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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR CAPUTO-HADAMARD SEQUENTIAL FRACTIONAL ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the existence and uniqueness of solutions for Hadamard-type sequential fractional order neutral functional differential equations. The Banach fixed point theorem, a nonlinear alternative of Leray-Schauder type and Krasnoselski fixed point theorem are used to obtain the desired results. Examples illustrating the main results are presented. An initial value integral condition case is also discussed.

1. Introduction

This work is concerned with the existence and uniqueness of solutions to the following initial value problem (IVP) of Caputo-Hadamard sequential fractional order neutral functional differential equations

$$D^{\alpha}[D^{\beta}y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J := [1, b], \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [1 - r, 1], \tag{1.2}$$

$$D^{\beta}y(1) = \eta \in \mathbb{R}, \tag{1.3}$$

where D^{α}, D^{β} are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$, $f, g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions and $\phi \in C([1 - r, 1], \mathbb{R})$. For any function y defined on $[1 - r, b]$ and any $t \in J$, we denote by y_t the element of $C_r := C([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Functional differential equations are found to be of central importance in many disciplines such as control theory, neural networks, epidemiology, etc. [17]. In analyzing the behavior of real populations, delay differential equations are regarded as effective tools. Since the delay terms can be finite as well as infinite in nature, one needs to study these two cases independently. Moreover, the delay terms may appear in the derivatives involved in the given equation. As it is difficult to formulate such a problem, an alternative approach is followed by considering neutral functional differential equations. On the other hand, fractional derivatives are capable to describe hereditary and memory effects in many processes and materials. So the study of neutral functional differential equations in presence of fractional

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derivatives constitutes an important area of research. For more details, see the text [27].

In recent years, there has been a significant development in fractional calculus, and initial and boundary value problems of fractional differential equations, see the monographs of Kilbas *et al.* [19], Lakshmikantham *et al.* [21], Miller and Ross [22], Podlubny [23], Samko *et al.* [24], Diethelm [11] and a series of papers [1, 2, 3, 4, 10, 12, 13, 18, 25, 26] and the references therein. One can notice that much of the work on the topic involves Riemann-Liouville and Caputo type fractional derivatives. Besides these derivatives, there is an other fractional derivative introduced by Hadamard in 1892 [16], which is known as Hadamard derivative and differs from aforementioned derivatives in the sense that the kernel of the integral in its definition contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [7, 8, 9] and references cited therein.

In [6], the authors studied an initial value problem (IVP) for Riemman-Liouville type fractional functional and neutral functional differential equations with infinite delay. Recently, initial value problems for fractional order Hadamard-type functional and neutral functional differential equations and inclusions were respectively investigated in [3, 5], while an IVP for retarded functional Caputo type fractional impulsive differential equations with variable moments was discussed in [14].

In this paper, we investigate a new class of Hadamard-type sequential fractional neutral functional differential equations. Our study is based on fixed point theorems due to Banach and Krasnoselskii [20], and nonlinear alternative of Leray-Schauder type [15].

The rest of this paper is organized as follows: in Section 2 we recall some useful preliminaries. In Section 3 we discuss the existence and uniqueness of solutions for the problem (1.1)-(1.3), while the existence results for the problem are presented in Section 4. Examples are constructed in Section 5 for illustrating the obtained results. Finally, a generalization involving initial value integral condition is described in Section 6.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts that we need in the sequel.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Also C_r is endowed with norm

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

Definition 2.1 ([19])**.** The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\Big(t\frac{d}{dt}\Big)^n \int_1^t \Big(\log\frac{t}{s}\Big)^{n-q-1}\frac{g(s)}{s}ds, \quad n-1<q<n,\ n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 ([19])**.** The Hadamard fractional integral of order q for a function g is defined as
$$I^q g(t)=\frac{1}{\Gamma(q)}\int_1^t \big(\log\frac{t}{s}\big)^{q-1}\frac{g(s)}{s}ds,\quad q>0,$$
provided the integral exists.

Lemma 2.3. *The function $y\in C^2([1-r,b],\mathbb{R})$ is a solution of the problem*
$$\begin{gathered} D^\alpha[D^\beta y(t)-g(t,y_t)]=f(t,y_t),\quad t\in J:=[1,b],\\ y(t)=\phi(t),\quad t\in[1-r,1],\\ D^\beta y(1)=\eta\in\mathbb{R}, \end{gathered}\tag{2.1}$$
if and only if
$$y(t)=\begin{cases}\phi(t), & \text{if } t\in[1-r,1],\\ \phi(1)+(\eta-g(1,\phi(1)))\frac{(\log t)^\beta}{\Gamma(\beta+1)} & \\ +\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{g(s,y_s)}{s}ds & \\ +\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds, & \text{if } t\in[1,b].\end{cases}\tag{2.2}$$

Proof. The solution of Hadamard differential equation in (2.1) can be written as
$$D^\beta y(t)-g(t,y_t)=\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{f(s,y_s)}{s}ds+c_1,\tag{2.3}$$
where $c_1\in\mathbb{R}$ is arbitrary constant. Using the condition $D^\beta y(1)=\eta$ we find that $c_1=\eta-g(1,\phi(1))$. Then we obtain
$$\begin{aligned} y(t)&=(\eta-g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\beta-1}\frac{g(s,y_s)}{s}ds\\ &\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds+c_2. \end{aligned}$$
From the above equation we find $c_2=\phi(1)$ and (2.2) is proved. The converse follows by direct computation. □

3. Existence and uniqueness result

In this section, we establish the existence and uniqueness of a solution for the IVP (1.1)–(1.3).

Definition 3.1. A function $y\in C^2([1-r,b],\mathbb{R})$, is said to be a solution of (1.1)–(1.3) if y satisfies the equation $D^\alpha[D^\beta y(t)-g(t,y_t)]=f(t,y_t)$ on J, the condition $y(t)=\phi(t)$ on $[1-r,1]$ and $D^\beta y(1)=\eta$.

The next theorem gives us a uniqueness result using the assumptions

(A1) there exists $\ell>0$ such that
$$|f(t,u)-f(t,v)|\le\ell\|u-v\|_C,\quad \text{for } t\in J \text{ and every } u,v\in C_r;$$
(A2) there exists a nonnegative constant k such that
$$|g(t,u)-g(t,v)|\le k\|u-v\|_C,\quad \text{for } t\in J \text{ and every } u,v\in C_r.$$

Theorem 3.2. *Assume that* (A1), (A2) *hold. If*

$$\frac{k(\log b)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}<1, \tag{3.1}$$

then there exists a unique solution for IVP (1.1)–(1.3) *on the interval* $[1-r,b]$.

Proof. Consider the operator $N: C([1-r,b],\mathbb{R})\to C([1-r,b],\mathbb{R})$ defined by

$$N(y)(t)=\begin{cases}\phi(t), & \text{if } t\in[1-r,1],\\ \phi(1)+(\eta-g(1,\phi))\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} & \\ +\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{g(s,y_s)}{s}ds & \\ +\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds, & \text{if } t\in J.\end{cases} \tag{3.2}$$

To show that the operator N is a contraction, let $y,z\in C([1-r,b],\mathbb{R})$. Then we have

$$\begin{aligned}|N(y)(t)-N(z)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{|g(s,y_s)-g(s,z_s)|}{s}ds\\ &\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{|f(s,y_s)-f(s,z_s)|}{s}ds\\ &\le \frac{k}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{\|y_s-z_s\|_C}{s}ds\\ &\quad+\frac{\ell}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\|y_s-z_s\|_C\,ds\\ &\le \frac{k(\log t)^{\alpha}}{\Gamma(\alpha+1)}\|y-z\|_{[1-r,b]}+\frac{\ell(\log t)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|y-z\|_{[1-r,b]}.\end{aligned}$$

Consequently we obtain

$$\|N(y)-N(z)\|_{[1-r,b]}\le\Big[\frac{k(\log b)^{\alpha}}{\Gamma(\alpha+1)}+\frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big]\|y-z\|_{[1-r,b]},$$

which, in view of (3.1), implies that N is a contraction. Hence N has a unique fixed point by Banach's contraction principle. This, in turn, shows that problem (1.1)–(1.3) has a unique solution on $[1-r,b]$. $\square$

4. Existence results

In this section, we establish our existence results for the IVP (1.1)–(1.3). The first result is based on Leray-Schauder nonlinear alternative.

Lemma 4.1 (Nonlinear alternative for single valued maps [15])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

For the next theorem we need the following assumptions:

(A3) $f,g:J\times C_r\to\mathbb{R}$ are continuous functions;

(A4) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C(J,\mathbb{R}^+)$ such that
$$|f(t,u)| \le p(t)\psi(\|u\|_C) \text{for each } (t,u) \in J \times C_r;$$
(A5) there exist constants $d_1 < \Gamma(\alpha+1)(\log b)^{-\alpha}$ and $d_2 \ge 0$ such that
$$|g(t,u)| \le d_1\|u\|_C + d_2, \quad t \in J,\ u \in C_r.$$
(A6) there exists a constant $M > 0$ such that
$$\frac{\Big(1 - \frac{d_1(\log b)^\alpha}{\Gamma(\alpha+1)}\Big)M}{M_0 + \frac{d_2(\log b)^\alpha}{\Gamma(\alpha+1)} + \psi(M)\|p\|_\infty \frac{1}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}} > 1,$$
where
$$M_0 = \|\phi\|_C + [|\eta| + d_1\|\phi\|_C + d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}.$$

Theorem 4.2. *Under assumptions* (A3)–(A6) *hold, IVP* (1.1)–(1.3) *has at least one solution on* $[1-r,b]$.

Proof. We shall show that the operator $N : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by (3.2) is continuous and completely continuous.

Step 1: N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([1-r,b],\mathbb{R})$. Then
$$\begin{aligned}
&|N(y_n)(t) - N(y)(t)|\\
&\le \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1}\frac{|g(s,y_{ns}) - g(s,y_s)|}{s}ds\\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}|f(s,y_{ns}) - f(s,y_s)|\frac{ds}{s}\\
&\le \frac{1}{\Gamma(\alpha)}\int_1^b \big(\log\frac{t}{s}\big)^{\alpha-1}\sup_{s\in[1,b]}|g(s,y_{ns}) - g(s,y_s)|\frac{ds}{s}\\
&\quad + \frac{1}{\Gamma(\alpha+\beta)}\int_1^b \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\sup_{s\in[1,b]}|f(s,y_{ns}) - f(s,y_s)|\frac{ds}{s}\\
&\le \frac{\|g(\cdot,y_{n\cdot}) - g(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha)}\int_1^b \big(\log\frac{t}{s}\big)^{\alpha-1}\frac{ds}{s}\\
&\quad + \frac{\|f(\cdot,y_{n\cdot}) - f(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha+\beta)}\int_1^b \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{ds}{s}\\
&\le \frac{(\log b)^\alpha\|g(\cdot,y_{n\cdot}) - g(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha+1)}\\
&\quad + \frac{(\log b)^{\alpha+\beta}\|f(\cdot,y_{n\cdot}) - f(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$
Since f, g are continuous functions, we have
$$\begin{aligned}
&\|N(y_n) - N(y)\|_\infty\\
&\le \frac{(\log b)^\alpha\|g(\cdot,y_{n\cdot}) - g(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha+\beta}\|f(\cdot,y_{n\cdot}) - f(\cdot,y_\cdot)\|_\infty}{\Gamma(\alpha+\beta+1)} \to 0
\end{aligned}$$
as $n \to \infty$.

Step 2: N maps bounded sets into bounded sets in $C([1-r,b],\mathbb{R})$. Indeed, it is sufficient to show that for any $\theta>0$ there exists a positive constant $\tilde{\ell}$ such that for each $y\in B_\theta=\{y\in C([1-r,b],\mathbb{R}):\|y\|_\infty\le\theta\}$, we have $\|N(y)\|_\infty\le\tilde{\ell}$. By (A4) and (A5), for each $t\in J$, we have

$$
\begin{aligned}
|N(y)(t)| &\le \|\phi\|_C+[|\eta|+d_1\|\phi\|_C+d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}|g(s,y_s)|\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}|f(s,y_s)|\frac{ds}{s}\\
&\le \|\phi\|_C+[|\eta|+d_1\|\phi\|_C+d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}\\
&\quad+\frac{d_1\|y\|_{[1-r,b]}+d_2}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{ds}{s}\\
&\quad+\frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{ds}{s}\\
&\le \|\phi\|_C+[|\eta|+d_1\|\phi\|_C+d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}\\
&\quad+\frac{d_1\|y\|_{[1-r,b]}+d_2}{\Gamma(\alpha+1)}(\log b)^\alpha+\frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|N(y)\|_\infty &\le \|\phi\|_C+[|\eta|+d_1\|\phi\|_C+d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}\\
&\quad+\frac{d_1\theta+d_2}{\Gamma(\alpha+1)}(\log b)^\alpha+\frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}:=\tilde{\ell}.
\end{aligned}
$$

Step 3: N maps bounded sets into equicontinuous sets of $C([1-r,b],\mathbb{R})$. Let $t_1,t_2\in J$, $t_1<t_2$, B_θ be a bounded set of $C([1-r,b],\mathbb{R})$ as in Step 2, and let $y\in B_\theta$. Then

$$
\begin{aligned}
&|N(y)(t_2)-N(y)(t_1)|\\
&\le \frac{|\eta|+d_1\|\phi\|_C+d_2}{\Gamma(\beta+1)}[(\log t_2)^\beta-(\log t_1)^\beta]\\
&\quad+\Big|\frac{1}{\Gamma(\alpha)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}\Big]g(s,y_s)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}g(s,y_s)\frac{ds}{s}\Big|\\
&\quad+\Big|\frac{1}{\Gamma(\alpha+\beta)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha+\beta-1}\Big]f(s,y_s)\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}f(s,y_s)\frac{ds}{s}\Big|\\
&\le \frac{|\eta|+d_1\|\phi\|_C+d_2}{\Gamma(\beta+1)}[(\log t_2)^\beta-(\log t_1)^\beta]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{d_1\theta+d_2}{\Gamma(\alpha)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha-1}\Big]\frac{ds}{s}\\
&+\frac{d_1\theta+d_2}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha-1}\frac{ds}{s}\\
&+\frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta)}\int_1^{t_1}\Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}-\Big(\log\frac{t_1}{s}\Big)^{\alpha+\beta-1}\Big]\frac{ds}{s}\\
&+\frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1}\frac{ds}{s}\\
\le\;&\frac{|\eta|+d_1\|\phi\|_C+d_2}{\Gamma(\beta+1)}[(\log t_2)^\beta-(\log t_1)^\beta]\\
&+\frac{d_1\theta+d_2}{\Gamma(\alpha+1)}[\big|(\log t_2)^\alpha-(\log t_1)^\alpha\big|+|\log t_2/t_1|^\alpha]\\
&+\frac{\psi(\theta)\|p\|_\infty}{\Gamma(\alpha+\beta+1)}[\big|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\big|+|\log t_2/t_1|^{\alpha+\beta}].
\end{aligned}
$$

As $t_1\to t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1<t_2\le 0$ and $t_1\le 0\le t_2$ is obvious.

As a consequence of Steps 1 to 3, it follows by the Arzelá-Ascoli theorem that $N: C([1-r,b],\mathbb{R})\to C([1-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: We show that there exists an open set $U\subseteq C([1-r,b],\mathbb{R})$ with $y\neq\lambda N(y)$ for $\lambda\in(0,1)$ and $y\in\partial U$. Let $y\in C([1-r,b],\mathbb{R})$ and $y=\lambda N(y)$ for some $0<\lambda<1$. Then, for each $t\in J$, we have

$$
\begin{aligned}
y(t)&=\lambda\Big(\phi(1)+(\eta-g(1,\phi(1)))\frac{(\log t)^\beta}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{g(s,y_s)}{s}ds\\
&\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds\Big).
\end{aligned}
$$

By our assumptions, for each $t\in J$, we obtain

$$
\begin{aligned}
|y(t)|&\le\|\phi\|_C+[|\eta|+d_1\|\phi\|_C+d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}\\
&\quad+\frac{d_1\|y\|_{[1-r,b]}+d_2}{\Gamma(\alpha)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha-1}\frac{ds}{s}\\
&\quad+\frac{1}{\Gamma(\alpha+\beta)}\int_1^t\big(\log\frac{t}{s}\big)^{\alpha+\beta-1}p(s)\psi(\|y_s\|_C)\frac{ds}{s}\\
&\le\|\phi\|_C+[|\eta|+d_1\|\phi\|_C+d_2]\frac{(\log b)^\beta}{\Gamma(\beta+1)}+\frac{d_1\|y\|_{[1-r,b]}+d_2}{\Gamma(\alpha+1)}(\log b)^\alpha\\
&\quad+\frac{\|p\|_\infty\psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta},
\end{aligned}
$$

which can be expressed as

$$
\frac{\Big(1-\frac{d_1(\log b)^\alpha}{\Gamma(\alpha+1)}\Big)\|y\|_{[1-r,b]}}{M_0+\frac{d_2(\log b)^\alpha}{\Gamma(\alpha+1)}+\psi(\|y\|_{[1-r,b]})\|p\|_\infty\frac{1}{\Gamma(\alpha+\beta+1)}(\log b)^{\alpha+\beta}}\le 1.
$$

In view of (A6), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $N : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N y$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.1), we deduce that N has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.1)-(1.3). This completes the proof. $\square$

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 4.3 (Krasnoselskii's fixed point theorem [20])**.** *Let S be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + Bx \in S$ whenever $x, y \in S$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in S$ such that $z = Az + Bz$.*

Theorem 4.4. *Assume that* (A2) *and* (A3) *hold. In addition we assume that*

(A7) $|f(t,x)| \leq \mu(t)$, $|g(t,x)| \leq \nu(t)$, *for all* $(t,x) \in J \times \mathbb{R}$, *and* $\mu, \nu \in C(J,\mathbb{R}^+)$.

Then problem (1.1)-(1.3) *has at least one solution on* $[1-r,b]$, *provided*

$$\frac{k(\log b)^\alpha}{\Gamma(\alpha+1)} < 1. \tag{4.1}$$

Proof. We define the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ by

$$\mathcal{G}_1 y(t) = \begin{cases} 0, & \text{if } t \in [1-r,1], \\ (\eta - g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)} \\ +\frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1}\frac{g(s,y_s)}{s}ds, & \text{if } t \in J. \end{cases} \tag{4.2}$$

$$\mathcal{G}_2 y(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r,1], \\ \phi(1) + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,y_s)}{s}ds, & \text{if } t \in J. \end{cases} \tag{4.3}$$

Setting $\sup_{t\in[1,b]}\mu(t) = \|\mu\|_\infty, \sup_{t\in[1,b]}\nu(t) = \|\nu\|_\infty$ and choosing

$$\rho \geq \|\phi\|_C + [|\eta| + \|\nu\|_\infty]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{\|\nu\|(\log b)^\alpha}{\Gamma(\alpha+1)} + \|\mu\|_\infty\frac{(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \tag{4.4}$$

we consider $B_\rho = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_\infty \leq \rho\}$. For any $y, z \in B_\rho$, we have

$$\begin{aligned}
&|\mathcal{G}_1 y(t) + \mathcal{G}_2 z(t)| \\
&\leq \sup_{t\in[1,b]}\Big\{(\eta - g(1,\phi))\frac{(\log t)^\beta}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\alpha)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha-1}\frac{g(s,y_s)}{s}ds \\
&\quad + \phi(1) + \frac{1}{\Gamma(\alpha+\beta)}\int_1^t \big(\log\frac{t}{s}\big)^{\alpha+\beta-1}\frac{f(s,z_s)}{s}ds\Big\} \\
&\leq \|\phi\|_C + [|\eta| + \|\nu\|_\infty]\frac{(\log b)^\beta}{\Gamma(\beta+1)} + \frac{\|\nu\|(\log b)^\alpha}{\Gamma(\alpha+1)} + \|\mu\|_\infty\frac{(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \\
&\leq \rho.
\end{aligned}$$

This shows that $\mathcal{G}_1 y + \mathcal{G}_2 z \in B_\rho$. Using (4.1) it is easy to see that $\mathcal{G}_1$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{G}_2$ is continuous. Also, $\mathcal{G}_2$ is uniformly bounded on B_ρ as

$$\|\mathcal{G}_2 y\| \le \|\phi\|_C + \|\mu\|_\infty \frac{(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}.$$

Now we prove the compactness of the operator $\mathcal{G}_2$. We define

$$\bar{f} = \sup_{(t,y)\in[1,b]\times B_\rho} |f(t,y)| < \infty,$$

and consequently, for $t_1, t_2 \in [1,b]$, $t_1 < t_2$, we have

$$\begin{aligned}
&|\mathcal{G}_2 y(t_2) - \mathcal{G}_2 y(t_1)| \\
&\le \frac{\bar{f}}{\Gamma(\alpha+\beta)} \int_1^{t_1} \Big| \Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1} - \Big(\log\frac{t_1}{s}\Big)^{\alpha+\beta-1} \Big| \frac{ds}{s} \\
&\quad + \frac{\bar{f}}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\alpha+\beta-1} \frac{ds}{s} \\
&\le \frac{\bar{f}}{\Gamma(\alpha+\beta+1)} \big[\big|(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta}\big| + |\log t_2/t_1|^{\alpha+\beta}\big],
\end{aligned}$$

which is independent of y and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{G}_2$ is equicontinuous. So $\mathcal{G}_2$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{G}_2$ is compact on B_ρ. Thus all the assumptions of Lemma 4.3 are satisfied. So the conclusion of Lemma 4.3 implies that the problem (1.1)-(1.3) has at least one solution on $[1-r,b]$ □

5. Examples

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation,

$$D^{1/2}\Big[D^{3/4}y(t) - \frac{1+e^{-t}}{8+e^t}\frac{\|y_t\|_C}{(1+\|y_t\|_C)}\Big] = \frac{\|y_t\|_C}{2(1+\|y_t\|_C)} + e^{-t}, \quad t\in J := [1,e], \tag{5.1}$$

$$y(t) = \phi(t), \quad t \in [1-r,1], \tag{5.2}$$

$$D^{3/4}y(1) = 1/2. \tag{5.3}$$

Let

$$f(t,x) = \frac{x}{2(1+x)}, \quad g(t,x) = \frac{1+e^{-t}}{8+e^t}\Big(\frac{x}{1+x}\Big), \quad (t,x) \in [1,e]\times[0,\infty).$$

For $x, y \in [0,\infty)$ and $t \in J$, we have

$$|f(t,x) - f(t,y)| = \frac{1}{2}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| = \frac{|x-y|}{2(1+x)(1+y)} \le \frac{1}{2}|x-y|,$$

and

$$\begin{aligned}
|g(t,x) - g(t,y)| &= \frac{1+e^{-t}}{8+e^t}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| = \frac{1+e^{-t}}{8+e^t}\frac{|x-y|}{(1+x)(1+y)} \\
&\le \frac{e+1}{e(e+8)}|x-y|.
\end{aligned}$$

Hence conditions (A1) and (A2) hold with $\ell = 1/2$ and $k = \frac{e+1}{e(e+8)}$ respectively. Since $\frac{k(\log b)^\alpha}{\Gamma(\alpha+1)} + \frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \approx 0.5853088 < 1$, therefore, by Theorem 3.2, problem (5.1)-(5.3) has a unique solution on $[1-r, e]$.

Also $|f(t,x)| \le (1+2e^{-t})/2 = \mu(t)$, $|g(t,x)| \le (1+e^{-t})/(8+e^t) = \nu(t)$ and $k(\log b)^\alpha/\Gamma(\alpha+1) = 2(e+1)/\sqrt{\pi}e(e+8) \approx 0.144005 < 1$. Clearly the assumptions of Theorem 4.4 are satisfied. Consequently, by the conclusion of Theorem 4.4, there exists a solution of the problem (5.1)-(5.3) on $[1-r, e]$.

6. Initial value integral condition case

The results of this paper can be extended to the case of an initial value integral condition of the form

$$D^\beta y(1) = \int_1^b h(s, y_s)ds, \tag{6.1}$$

where $h : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ is a given function. In this case η will be replaced with $\int_1^b h(s,y_s)ds$ in (3.2) and the statement of the existence and uniqueness result for the problem (1.1)–(1.2)–(6.1) can be formulated as follows.

Theorem 6.1. *Assume that the conditions* (A1) *and* (A2) *hold. Further, we suppose that*

(A8) *there exists a nonnegative constant m such that*

$$|h(t,u) - h(t,v)| \le m\|u-v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C_r.$$

Then the problem (1.1)-(1.2)-(6.1) *has a unique solution on* $[1-r, b]$ *if*

$$\frac{m(b-1)(\log b)^\beta}{\Gamma(\beta+1)} + \frac{k(\log b)^\alpha}{\Gamma(\alpha+1)} + \frac{\ell(\log b)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} < 1.$$

We do not provide the proof of the above theorem as it is similar to that of Theorem 3.2.

The analog form of the existence results: Theorems 4.2 and 4.4 for the problem (1.1)-(1.2)-(6.1) can be constructed in a similar manner.

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