \to +\infty$ ensures that s^* is also global. Therefore,

$$q_{\Phi_2} = \sup_{s>0}\frac{s\phi_2(s)}{\Phi_2(s)} = \max_{s>0}\frac{s\phi_2(s)}{\Phi_2(s)} = \frac{p}{1-g_\mu(s^*)} < +\infty.$$

By Lemma 2.1, Φ_2 satisfies a Δ_2-condition. Bound (4.10) implies that estimate (2.2) is also fulfilled in this case (i.e. $\overline{\Phi_2}$ satisfies a Δ_2-condition). Furthermore, if we choose $r=p$ then

$$0 < \liminf_{s\to 0^+}\frac{\phi_2(s)}{s^{r-1}} = \limsup_{s\to 0^+}\frac{\phi_2(s)}{s^{r-1}} = \lim_{s\to 0^+}\frac{\phi_2(s)}{s^{r-1}} = \log(1+\mu) < +\infty.$$

Hence conditions (2.5) are fulfilled. Theorem 4.4 yields $\mathscr{D}^{1,\Phi_2}(\mathbb{R}^N) = \mathscr{D}^{1,\Phi_2}_{\mathrm{o}}(\mathbb{R}^N)$.

Example 4.7. This example proves that there exists an N-function Φ for which the corresponding Orlicz-Sobolev space $W^1L_\Phi(\mathbb{R}^N)$ is in general a proper subset of $\mathscr{D}^{1,\Phi}(\mathbb{R}^N)$. Consider $p>1$ and set the real homeomorphism $\phi(t) = |t|^{p-2}t$. Let us define a function

$$u(x) = (1+\|x\|^2)^{-s}$$

where $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^N$ and s is a positive quantity to be fixed later. It is easy to see that

$$|\nabla u(x)| = \frac{2s\|x\|}{(1+\|x\|^2)^{s+1}}.$$

We take spherical coordinates $\mathbf{F} : (x_1, \dots, x_N) \to (\rho, \varphi_1, \dots, \varphi_{N-1})$ in $\mathbb{R}^N$ defined by

$$\begin{gathered} x_1 = \rho \cos \varphi_1 \\ x_i = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{i-1} \cos \varphi_i, \quad i = 2, \dots, N-1 \\ x_N = \rho \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{N-2} \sin \varphi_{N-1} \end{gathered}$$

where $\rho = (x_1^2 + \dots + x_N^2)^{1/2}$ and $\varphi_i \in [0, \pi]$ for $i = 1, \dots, N-2$ and $\varphi_{N-1} \in [0, 2\pi]$. A simple computation yields the Jacobian:

$$\begin{aligned} \mathbf{J_F}(\rho, \varphi_1, \dots, \varphi_{N-1}) &= \frac{\partial(x_1, x_2, \dots, x_N)}{\partial(\rho, \varphi_1, \dots, \varphi_{N-1})} \\ &= \rho^{N-1} (\sin \varphi_1)^{N-2} (\sin \varphi_2)^{N-3} \dots (\sin \varphi_{N-3})^2 \sin \varphi_{N-2}. \end{aligned}$$

Let us define the integral

$$I := \int_{\mathbb{R}^N \setminus B_1(0)} \frac{dx}{(1 + \|x\|^2)^{sr}}$$

where $1 < r < N$. (Obviously, $u^r \in L^1(\mathbb{R}^N)$ if and only if I is finite). Change to spherical coordinates and further integration yields

$$\begin{aligned} I &= \int_1^{+\infty} \int_0^{2\pi} \int_0^{\pi} \dots \int_0^{\pi} \frac{\mathbf{J_F}(\rho, \varphi_1, \dots, \varphi_{N-1})}{(1 + \rho^2)^{sr}} \, d\varphi_1 \dots d\varphi_{N-2} \, d\varphi_{N-1} \, d\rho \\ &= \mathscr{C} \int_1^{+\infty} \frac{\rho^{N-1}}{(1 + \rho^2)^{sr}} \, d\rho \end{aligned}$$

where $\mathscr{C}$ depends on $\int_0^{\pi} \sin^k \varphi_{N-k-1} \, d\varphi_{N-k-1}$, for all index $k = 1, \dots, N-2$. The limit comparison test for improper integrals yields

$$\int_1^{+\infty} \frac{\rho^{N-1}}{(1 + \rho^2)^{sr}} \, d\rho < +\infty$$

if and only if $N < 2sr$. If we set $r = p$ in the latter inequality, we obtain that convergence of the integral is equivalent to the condition $s > N/2p$. Thus if $s \le N/2p$ we get $u \notin L^p(\mathbb{R}^N)$. Likewise, in the particular case $r = p^* = Np/(N-p)$, convergence of the integral means $s > (N-p)/2p$. Therefore,

$$u \notin L^p(\mathbb{R}^N) \text{ and } u \in L^{p^*}(\mathbb{R}^N) \text{ if and only if } s \in \Big(\frac{N-p}{2p}, \frac{N}{2p} \Big].$$

The same argument we employed above proves that

$$J := \int_{\mathbb{R}^N \setminus B_1(0)} |\nabla u|^p \, dx = (2s)^p \mathscr{C} \int_1^{+\infty} \frac{\rho^{N+p-1}}{(1 + \rho^2)^{(s+1)p}} \, d\rho.$$

Hence, the integral J is finite if and only if $N + p - 2sp - 2p < 0$. That is,

$$|\nabla u| \in L^p(\mathbb{R}^N) \text{ if and only if } s \in \Big(\frac{N-p}{2p}, +\infty \Big).$$

We conclude that $u \in \mathscr{D}^{1,\Phi}(\mathbb{R}^N)$ and $u \notin W^1 L_\Phi(\mathbb{R}^N)$ (with $\Phi(t) = |t|^p/p$) provided the parameter $s \in \big((N-p)/2p, N/2p \big]$.

5. Application

In this section the number p_Φ defined in (2.2) plays a paramount role. We prove existence of nontrivial and non-negative solutions of equation (1.4) under the assumptions made at the beginning of Section 4. Additionally we will require the following hypotheses:

(H0) Condition (2.2) is fulfilled (i.e. $\overline{\Phi}$ satisfies a Δ_2-condition);
(H1) $q_\Phi < N$ and $q_\Phi < p_\Phi^* = p_\Phi\, N/(N-p_\Phi)$ (the conjugate exponent);
(H2) $g \in L^{q_\Phi^*/(q_\Phi^*-p_\Phi)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and the positive part $g^+ \not\equiv 0$.

We define functionals

$$I(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx \quad \text{and} \quad G(u) = \int_{\mathbb{R}^N} g(x)\Phi(u)dx.$$

Since Φ satisfies a global Δ_2-condition, the functional I is well-defined on $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ and real-valued there. Further, [10, Lemma A.3] ensures that I is of class C^1 with Fréchet derivative

$$I'(u)(v) = \int_{\mathbb{R}^N} \phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot \nabla v dx.$$

Application of the same lemma (with the term $f(x,t) = g(x)\phi(t)$ in (1.2)) shows that G is real-valued on $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ and that $G : \mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N) \to \mathbb{R}$ is of class C^1 as well with Fréchet derivative

$$G'(u)(v) = \int_{\mathbb{R}^N} g(x)\phi(u)v dx$$

where $u, v \in \mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$.

Proposition 5.1. *Let $\{u_n\}$ be a sequence in $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ (weak convergence). Then there exists a subsequence denoted again by $\{u_n\}$ such that $G(u_n) \to G(u)$.*

Proof. By definition there exists $d' > 0$ such that $\|u_n\|_{\Phi_*} \le \mathscr{C}(N)\, \|u_n\|_{\mathrm{o},\Phi} \le d'$ for all $n \in \mathbb{N}$, where $\mathscr{C}(N)$ is the constant in (4.4). Choose $R > 0$ and let B_R be a ball of radius R centered at 0. For each natural number n we have $G(u_n) - G(u) = I_n^R + J_n^R$, where

$$I_n^R = \int_{B_R} g(x)\left(\Phi(u_n) - \Phi(u)\right)dx, \quad J_n^R = \int_{\mathbb{R}^N\setminus B_R} g(x)\left(\Phi(u_n) - \Phi(u)\right)dx.$$

Let us define $A_{R,n} = \{x \in \mathbb{R}^N\setminus B_R : 0 \le u_n(x) \le 1\}$ and $C_{R,n} = \{x \in \mathbb{R}^N\setminus B_R : u_n(x) \ge 1\}$. Let $\sigma = q_\Phi^*/(q_\Phi^* - p_\Phi)$. Items (a) and (b) in Proposition 2.3 applied with $\rho = u_n \in A_{R,n}$ and $t = 1$ yield

$$|\Phi(u_n)|^{q_\Phi^*/p_\Phi} \le |u_n^{p_\Phi}\Phi(1)|^{q_\Phi^*/p_\Phi} = |u_n|^{q_\Phi^*}(\Phi(1))^{q_\Phi^*/p_\Phi} \le \frac{(\Phi(1))^{q_\Phi^*/p_\Phi}}{\Phi_*(1)}\Phi_*(u_n).$$

Hence Holder's inequality produces

$$\begin{aligned}\int_{A_{R,n}} |g\Phi(u_n)|\,dx &\le \Phi(1)\Big(\int_{A_{R,n}} |g|^\sigma\,dx\Big)^{1/\sigma}\Big(\int_{A_{R,n}} |u_n|^{q_\Phi^*}\,dx\Big)^{p_\Phi/q_\Phi^*}\\ &\le C_1\Big(\int_{\mathbb{R}^N\setminus B_R} |g|^\sigma\,dx\Big)^{1/\sigma}\Big(\int_{\mathbb{R}^N} \Phi_*(u_n)\,dx\Big)^{p_\Phi/q_\Phi^*}\end{aligned}$$

where $C_1 = \Phi(1)/(\Phi_*(1))^{p_\Phi/q^*_\Phi}$. Since $\sigma \le p^*_\Phi/(p^*_\Phi - q_\Phi)$ by interpolation we have $g \in L^{p^*_\Phi/(p^*_\Phi - q_\Phi)}(\mathbb{R}^N)$ as well. If $u \in C_{R,n}$ then analogue arguments as the ones used above yield

$$\int_{C_{R,n}} |g\Phi(u_n)|\,dx \le C_2\Big(\int_{\mathbb{R}^N\setminus B_R} |g|^{\sigma^*}\,dx\Big)^{1/\sigma^*}\Big(\int_{\mathbb{R}^N} \Phi_*(u_n)\,dx\Big)^{q_\Phi/p^*_\Phi}$$

where $\sigma^* = p^*_\Phi/(p^*_\Phi - q_\Phi)$ and $C_2 > 0$. Since $\|u_n\|_{\Phi_*} \le d'$ the integral $\int_{\mathbb{R}^N} \Phi_*(u_n)\,dx$ is bounded and then the two inequalities above imply

$$\int_{\mathbb{R}^N} |g\Phi(u_n)|\,dx < +\infty.$$

Thus, given $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_0}} |g\Phi(u_n)|\,dx < \varepsilon/4.$$

One can similarly prove that

$$\int_{\mathbb{R}^N\setminus B_{R_1}} |g\Phi(u)|\,dx < \varepsilon/4$$

for R_1 large enough. Thus, if $R_2 = \max\{R_0, R_1\}$ then we have $|J_n^{R_2}| < \varepsilon/2$ for $n \in \mathbb{N}$.

Let us study now $I_n^{R_2}$. Since the injection $L_{\Phi^*}(B_{R_2}) \hookrightarrow L_\Phi(B_{R_2})$ is continuous (see [1, Theorem 8.16]) the inclusions (4.5) yield $u_n, u \in W^1L_\Phi(B_{R_2})$ and hence there exist $d, \tilde{d} > 0$ such that

$$|||u_n|||_{B_{R_2}} \le d\|u_n\|_{\mathrm{o},\Phi} \le \tilde{d}$$

for all $n \in \mathbb{N}$ where $|||\cdot|||_{B_{R_2}}$ is the norm (3.2) on the ball B_{R_2}. Since the imbedding $W^1L_\Phi(B_{R_2}) \hookrightarrow L_\Phi(B_{R_2})$ is compact (see [11, Theorem 2.2]) we have $u_n \to u$ in $L_\Phi(B_{R_2})$. Thus, passing to a subsequence (denoted by $\{u_n\}$ again) we can further assume that $u_n \to u$, a.e. in B_{R_2} and that there exists $w \in L_\Phi(B_{R_2})$ such that $|u_n| \le w$, a.e. in B_{R_2}, for all $n \in \mathbb{N}$. By Lebesgue's dominated convergence on B_{R_2},

$$\lim_{n\to+\infty} \int_{B_{R_2}} |\Phi(u_n) - \Phi(u)|\,dx = 0.$$

Thus, for n sufficiently large, $|I_n^{R_2}| \le \|g\|_\infty \|\Phi(u_n) - \Phi(u)\|_{L^1(B_{R_2})} \le \varepsilon/2$. Since $|G(u_n) - G(u)| \le |I_n^{R_2}| + |J_n^{R_2}|$ the result is proved. $\square$

Lemma 5.2 (Lagrange multipliers rule [6])**.** *Let $v_0 \in \mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N)$ be such that $G'(v_0) \ne 0$. If I has a local minimum at v_0 with respect to the set $\{v : G(v) = G(v_0)\}$ then there exists $\lambda \in \mathbb{R}$ such that $I'(v_0) = \lambda G'(v_0)$.*

Lagrange multipliers rule motivates the following definition. A pair $(\lambda, u) \in \mathbb{R} \times \mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N)$ is a solution of (1.4) if $\phi(|\nabla u|) \in L_{\overline{\Phi}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla\theta\,dx = \lambda\int_{\mathbb{R}^N} g(x)\phi(u)\,\theta\,dx$$

for all $\theta \in \mathscr{D}^{1,\Phi}_{\mathrm{o}}(\mathbb{R}^N)$. If (λ, u) is a solution of (1.4) and $u \not\equiv 0$ we call λ an eigenvalue of (1.4) with corresponding eigenfunction u. That is, λ is the eigenvalue associated to the eigenfunction u. Note that the inclusion on the right in (4.5) ensures that any solution u belongs to $L_{\Phi_*}(\mathbb{R}^N)$ and $|\nabla u| \in L_\Phi(\mathbb{R}^N)$.

Theorem 5.3. *The optimization problem*

$$\inf_{G(u)=\mu>0} I(u)$$

has a nontrivial solution $u_\mu \in \mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$. *Define the nonzero number*

$$\lambda_\mu = \frac{\int_{\mathbb{R}^N} \phi(|\nabla u_\mu|)|\nabla u_\mu| dx}{\int_{\mathbb{R}^N} g(x)\phi(u_\mu)u_\mu \, dx}. \tag{5.1}$$

Then u_μ *is a non-negative eigenfunction of equation* (1.4) *with associated eigenvalue* $\lambda = \lambda_\mu$.

Proof. The first part is motivated by the ideas in the proof of [16, Theorem 3.1]. Compare also with the proof of [19, Theorem 2.2]. We prove that for any $\mu > 0$, the set $\mathcal{M}_\mu = \{u \in \mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N) : G(u) = \mu\}$ is not empty. Since $G(0) = 0$, by continuity of G, it will be sufficient to find $\overline{u} \in \mathcal{D}(\mathbb{R}^N)$ such that $G(\overline{u}) \geq \mu$.

Since $g^+ \not\equiv 0$ in $\mathbb{R}^N$ there exists a compact subset K of $\mathbb{R}^N$, with $\text{meas}(K) > 0$, such that $g > 0$ on K. If $r \in \mathbb{R}$ we define $u_r(x) = r\chi_K(x)$ where $\chi_K : \mathbb{R}^N \to \mathbb{R}$ is the characteristic function

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \in K^c. \end{cases}$$

We choose $r_0 > 0$ such that the number $\mu_0 = G(u_{r_0}) - \mu = \Phi(r_0) \int_K g \, dx - \mu$ be strictly positive. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain such that $K \subset \Omega$. Since the function

$$u \in L_\Phi(\Omega) \mapsto \Phi(u) \in L^1(\Omega)$$

is continuous, we have that $\Phi(u_\varepsilon)$ converges to $\Phi(r_0\chi_K)$ in $L^1(\Omega)$, as $\varepsilon \to 0^+$ where $u_\varepsilon \in \mathcal{D}(\Omega)$ is the regularized function of $r_0\chi_K$ and $\mathcal{D}(\Omega)$ denotes the space of C^∞-functions with compact support in Ω. Hölder's inequality yields $G(u_\varepsilon) \to \mu + \mu_0$ and hence we can choose ε_0 sufficiently small such that $G(\overline{u}) = G(u_{\varepsilon_0}) \geq \mu$.

Denote by $\beta = \inf_{\mathcal{M}_\mu} I$ and let $\{u_n\}$ be a sequence in $\mathcal{M}_\mu$ such that

$$\lim_{n\to+\infty} I(u_n) = \beta.$$

Hence, there exists $\mathcal{C} > 1$ such that for each $n \in \mathbb{N}$,

$$I(u_n) = \int_{\mathbb{R}^N} \Phi(|\nabla u_n|) dx \leq \mathcal{C}.$$

Since $\Phi(u/t) \leq \Phi(u)/t$ for $t \geq 1$ (convexity), we get

$$\int_{\mathbb{R}^N} \Phi\Big(\frac{|\nabla u_n|}{\mathcal{C}}\Big) dx \leq \int_{\mathbb{R}^N} \frac{\Phi(|\nabla u_n|)}{\mathcal{C}} dx \leq 1$$

and by definition of the Luxemburg norm, $\|u_n\|_{{\rm o},\Phi} = \||\nabla u_n|\|_\Phi \leq \mathcal{C}$. That is, the minimizing sequence $\{u_n\}$ is bounded in $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$. Inclusions (4.5) imply that this space is a closed subspace of $L_{\Phi_*}(\mathbb{R}^N)$. Corollary 3.2 proves that $\mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$ is itself reflexive. Then there exists $u_\mu \in \mathscr{D}_{\rm o}^{1,\Phi}(\mathbb{R}^N)$ and a subsequence in $\mathcal{M}_\mu$, denoted again by $\{u_n\}$, such that $u_n \rightharpoonup u_\mu$ in the weak topology. As the function G is sequentially continuous with respect to this weak topology, Proposition 5.1 yields

$$G(u_\mu) = \lim_{n\to+\infty} G(u_n) = \mu$$

and hence $u_\mu \in \mathcal{M}_\mu$. Since the convex functional I is continuously Fréchet-differentiable on $\mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$ we obtain by [4, Corollary III.8],

$$\beta \le I(u_\mu) \le \liminf_{n\to+\infty} I(u_n) = \beta$$

which is what we wanted to prove.

On the other hand, as $|g\Phi(u_\mu)| \le |g\phi(u_\mu)u_\mu|$ and $\mu \ne 0$, we obtain both $g\phi(u_\mu)u_\mu \not\equiv 0$ and $g\phi(u_\mu) \not\equiv 0$ in $\mathbb{R}^N$. The latter implies that there exists $K' \subseteq \mathbb{R}^N$, with $\mathrm{meas}(K') > 0$, such that $g\phi(u_\mu) \ne 0$ on K' and the sign of $g\phi(u_\mu)$ on K' is constant. Thus, for a suitable $r \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} g(x)\phi(u_\mu) r\chi_{K'}\,dx > \int_{\mathbb{R}^N} g(x)\phi(u_\mu)u_\mu\,dx$$

where $\chi_{K'}$ is the characteristic function on K'. Since $g\phi(u_\mu) \in L_{\overline{\Phi}_*}(\mathbb{R}^N)$ and as the regularized function $(r\,\chi_{K'})_\varepsilon \in \mathcal{D}(\mathbb{R}^N)$ converges to $r\chi_{K'}$ in $L_{\Phi_*}(\mathbb{R}^N)$,

$$G'(u_\mu)(u_1) = \int_{\mathbb{R}^N} g(x)\phi(u_\mu)u_1\,dx > \int_{\mathbb{R}^N} g(x)\phi(u_\mu)u_\mu\,dx = G'(u_\mu)(u_\mu)$$

where $u_1 = (r\,\chi_{K'})_\varepsilon$ for $\varepsilon > 0$ sufficiently small. Notice that $G'(u_\mu) \not\equiv 0$ (otherwise, $0 > G'(u_\mu)(u_\mu) = 0$ in the above strict inequality). By Lemma 5.2 there exists $\lambda_u \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \phi(|\nabla u_\mu|)\frac{\nabla u_\mu}{|\nabla u_\mu|}\cdot \nabla u\,dx = \lambda_\mu \int_{\mathbb{R}^N} g(x)\phi(u_\mu)\,u\,dx \tag{5.2}$$

for all $u \in \mathscr{D}_{\mathrm{o}}^{1,\Phi}(\mathbb{R}^N)$. Thus, u_μ is a weak solution of (1.4). We then set $u = u_\mu$ in (5.2) and we obtain the value of the eigenvalue in (5.1).

Since Φ is even it is clear that $G(|u_\mu|) = G(u_\mu)$. Moreover, the chain rule implies $|\nabla|u_\mu|| = |\nabla u_\mu|$ and hence the equivalence $I(|u_\mu|) = I(u_\mu)$ follows as well. Therefore, we can take $u_\mu(x) \ge 0$ for a.e. $x \in \mathbb{R}^N$. The proof of the theorem is complete. □

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