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GROWTH OF MEROMORPHIC SOLUTIONS TO HOMOGENEOUS AND NON-HOMOGENEOUS LINEAR (DIFFERENTIAL-)DIFFERENCE EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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Abstract. In this article, we study the growth of meromorphic solutions of homogeneous and non-homogeneous linear difference equations and linear differential-difference equations. When there exists only one coefficient having the maximal iterated order or having the maximal iterated type among those having the maximal iterated order, and the above coefficient satisfies certain conditions on its poles, we obtain estimates on the lower bound of the iterated order of the meromorphic solutions. The case $p = 1$ is also discussed and corresponding results are obtained by strengthening some conditions.

1. Introduction and statement of main results

Throughout this article, we use the standard notation and basic results of Nevanlinna's value distribution theory (see e.g. [7, 9, 17]). In addition, we use $\sigma(f)$, $\tau(f)$, $\lambda(1/f)$ to denote respectively the order, the type, and the exponent of convergence of the poles of a meromorphic function $f(z)$ in the complex plane. For $p \in N_+$, we introduce the definitions of the iterated order, the iterated type and the iterated exponent of convergence of the poles of $f(z)$ as follows:

$$\sigma_p(f) = \limsup_{r\to\infty} \frac{\log_p T(r,f)}{\log r}, \quad \tau_p(f) = \limsup_{r\to\infty} \frac{\log_{p-1} T(r,f)}{r^{\sigma_p(f)}},$$
$$\lambda_p(\frac{1}{f}) = \limsup_{r\to\infty} \frac{\log_p N(r,f)}{\log r}$$

(see e.g. [8, 15]). In particular, $\sigma_1(f) = \sigma(f)$, $\tau_1(f) = \tau(f)$, $\lambda_1(1/f) = \lambda(1/f)$.

Recently, the properties of meromorphic solutions of complex difference equations have become a subject of great interest from the viewpoint of Nevanlinna theory and its difference analogues. By this important tool, many scholars investigated the homogeneous linear difference equation

$$A_k(z)f(z+c_k) + \dots + A_1(z)f(z+c_1) + A_0(z)f(z) = 0 \tag{1.1}$$

and its special case

$$A_k(z)f(z+k) + \dots + A_1(z)f(z+1) + A_0(z)f(z) = 0, \tag{1.2}$$

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where $k \in N_+, c_i(i = 1, \dots, k)$ are distinct non-zero complex constants, and obtained many results on the growth and value distribution of meromorphic solutions of (1.1) or (1.2) (see e.g. [2, 3, 4, 10, 11, 12, 13, 14, 18]).

When the coefficients of (1.1) or (1.2) are entire functions of finite order, Chiang-Feng [4] and Laine-Yang [10] obtained the following two theorems, respectively.

Theorem 1.1 ([4]). *Let $A_j(z)$ $(j = 0, 1, \dots, k)$ be entire functions such that there exists an integer $l(0 \le l \le k)$ such that*

$$\sigma(A_l) > \max_{0 \le j \le k,\, j \ne l} \{\sigma(A_j)\}.$$

If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution to (1.2), then we have $\sigma(f) \ge \sigma(A_l) + 1$.

Theorem 1.2 ([10]). *Let $A_j(z)$ $(j = 0, 1, \dots, k)$ be entire functions of finite order such that among those having the maximal order $\sigma = \max_{0 \le j \le k}\{\sigma(A_j)\}$, exactly one has its type strictly greater than the others. Then for any meromorphic solution $f(z)$ $(\not\equiv 0)$ of (1.1), we have $\sigma(f) \ge \sigma + 1$.*

When there exists more than one coefficient having the infinite order among entire functions of (1.2), Liu-Mao [13] obtained the following theorem.

Theorem 1.3 ([13]). *Let $A_j(z)(j = 0, 1, \dots, k)$ be entire functions. If there exists an integer l $(0 \le l \le k)$ such that*

$$\max\{\sigma_2(A_j) : j = 0, 1, \dots, k, j \ne l\} \le \sigma_2(A_l) \quad (0 < \sigma_2(A_l) < \infty),$$
$$\max\{\tau_2(A_j) : \sigma_2(A_j) = \sigma_2(A_l), j = 0, 1, \dots, k, j \ne l\} < \tau_2(A_l) \quad (0 < \tau_2(A_l) < \infty),$$

then every meromorphic solution $f(z)(\not\equiv 0)$ of (1.2) satisfies $\sigma(f) = \infty$ and $\sigma_2(f) \ge \sigma_2(A_l)$.

Liu-Mao[13] considered the hyper-order of meromorphic solutions of the non-homogeneous linear difference equation

$$A_k(z)f(z + k) + \dots + A_1(z)f(z + 1) + A_0(z)f(z) = F(z), \tag{1.3}$$

where $k \in N_+$, and obtained the following theorem.

Theorem 1.4 ([13]). *Let $A_j(z)(j = 0, 1, \dots, k)$ satisfy the hypothesis of Theorem 1.3, and $F(z)$ $(\not\equiv 0)$ be an entire function.*

(i) If $\sigma_2(F) < \sigma_2(A_l)$, or $\sigma_2(F) = \sigma_2(A_l)$ and $\tau_2(F) < \tau_2(A_l)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of (1.3) satisfies $\sigma(f) = \infty$ and $\sigma_2(f) \ge \sigma_2(A_l)$.

(ii) If $\sigma_2(F) > \sigma_2(A_l)$, then every meromorphic solution $f(z)(\not\equiv 0)$ of (1.3) satisfies $\sigma(f) = \infty$ and $\sigma_2(f) \ge \sigma_2(F)$.

Note that in Theorems 1.1–1.4, the coefficients of (1.1)-(1.3) are entire functions. Naturally, a question arises: When the coefficients are meromorphic functions, the above conclusions hold yet? The main aim of our article is to answer the question for both the case of the homogeneous equation (1.1) and the case of the non-homogeneous equation

$$A_k(z)f(z + c_k) + \dots + A_1(z)f(z + c_1) + A_0(z)f(z) = F(z), \tag{1.4}$$

where $k \in N_+$, $c_i(i = 1, \dots, k)$ are distinct non-zero complex constants, and obtain the following results.

Firstly, we obtain the following Theorem 1.5 when $p \ge 2$.

Theorem 1.5. *Let $p \in N_+ \setminus \{1\}, A_j(z)(j = 0, 1, \dots, k)$ and $F(z)$ be meromorphic functions. If there exists an integer $l(0 \le l \le k)$ such that $A_l(z)$ satisfies*

$$\lambda_p(\frac{1}{A_l}) < \sigma_p(A_l) < \infty,$$
$$\max\{\sigma_p(A_j) : j = 0, 1, \dots, k, j \neq l\} \le \sigma_p(A_l),$$
$$\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_l), j = 0, 1, \dots, k, j \neq l\} < \tau_p(A_l) < \infty.$$

(i) If $\sigma_p(F) < \sigma_p(A_l)$, or $\sigma_p(F) = \sigma_p(A_l)$ and $\tau_p(F) \neq \tau_p(A_l)$, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of (1.4) satisfies $\sigma_p(f) \ge \sigma_p(A_l)$.

(ii) If $\sigma_p(F) > \sigma_p(A_l)$, then every meromorphic solution $f(z)$ of (1.4) satisfies $\sigma_p(f) \ge \sigma_p(F)$.

For $p = 1$, Latreuch-Belaïdi [11] considered the case of the homogeneous equation (1.2), and obtained the following theorem.

Theorem 1.6 ([11])**.** *Let $A_j(z)(j = 0, 1, \dots, k)$ be meromorphic functions such that $\lambda(\frac{1}{A_l}) < \sigma(A_l) = \sigma(0 < \sigma < \infty)$ and $\tau(A_l) = \tau(0 < \tau < \infty)$. Suppose that*

$$\max\{\sigma(A_j) : j = 0, 1, \dots, k, j \neq l\} \le \sigma \quad \text{and} \quad \sum_{\sigma(A_j)=\sigma,\, j\neq l} \tau(A_j) < \tau.$$

If $f(z)$ ($\not\equiv 0$) is a meromorphic solution of (1.2), then $\sigma(f) \ge \sigma(A_l) + 1$.

Further, we consider the case of the non-homogeneous equation (1.4), and obtain the following theorem.

Theorem 1.7. *Let $A_j(z)$ $(j = 0, 1, \dots, k)$ and $F(z)$ be meromorphic functions. If there exists an integer $l(0 \le l \le k)$ such that $A_l(z)$ satisfies*

$$\lambda(\frac{1}{A_l}) < \sigma(A_l) < \infty,$$
$$\max\{\sigma(A_j) : j = 0, 1, \dots, k, j \neq l\} \le \sigma(A_l),$$
$$\sum_{\sigma(A_j)=\sigma(A_l),\, j\neq l} \tau(A_j) < \tau(A_l) < \infty.$$

(i) If $\sigma(F) < \sigma(A_l)$, or $\sigma(F) = \sigma(A_l)$ and $\sum_{\sigma(A_j)=\sigma(A_l),\, j\neq l} \tau(A_j) + \tau(F) < \tau(A_l)$, or $\sigma(F) = \sigma(A_l)$ and $\sum_{\sigma(A_j)=\sigma(A_l)} \tau(A_j) < \tau(F)$, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of (1.4) satisfies $\sigma(f) \ge \sigma(A_l)$. Further, if $F(z) \equiv 0$, then $\sigma(f) \ge \sigma(A_l) + 1$.

(ii) If $\sigma(F) > \sigma(A_l)$, then every meromorphic solution $f(z)$ of (1.4) satisfies $\sigma(f) \ge \sigma(F)$.

Remark 1.8. From the proof of Theorem 1.5, we can see that the condition $\lambda_p(\frac{1}{A_l}) < \sigma_p(A_l)$ in Theorem 1.5 can be replaced by $\delta(\infty, A_l) > 0$; but from the proof of Theorem 1.7, we can see that the condition $\lambda(\frac{1}{A_l}) < \sigma(A_l)$ in Theorem 1.7 may not be replaced by $\delta(\infty, A_l) > 0$.

Next, on the base of complex linear difference equations (1.1)-(1.4), we proceed in this way by combining the reasoning methods from both complex differential equations and complex difference equations, that is, we study the more general complex linear differential-difference equations

$$\sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z + c_i) = 0, \tag{1.5}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i)=F(z), \tag{1.6}$$

where $n,m\in N_+, c_i(i=0,1,\dots,n)$ are distinct complex constants.

Wu-Zheng [16] investigated the growth of meromorphic solutions of the homogeneous linear differential-difference equation (1.5) and obtained the following theorem.

Theorem 1.9 ([16]). *Let $A_{ij}(z)$ $(i=0,1,\dots,n, j=0,1,\dots,m)$ be meromorphic functions such that there exists an integer $l(0\le l\le k)$ satisfying*

$$\max\{\sigma(A_{ij}),(i,j)\neq(l,0)\}<\sigma(A_{l0})<\infty \quad \text{and} \quad \delta(\infty,A_{l0})>0.$$

If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution of (1.5), then we have $\sigma(f)\ge\sigma(A_{l0})+1$.

Similar to Theorem 1.7, we consider the non-homogeneous linear differential-difference equations (1.6) and obtain the following theorem.

Theorem 1.10. *Let $A_{ij}(z)$ $(i=0,1,\dots,n, j=0,1,\dots m)$ and $F(z)$ be meromorphic functions. If there exists an integer $l(0\le l\le n)$ such that*

$$\max\{\sigma(A_{ij}),(i,j)\neq(l,0)\}<\sigma(A_{l0})<\infty \quad \text{and} \quad \delta(\infty,A_{l0})>0.$$

(i) If $\sigma(F)<\sigma(A_{l0})$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (1.6) satisfies $\sigma(f)\ge\sigma(A_{l0})$. Further, if $F(z)\equiv 0$, then $\sigma(f)\ge\sigma(A_{l0})+1$.

(ii) If $\sigma(F)>\sigma(A_{l0})$, then every meromorphic solution $f(z)$ of (1.6) satisfies $\sigma(f)\ge\sigma(F)$.

For the homogeneous or non-homogeneous linear differential-difference equation (1.6), we obtain the following theorem under some different conditions from Theorem 1.10.

Theorem 1.11. *Let $A_{ij}(z)$ $(i=0,1,\dots,n, j=0,1,\dots,m)$ and $F(z)$ be meromorphic functions. If there exists an integer $l(0\le l\le n)$ such that $A_{l0}(z)$ satisfies*

$$\lambda(\frac{1}{A_{l0}})<\sigma(A_{l0})<\infty,$$
$$\max\{\sigma(A_{ij}),(i,j)\neq(l,0)\}\le\sigma(A_{l0}),$$
$$\sum_{\sigma(A_{ij})=\sigma(A_{l0}),\,(i,j)\neq(l,0)}\tau(A_{ij})<\tau(A_{l0})<\infty.$$

(i) If $\sigma(F)<\sigma(A_{l0})$, or $\sigma(F)=\sigma(A_{l0})$ and $\sum_{\sigma(A_{ij})=\sigma(A_{l0}),\,(i,j)\neq(l,0)}\tau(A_{ij})+\tau(F)<\tau(A_{l0})$, or $\sigma(F)=\sigma(A_{l0})$ and $\sum_{\sigma(A_{ij})=\sigma(A_{l0})}\tau(A_{ij})<\tau(F)$, then every meromorphic solution $f(z)$ $(\not\equiv 0)$ of (1.6) satisfies $\sigma(f)\ge\sigma(A_{l0})$. Further, if $F(z)\equiv 0$, then $\sigma(f)\ge\sigma(A_{l0})+1$.

(ii) If $\sigma(F)>\sigma(A_{l0})$, then every meromorphic solution $f(z)$ of (1.6) satisfies $\sigma(f)\ge\sigma(F)$.

Further, we generalize Theorems 1.10 and 1.11 into the iterated case, and obtain the following theorem.

Theorem 1.12. *Let $p\in N_+\setminus\{1\}$, $A_{ij}(z)(i=0,1,\dots,n, j=0,1,\dots,m)$ and $F(z)$ be meromorphic functions. If there exists an integer $l(0\le l\le n)$ such that $A_{l0}(z)$ satisfies*

$$\lambda_p(\frac{1}{A_{l0}})<\sigma_p(A_{l0})<\infty,$$

$$\max\{\sigma_p(A_{ij}) : (i,j) \neq (l,0)\} \leq \sigma_p(A_{l0}),$$
$$\max\{\tau_p(A_{ij}) : \sigma_p(A_{ij}) = \sigma_p(A_{l0}), (i,j) \neq (l,0)\} < \tau_p(A_{l0}) < \infty.$$

(i) If $\sigma_p(F) < \sigma_p(A_{l0})$, or $\sigma_p(F) = \sigma_p(A_{l0})$ and $\tau_p(F) \neq \tau_p(A_{l0})$, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of (1.6) satisfies $\sigma_p(f) \geq \sigma_p(A_{l0})$.

(ii) If $\sigma_p(F) > \sigma_p(A_{l0})$, then every meromorphic solution $f(z)$ of (1.6) satisfies $\sigma_p(f) \geq \sigma_p(F)$.

Remark 1.13. From the proofs of Theorems 1.10 and 1.12, we can see respectively that the condition $\delta(\infty, A_{l0}) > 0$ in Theorem 1.10 can be replaced by $\lambda(\frac{1}{A_{l0}}) < \sigma(A_{l0})$, and that the condition $\lambda_p(\frac{1}{A_{l0}}) < \sigma_p(A_{l0})$ in Theorem 1.12 can be replaced by $\delta(\infty, A_{l0}) > 0$; but from the proof of Theorem 1.11, we can see that the condition $\lambda(\frac{1}{A_{l0}}) < \sigma(A_{l0})$ may not be replaced by $\delta(\infty, A_{l0}) > 0$.

2. Preliminary lemmas

Lemma 2.1 ([6])**.** *Let $f(z)$ be a non-constant meromorphic function, $c \in \mathbb{C}, \delta < 1$, and $\varepsilon > 0$, then*
$$m(r, \frac{f(z+c)}{f(z)}) = o(\frac{T(r+|c|, f)^{1+\varepsilon}}{r^\delta})$$
for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

Remark 2.2 ([5])**.** Let $f(z)$ be a meromorphic function, c be a non-zero complex constant, then we have that for $r \to \infty$,
$$(1+o(1))T(r-|c|, f) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f).$$
It follows that for $p \in N_+$, $\sigma_p(f(z+c)) = \sigma_p(f), \mu_p(f(z+c)) = \mu_p(f)$.

Lemma 2.1 and Remark 2.2 result in the following lemma.

Lemma 2.3 ([6])**.** *Let $f(z)$ be a non-constant meromorphic function, $c, h \in \mathbb{C}, c \neq h, \delta < 1, \varepsilon > 0$, then*
$$m(r, \frac{f(z+c)}{f(z+h)}) = o(\frac{T(r+|c-h|+|h|, f)^{1+\varepsilon}}{r^\delta})$$
for all r outside of a possible exceptional set E with finite logarithmic measure $\int_E \frac{dr}{r} < \infty$.

Lemma 2.4 ([1])**.** *Let $f(z)$ be a meromorphic function with $0 < \sigma_p(f) < \infty$ and $0 < \tau_p(f) < \infty$, then for any given $\beta < \tau_p(f)$, there exists a subset E of $[1, +\infty)$ that has infinite logarithmic measure such that $\log_{p-1} T(r, f) > \beta r^{\sigma_p(f)}$ holds for all $r \in E$.*

Lemma 2.5 ([4])**.** *Let c_1, c_2 be two complex numbers such that $c_1 \neq c_2$ and let $f(z)$ be a finite order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*
$$m(r, \frac{f(z+c_1)}{f(z+c_2)}) = O(r^{\sigma-1+\varepsilon}).$$

3. Proofs of main results

Proof of Theorem 1.5. (i) We divide (1.4) by $f(z+c_l)$ to get

$$-A_l(z)=\sum_{j=0,\,j\neq l}^{k}A_j(z)\frac{f(z+c_j)}{f(z+c_l)}-\frac{F(z)}{f(z+c_l)}. \tag{3.1}$$

It follows from (3.1), Lemma 2.3 and Remark 2.2 that for any given $\varepsilon>0$, we have

$$\begin{aligned}
T(r,A_l)&=m(r,A_l)+N(r,A_l)\\
&\le\sum_{j=0,\,j\neq l}^{k}m(r,A_j)+\sum_{j=0,\,j\neq l}^{k}m(r,\frac{f(z+c_j)}{f(z+c_l)})\\
&\quad+m(r,F)+m(r,\frac{1}{f(z+c_l)})+N(r,A_l)+O(1)\\
&\le\sum_{j=0,\,j\neq l}^{k}T(r,A_j)+o(T(r+3c),f)^{1+\varepsilon})\\
&\quad+T(r,F)+(1+o(1))T(r+|c_l|,f)+N(r,A_l)+O(1),
\end{aligned}\tag{3.2}$$

where $c=\max_{1\le j\le k}\{|c_j|\}, r\notin E_1, m_lE_1<\infty, r\to\infty$.

It follows from Lemma 2.4 that for the above ε, there exists a subset E_2 with infinite logarithmic measure such that for all $r\in E_2$ and $r\to\infty$, we have

$$T(r,A_l)>\exp_{p-1}\{(\tau_p(A_l)-\varepsilon)r^{\sigma_p(A_l)}\}. \tag{3.3}$$

Denote

$$\sigma_1=\max_{0\le j\le k}\{\sigma_p(A_j):\sigma_p(A_j)<\sigma_p(A_l)\},\quad \tau_1=\max_{j\neq l}\{\tau_p(A_j):\sigma_p(A_j)=\sigma_p(A_l)\}.$$

If $\sigma_p(A_j)<\sigma_p(A_l)$, then for the above ε and sufficiently large r, we have

$$T(r,A_j)\le\exp_{p-1}\{r^{\sigma_1+\varepsilon}\}. \tag{3.4}$$

If $\sigma_p(A_j)=\sigma_p(A_l), j\neq l$, then for the above ε and sufficiently large r, we have

$$T(r,A_j)\le\exp_{p-1}\{(\tau_1+\varepsilon)r^{\sigma_p(A_l)}\},\quad j\neq l. \tag{3.5}$$

By the definition of $\lambda_p(1/A_l)$, we have that for the above ε and sufficiently large r,

$$N(r,A_l)\le\exp_{p-1}\{r^{\lambda_p(\frac{1}{A_l})+\varepsilon}\}. \tag{3.6}$$

If $\sigma_p(F)<\sigma_p(A_l)$, then for the above ε and sufficiently large r, we have

$$T(r,F)\le\exp_{p-1}\{r^{\sigma_p(F)+\varepsilon}\}. \tag{3.7}$$

Now, we choose sufficiently small ε satisfying $0<2\varepsilon<\min\{\sigma_p(A_l)-\sigma_1,\tau_p(A_l)-\tau_1,$ $\sigma_p(A_l)-\lambda_p(\frac{1}{A_l}),\sigma_p(A_l)-\sigma_p(F)\}$, and deduce from (3.2)-(3.7) that for $r\in E_2\setminus E_1$ and $r\to\infty$, we have

$$\begin{aligned}
&\exp_{p-1}\{(\tau_p(A_l)-\varepsilon)r^{\sigma_p(A_l)}\}\\
&<O(\exp_{p-1}\{r^{\sigma_1+\varepsilon}\})+O(\exp_{p-1}\{(\tau_1+\varepsilon)r^{\sigma_p(A_l)}\})+3T(2r,f)^2\\
&\quad+\exp_{p-1}\{r^{\sigma_p(F)+\varepsilon}\}+\exp_{p-1}\{r^{\lambda_p(\frac{1}{A_l})+\varepsilon}\}.
\end{aligned}\tag{3.8}$$

It follows by (3.8) that $\sigma_p(f)\ge\sigma_p(A_l)$.

If $\sigma_p(F)=\sigma_p(A_l)$ and $\tau_p(F)<\tau_p(A_l)$, then for the above ε and sufficiently large r, we have

$$T(r,F)\le \exp_{p-1}\{(\tau_p(F)+\varepsilon)r^{\sigma_p(A_l)}\}. \tag{3.9}$$

Now, we choose sufficiently small ε satisfying

$$0<2\varepsilon<\min\{\sigma_p(A_l)-\sigma_1,\tau_p(A_l)-\tau_1,\sigma_p(A_l)-\lambda_p(\frac{1}{A_l}),\tau_p(A_l)-\tau_p(F)\},$$

and deduce from (3.2)-(3.6) and (3.9) that for $r\in E_2\setminus E_1$ and $r\to\infty$, we have

$$\begin{aligned}
&\exp_{p-1}\{(\tau_p(A_l)-\varepsilon)r^{\sigma_p(A_l)}\}\\
&<O(\exp_{p-1}\{r^{\sigma_1+\varepsilon}\})+O(\exp_{p-1}\{(\tau_1+\varepsilon)r^{\sigma_p(A_l)}\})\\
&\quad+3T(2r,f)^2+\exp_{p-1}\{(\tau_p(F)+\varepsilon)r^{\sigma_p(A_l)}\}+\exp_{p-1}\{r^{\lambda_p(\frac{1}{A_l})+\varepsilon}\}.
\end{aligned} \tag{3.10}$$

It follows by (3.10) that $\sigma_p(f)\ge\sigma_p(A_l)$.

If $\sigma_p(F)=\sigma_p(A_l)$ and $\tau_p(F)>\tau_p(A_l)$, then by Lemma 2.4, for the above ε, there exists a subset E_3 with infinite logarithmic measure such that for all $r\in E_3$ and $r\to\infty$, we have

$$T(r,F)>\exp_{p-1}\{(\tau_p(F)-\varepsilon)r^{\sigma_p(A_l)}\}. \tag{3.11}$$

By the definition of $\tau_p(A_l)$, we have that for the above ε and sufficiently large r,

$$T(r,A_l)\le \exp_{p-1}\{(\tau_p(A_l)+\varepsilon)r^{\sigma_p(A_l)}\}. \tag{3.12}$$

From (1.4) and Remark 2.2 it follows that for sufficiently large r,

$$T(r,F)\le\sum_{j=0,\,j\ne l}^{k}T(r,A_j)+T(r,A_l)+(k+2)T(2r,f). \tag{3.13}$$

Now, we choose sufficiently small ε satisfying $0<2\varepsilon<\min\{\sigma_p(A_l)-\sigma_1,\tau_p(A_l)-\tau_1,\tau_p(F)-\tau_p(A_l)\}$, and from (3.4), (3.5) and (3.11)-(3.13) deduce that for $r\in E_3\setminus E_1$ and $r\to\infty$, we have

$$\begin{aligned}
&\exp_{p-1}\{(\tau_p(F)-\varepsilon)r^{\sigma_p(A_l)}\}\\
&<O(\exp_{p-1}\{r^{\sigma_1+\varepsilon}\})+O(\exp_{p-1}\{(\tau_1+\varepsilon)r^{\sigma_p(A_l)}\})\\
&\quad+\exp_{p-1}\{(\tau_p(A_l)+\varepsilon)r^{\sigma_p(A_l)}\}+(k+2)T(2r,f).
\end{aligned} \tag{3.14}$$

It follows by (3.14) that $\sigma_p(f)\ge\sigma_p(A_l)$.

(ii) If $\sigma_p(F)>\sigma_p(A_l)$, then we may suppose that $\sigma_p(f)<\sigma_p(F)$ on the contrary. By (1.4) and Remark 2.2, we obtain

$$\sigma_p(A_k(z)f(z+c_k)+\dots+A_1(z)f(z+c_1)+A_0(z)f(z))<\sigma_p(F),$$

a contradiction. Hence, we have $\sigma_p(f)\ge\sigma_p(F)$. The proof is complete. □

Proof of Theorem 1.7. (i) If $f(z)$ has infinite order, then the result holds yet. Now, we suppose that $f(z)$ has finite order. From (3.1), Lemma 2.5 and Remark 2.2 it follows that for any given $\varepsilon > 0$ and sufficiently large r, we have

$$\begin{aligned}
&T(r, A_l)\\
&= m(r, A_l) + N(r, A_l)\\
&\le \sum_{j=0,\, j\neq l}^{k} m(r, A_j) + \sum_{j=0,\, j\neq l}^{k} m(r, \frac{f(z+c_j)}{f(z+c_l)})\\
&\quad + m(r, F) + m(r, \frac{1}{f(z+c_l)}) + N(r, A_l) + O(1)\\
&\le \sum_{j=0,\, j\neq l}^{k} T(r, A_j) + O(r^{\sigma(f)-1+\varepsilon})\\
&\quad + T(r, F) + (1+o(1))T(r+|c_l|, f) + N(r, A_l) + O(1)\\
&\le \sum_{j=0,\, j\neq l}^{k} T(r, A_j) + O(r^{\sigma(f)-1+\varepsilon}) + T(r, F) + O(r^{\sigma(f)+\varepsilon}) + N(r, A_l)\\
&\le \sum_{j=0,\, j\neq l}^{k} T(r, A_j) + O(r^{\sigma(f)+\varepsilon}) + T(r, F) + N(r, A_l).
\end{aligned} \tag{3.15}$$

From Lemma 2.4 it follows that for the above ε, there exists a subset E_4 with infinite logarithmic measure such that for all $r \in E_4$ and $r \to \infty$, we have

$$T(r, A_l) > (\tau(A_l) - \varepsilon) r^{\sigma(A_l)}. \tag{3.16}$$

Denote

$$\sigma_2 = \max_{0\le j\le k}\{\sigma(A_j) : \sigma(A_j) < \sigma(A_l)\}, \quad \tau_2 = \sum_{\sigma(A_j)=\sigma(A_l),\, j\neq l} \tau(A_j).$$

If $\sigma(A_j) < \sigma(A_l)$, then for the above ε and sufficiently large r, we have

$$T(r, A_j) \le r^{\sigma_2+\varepsilon}. \tag{3.17}$$

If $\sigma(A_j) = \sigma(A_l), j \neq l$, then for the above ε and sufficiently large r, we have

$$T(r, A_j) \le (\tau(A_j) + \varepsilon) r^{\sigma(A_l)}, \quad j \neq l. \tag{3.18}$$

By the definition of $\lambda(\frac{1}{A_l})$, we have that for the above ε and sufficiently large r,

$$N(r, A_l) < r^{\lambda(\frac{1}{A_l})+\varepsilon}. \tag{3.19}$$

If $\sigma(F) < \sigma(A_l)$, then for the above ε and sufficiently large r, we have

$$T(r, F) \le r^{\sigma(F)+\varepsilon}. \tag{3.20}$$

Now, we may choose sufficiently small ε satisfying $0 < (k+2)\varepsilon < \min\{\sigma(A_l) - \lambda(\frac{1}{A_l}), \sigma(A_l) - \sigma_2, \sigma(A_l) - \sigma(F), \tau(A_l) - \tau_2\}$, and deduce from (3.15)-(3.20) that for $r \in E_4$ and $r \to \infty$, we have

$$(\tau(A_l) - \tau_2 - (k+1)\varepsilon) r^{\sigma(A_l)} < O(r^{\sigma_2+\varepsilon}) + r^{\sigma(F)+\varepsilon} + r^{\lambda(\frac{1}{A_l})+\varepsilon} + O(r^{\sigma(f)+\varepsilon}). \tag{3.21}$$

It follows by (3.21) that $\sigma(f) \ge \sigma(A_l)$.

If $\sigma(F) = \sigma(A_l)$ and $\tau_2 + \tau(F) < \tau(A_l)$, then for the above ε and sufficiently large r, we have

$$T(r, F) \le (\tau(F) + \varepsilon) r^{\sigma(A_l)}. \tag{3.22}$$

Now, we may choose sufficiently small ε satisfying $0 < (k+3)\varepsilon < \min\{\sigma(A_l) - \lambda(\frac{1}{A_l}), \sigma(A_l) - \sigma_2, \tau(A_l) - \tau(F) - \tau_2\}$, and deduce from (3.15)-(3.19) and (3.22) that for $r \in E_4$ and $r \to \infty$, we have

$$(\tau(A_l) - \tau(F) - \tau_2 - (k+2)\varepsilon)r^{\sigma(A_l)} < O(r^{\sigma_2+\varepsilon}) + r^{\lambda(\frac{1}{A_l})+\varepsilon} + O(r^{\sigma(f)+\varepsilon}). \quad (3.23)$$

It follows by (3.23) that $\sigma(f) \geq \sigma(A_l)$.

If $\sigma(F) = \sigma(A_l)$ and $\tau_2 + \tau(A_l) < \tau(F)$, then by Lemma 2.4, for the above ε, there exists a subset E_5 with infinite logarithmic measure such that for all $r \in E_5$ and $r \to \infty$, we have

$$T(r, F) > (\tau(F) - \varepsilon)r^{\sigma(A_l)}. \quad (3.24)$$

By the definition of $\tau(A_l)$, we have that for the above ε and sufficiently large r,

$$T(r, A_l) \leq (\tau(A_l) + \varepsilon)r^{\sigma(A_l)}. \quad (3.25)$$

Now, we may choose sufficiently small ε satisfying $0 < (k+3)\varepsilon < \min\{\sigma(A_l) - \sigma_2, \tau(F) - \tau(A_l) - \tau_2\}$, and deduce from (3.13), (3.17)-(3.18) and (3.24)-(3.25) that for $r \in E_5$ and $r \to \infty$, we have

$$(\tau(F) - \tau(A_l) - \tau_2 - (k+2)\varepsilon)r^{\sigma(A_l)} < O(r^{\sigma_2+\varepsilon}) + O(r^{\sigma(f)+\varepsilon}). \quad (3.26)$$

It follows by (3.26) that $\sigma(f) \geq \sigma(A_l)$.

Further, if $F(z) \equiv 0$, then by using a similar reasoning method as the one in Theorem 1.6, we have $\sigma(f) \geq \sigma(A_l) + 1$.

(ii) If $\sigma(F) > \sigma(A_l)$, then we may suppose that $\sigma(f) < \sigma(F)$ on the contrary. By (1.4) and Remark 2.2, we obtain

$$\sigma(A_k(z)f(z+c_k) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z)) < \sigma(F),$$

a contradiction. Hence, we have $\sigma(f) \geq \sigma(F)$. The proof is complete. □

Proof of Theorem 1.10. (i) If $f(z)$ has infinite order, then the result holds. Now, we suppose that $f(z)$ has finite order. We divide (1.6) by $f(z+c_l)$ to obtain

$$\begin{aligned} -A_{l0}(z) &= \sum_{i=0,\, i\neq l}^{n} \sum_{j=0}^{m} A_{ij}(z)\frac{f^{(j)}(z+c_i)}{f(z+c_i)}\frac{f(z+c_i)}{f(z+c_l)} \\ &\quad + \sum_{j=1}^{m} A_{lj}(z)\frac{f^{(j)}(z+c_l)}{f(z+c_l)} - \frac{F(z)}{f(z+c_l)}. \end{aligned} \quad (3.27)$$

By (3.27) and Remark 2.2, for sufficiently large r, we have

$$\begin{aligned} &m(r, A_{l0}) \\ &\leq \sum_{i=0,\, i\neq l}^{n} \sum_{j=0}^{m} m(r, A_{ij}) + \sum_{j=1}^{m} m(r, A_{lj}) + \sum_{i=0}^{n}\sum_{j=1}^{m} m(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}) \\ &\quad + \sum_{i=0,\, i\neq l}^{n} m(r, \frac{f(z+c_i)}{f(z+c_l)}) + m(r, \frac{F(z)}{f(z+c_l)}) + O(1) \\ &\leq \sum_{i=0,\, i\neq l}^{n} \sum_{j=0}^{m} T(r, A_{ij}) + \sum_{j=1}^{m} T(r, A_{lj}) + \sum_{i=0}^{n}\sum_{j=1}^{m} m(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}) \\ &\quad + \sum_{i=0,\, i\neq l}^{n} m(r, \frac{f(z+c_i)}{f(z+c_l)}) + T(r, F) + (1+o(1))T(r+|c_l|, f) + O(1). \end{aligned} \quad (3.28)$$

By Lemma 2.5 it follows that for any given $\varepsilon > 0$, we have

$$m(r, \frac{f(z+c_i)}{f(z+c_l)}) = O(r^{\sigma(f)-1+\varepsilon}), \quad i = 0, 1, \dots, n, \ i \neq l. \tag{3.29}$$

The logarithmic derivative lemma and Remark 2.2 result in that for sufficiently large r, we have

$$m(r, \frac{f^{(j)}(z+c_i)}{f(z+c_i)}) = O(\log r), \quad i = 0, 1, \dots, n, \ j = 1, 2, \dots, m. \tag{3.30}$$

Set $\delta = \delta(\infty, A_{l0}) > 0$, then for sufficiently large r, we have

$$m(r, A_{l0}) \geq \frac{\delta}{2} T(r, A_{l0}). \tag{3.31}$$

Substituting (3.29)-(3.31) into (3.28) yields that for sufficiently large r, we have

$$\begin{aligned} \frac{\delta}{2} T(r, A_{l0}) \leq & \sum_{i=0,\, i \neq l}^{n} \sum_{j=0}^{m} T(r, A_{ij}) + \sum_{j=1}^{m} T(r, A_{lj}) + T(r, F) \\ & + O(\log r) + O(r^{\sigma(f)-1+\varepsilon}) + 2T(2r, f). \end{aligned} \tag{3.32}$$

Then (3.32) results in

$$\sigma(A_{l0}) \leq \max_{(i,j) \neq (l,0)} \{\sigma(f), \sigma(f) - 1 + \varepsilon, \sigma(A_{ij}), \sigma(F)\}. \tag{3.33}$$

If $\sigma(F) < \sigma(A_{l0})$, then by (3.33) and the fact $\sigma(A_{ij}) < \sigma(A_{l0}), (i,j) \neq (l,0)$, we have $\sigma(f) \geq \sigma(A_{l0})$.

Further, if $F(z) \equiv 0$, then by Theorem 1.9 we have $\sigma(f) \geq \sigma(A_{l0}) + 1$.

(ii) If $\sigma(F) > \sigma(A_{l0})$, then we may suppose that $\sigma(f) < \sigma(F)$ on the contrary. By (1.6) and Remark 2.2, we obtain

$$\sigma\Big(\sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z+c_i) \Big) < \sigma(F),$$

a contradiction. Hence, we have $\sigma(f) \geq \sigma(F)$. The proof is complete. □

Proof of Theorem 1.11. (i) If $f(z)$ has infinite order, then the result holds. Now, we suppose that $f(z)$ has finite order. If $\sigma(F) < \sigma(A_{l0})$, or $\sigma(F) = \sigma(A_{l0})$ and

$$\sum_{\sigma(A_{ij}) = \sigma(A_{l0}),\, (i,j) \neq (l,0)} \tau(A_{ij}) + \tau(F) < \tau(A_{l0}),$$

then by (3.27) and Remark 2.2, we have that for sufficiently large r,

$$\begin{aligned}
&T(r,A_{l0})\\
&= m(r,A_{l0})+N(r,A_{l0})\\
&\le \sum_{i=0,\,i\neq l}^{n}\sum_{j=0}^{m} m(r,A_{ij})+\sum_{j=1}^{m} m(r,A_{lj})+\sum_{i=0}^{n}\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)})\\
&\quad+\sum_{i=0,\,i\neq l}^{n} m(r,\frac{f(z+c_i)}{f(z+c_l)})+m(r,\frac{F(z)}{f(z+c_l)})+N(r,A_{l0})+O(1)\\
&\le \sum_{i=0,\,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij})+\sum_{j=1}^{m} T(r,A_{lj})\\
&\quad+\sum_{i=0}^{n}\sum_{j=1}^{m} m(r,\frac{f^{(j)}(z+c_i)}{f(z+c_i)})+\sum_{i=0,\,i\neq l}^{n} m(r,\frac{f(z+c_i)}{f(z+c_l)})\\
&\quad+T(r,F)+(1+o(1))T(r+|c_l|,f)+N(r,A_{l0})+O(1).
\end{aligned} \tag{3.34}$$

Also (3.29) and (3.30) hold. Then by using a similar reasoning as in (3.16)-(3.23) in the proof of Theorem 1.7, we have $\sigma(f)\ge\sigma(A_{l0})$.

If $\sigma(F)=\sigma(A_{l0})$ and

$$\sum_{\sigma(A_{ij})=\sigma(A_{l0})}\tau(A_{ij})<\tau(F),$$

then by (1.6), Remark 2.2 and $T(r,f^{(n)})\le(n+1)T(r,f)+S(r,f), n\in N_+$, we have that for sufficiently large r,

$$\begin{aligned}
T(r,F)&\le \sum_{(i,j)\neq(l,0)} T(r,A_{ij})+T(r,A_{l0})+\sum_{i=0}^{n}\sum_{j=0}^{m} T(r,f^{(j)}(z+c_i))\\
&\le \sum_{(i,j)\neq(l,0)} T(r,A_{ij})+T(r,A_{l0})+O(T(2r,f))+S(r,f).
\end{aligned} \tag{3.35}$$

Then by using a similar reasoning method as (3.24)-(3.26) in Theorem 1.7, we have $\sigma(f)\ge\sigma(A_{l0})$.

Further, if $F(z)\equiv 0$, then by (1.5), (3.29) and (3.30) it follows that

$$\begin{aligned}
T(r,A_{l0})\le &\sum_{i=0,\,i\neq l}^{n}\sum_{j=0}^{m} T(r,A_{ij})+\sum_{j=1}^{m} T(r,A_{lj})\\
&+O(r^{\sigma(f)-1+\varepsilon})+O(\log r)+N(r,A_{l0}).
\end{aligned} \tag{3.36}$$

From (3.36) it follows that $\sigma(f)\ge\sigma(A_{l0})+1$.

(ii) If $\sigma(F)>\sigma(A_{l0})$, then we may suppose that $\sigma(f)<\sigma(F)$ on the contrary. By (1.6) and Remark 2.2, we obtain

$$\sigma(\sum_{i=0}^{n}\sum_{j=0}^{m} A_{ij}(z)f^{(j)}(z+c_i))<\sigma(F),$$

a contradiction. Hence, we have $\sigma(f)\ge\sigma(F)$. The proof is complete. □

Proof of Theorem 1.12. (i) If $\sigma_p(F) < \sigma_p(A_{l0})$, or $\sigma_p(F) = \sigma_p(A_{l0})$ and $\tau_p(F) < \tau_p(A_{l0})$, then by (1.6) it follows that (3.34) holds. By the logarithmic derivative lemma and Lemma 2.3, we may rewrite (3.34) as

$$\begin{aligned} T(r, A_{l0}) \leq & \sum_{i=0,\, i\neq l}^{n} \sum_{j=0}^{m} T(r, A_{ij}) + \sum_{j=1}^{m} T(r, A_{lj}) + S(r, f) \\ & + 3T(2r, f)^2 + T(r, F) + N(r, A_{l0}) + O(1). \end{aligned} \tag{3.37}$$

Then by using a similar reasoning method as (3.3)-(3.10) in Theorem 1.5, we have $\sigma_p(f) \geq \sigma_p(A_{l0})$.

If $\sigma_p(F) = \sigma_p(A_{l0})$ and $\tau_p(F) > \tau_p(A_{l0})$, then (3.35) holds. Then by using a similar reasoning method as (3.11)-(3.14) in Theorem 1.5, we have $\sigma_p(f) \geq \sigma_p(A_{l0})$.

(ii) If $\sigma_p(F) > \sigma_p(A_{l0})$, then we may suppose that $\sigma_p(f) < \sigma_p(F)$ on the contrary. By (1.6) and Remark 2.2, we obtain

$$\sigma_p(\sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij}(z) f^{(j)}(z + c_i)) < \sigma_p(F),$$

a contradiction. Hence, we have $\sigma_p(f) \geq \sigma_p(F)$. The proof is complete. $\square$

4. Examples

The following examples show that the equalities in Theorems 1.7, 1.10 and 1.11 can be achieved, that is, these results are sharp.

Example 4.1. For Theorem 1.7, we consider the meromorphic functions

$$f(z) = e^{3z^2} \tan z \quad \text{and} \quad g(z) = e^{z^3} \tan z.$$

Case 1. $\sigma(F) < \sigma(A_l)$ and $F(z) \not\equiv 0$. Then $f(z)$ satisfies the difference equation

$$A_2(z) f(z + 2\pi) + A_1(z) f(z + \pi) + A_0(z) f(z) = F(z), \tag{4.1}$$

where

$$\begin{gathered} A_2(z) = e^{-3z^2} \cot z, \quad A_1(z) = e^{z^2 - 6\pi z - 3\pi^2}, \\ A_0(z) = -e^{z^2}, \quad F(z) = e^{12\pi z + 12\pi^2}. \end{gathered}$$

Clearly, $A_j(z)$, $j = 0, 1, 2$ and $F(z)$ satisfy

$$\begin{gathered} \lambda(\frac{1}{A_2}) = 1 < 2 = \sigma(A_2), \\ \sigma(F) = 1 < 2 = \max\{\sigma(A_0), \sigma(A_1)\} = \sigma(A_2), \\ \tau(A_0) + \tau(A_1) = \frac{1}{\pi} + \frac{1}{\pi} = \frac{2}{\pi} < \frac{3}{\pi} = \tau(A_2), \end{gathered}$$

where $l = k = 2$. Then $f(z)$ satisfies $\sigma(f) = \sigma(A_2) = 2$.

Case 2. $\sigma(F) = \sigma(A_l)$ and $\sum_{\sigma(A_j) = \sigma(A_l),\, j \neq l} \tau(A_j) + \tau(F) < \tau(A_l)$. Then $f(z)$ satisfies (4.1), where

$$\begin{gathered} A_2(z) = e^{-(7z^2 + 12\pi z + 12\pi^2)} \cot z, \quad A_1(z) = e^{z^2 - 6\pi z - 3\pi^2}, \\ A_0(z) = -e^{z^2}, \quad F(z) = e^{-4z^2}. \end{gathered}$$

Clearly, $A_j(z)$, $j=0,1,2$ and $F(z)$ satisfy

$$\lambda(\frac{1}{A_2})=1<2=\sigma(A_2),$$
$$\sigma(F)=2=\max\{\sigma(A_0),\sigma(A_1)\}=\sigma(A_2),$$
$$\tau(A_0)+\tau(A_1)+\tau(F)=\frac{1}{\pi}+\frac{1}{\pi}+\frac{4}{\pi}=\frac{6}{\pi}<\frac{7}{\pi}=\tau(A_2),$$

where $l=k=2$. Then $f(z)$ satisfies $\sigma(f)=\sigma(A_2)=2$.

Case 3. $\sigma(F)=\sigma(A_l)$. Then $f(z)$ satisfies (4.1), where

$$A_2(z)=e^{z^2-12\pi z-12\pi^2}\cot z,\quad A_1(z)=e^{-(z^2+6\pi z+3\pi^2)},$$
$$A_0(z)=-e^{-z^2},\quad F(z)=e^{4z^2}.$$

Clearly, $A_j(z)$, $j=0,1,2$ and $F(z)$ satisfy

$$\lambda(\frac{1}{A_2})=1<2=\sigma(A_2),$$
$$\sigma(F)=2=\max\{\sigma(A_0),\sigma(A_1)\}=\sigma(A_2),$$
$$\tau(A_0)+\tau(A_1)+\tau(A_2)=\frac{1}{\pi}+\frac{1}{\pi}+\frac{1}{\pi}=\frac{3}{\pi}<\frac{4}{\pi}=\tau(F),$$

where $l=k=2$. Then $f(z)$ satisfies $\sigma(f)=\sigma(A_2)=2$.

Case 4. $F(z)\equiv 0$. Then $g(z)$ satisfies the difference equation

$$A_2(z)g(z+\frac{3}{2}\pi)+A_1(z)g(z+\pi)+A_0(z)g(z)=0, \tag{4.2}$$

where

$$A_2(z)=e^{-(\frac{9}{2}\pi z^2+\frac{27}{4}\pi^2 z+\frac{27}{8}\pi^3)}\tan^2 z,$$
$$A_1(z)=2e^{-(3\pi z^2+3\pi^2 z+\pi^3)},\quad A_0(z)=-1.$$

Clearly, $A_j(z)$, $j=0,1,2$ satisfy

$$\lambda(\frac{1}{A_2})=1<2=\sigma(A_2),$$
$$\max\{\sigma(A_0),\sigma(A_1)\}=2=\sigma(A_2),\quad \tau(A_1)=3<\frac{9}{2}=\tau(A_2),$$

where $l=k=2$. Then $g(z)$ satisfies $\sigma(g)=3=\sigma(A_2)+1$.

Case 5. $\sigma(F)>\sigma(A_l)$. Then $g(z)$ satisfies (4.1), where

$$A_2(z)=e^{-(6\pi z^2+12\pi^2 z+8\pi^3)}\cot z,\quad A_1(z)=e^{-(3\pi z^2+3\pi^2 z+\pi^3)},$$
$$A_0(z)=-1,\quad F(z)=e^{z^3}.$$

Clearly, $A_j(z)$, $j=0,1,2$ and $F(z)$ satisfy

$$\sigma(F)=3>2=\max\{\sigma(A_0),\sigma(A_1)\}=\sigma(A_2),$$

where $l=k=2$. Then $g(z)$ satisfies $\sigma(g)=\sigma(F)=3$. Moreover, $A_j(z)$, $j=0,1,2$ satisfy

$$\lambda(\frac{1}{A_2})=1<2=\sigma(A_2),\quad \tau(A_1)=3<6=\tau(A_2),$$

these two conditions are not necessary for Case 5.

Example 4.2. For Theorem 1.10, we consider the meromorphic functions
$$f(z) = e^{3z^2}\tan z \quad \text{and} \quad g(z) = e^{z^3}\tan z.$$
Case 1. $\sigma(F) < \sigma(A_{l0})$ and $F(z) \not\equiv 0$. Then $f(z)$ satisfies the differential-difference equation
$$\begin{aligned} &A_{21}(z)f'(z+2\pi) + A_{20}(z)f(z+2\pi) + A_{11}(z)f'(z+\pi)\\ &+ A_{10}(z)f(z+\pi) + A_{01}(z)f'(z) + A_{00}(z)f(z) = F(z), \end{aligned} \tag{4.3}$$
where
$$A_{21}(z) = 2e^{-(12\pi z+12\pi^2)}, \quad A_{20}(z) = -9\pi e^{-(12\pi z+12\pi^2)},$$
$$A_{11}(z) = -e^{-(6\pi z+3\pi^2)}, \quad A_{10}(z) = e^{-3z^2}\cot z,$$
$$A_{01}(z) = -1, \quad A_{00}(z) = -9\pi, \quad F(z) = e^{6\pi z+3\pi^2}.$$
Clearly, $A_{ij}(z)$, $i = 0,1,2$, $j = 0,1$ and $F(z)$ satisfy
$$\delta(\infty, A_{10}) = 1 > 0,$$
$$\sigma(F) = 1 = \max\{\sigma(A_{ij}), (i,j) \neq (1,0)\} < 2 = \sigma(A_{10}),$$
where $n = 2$, $m = l = 1$. Then $f(z)$ satisfies $\sigma(f) = \sigma(A_{10}) = 2$.
Case 2.$F(z) \equiv 0$. Then $g(z)$ satisfies the differential-difference equation
$$\begin{aligned} &A_{21}(z)g'(z+2\pi) + A_{20}(z)g(z+2\pi) + A_{11}(z)g'(z+\pi)\\ &+ A_{10}(z)g(z+\pi) + A_{01}(z)g'(z) + A_{00}(z)g(z) = 0, \end{aligned} \tag{4.4}$$
where
$$A_{21}(z) = A_{20}(z) = A_{11}(z) = A_{00}(z) \equiv 0,$$
$$A_{10}(z) = e^{-(3\pi z^2+3\pi^2 z+\pi^3)}(3z^2\tan z + \sec^2 z), \quad A_{01}(z) = -\tan z.$$
Clearly, $A_{ij}(z)$, $i = 0,1,2$, $j = 0,1$ and $F(z)$ satisfy
$$\delta(\infty, A_{10}) = 1 > 0,$$
$$\max\{\sigma(A_{ij}), (i,j) \neq (1,0)\} = 1 < 2 = \sigma(A_{10}),$$
where $n = m = l = 1$. Then $g(z)$ satisfies $\sigma(g) = 3 = \sigma(A_{10}) + 1$.
Case 3. $\sigma(F) > \sigma(A_{l0})$. Then $g(z)$ satisfies (4.3), where
$$A_{21}(z) = -\tan z, \quad A_{20}(z) = 3(z+2\pi)^2\tan z + \sec^2 z,$$
$$A_{11}(z) \equiv 0, \quad A_{10}(z) = (3z^2 + \cot z\sec^2 z)e^{-(3\pi z^2+3\pi^2 z+\pi^3)},$$
$$A_{01}(z) = 1, \quad A_{00}(z) = 3z^2 + \cot z\sec^2 z,$$
$$F(z) = 3(3z^2\tan z + \sec^2 z)e^{z^3}.$$
Clearly, $A_{ij}(z)$, $i = 0,1,2$, $j = 0,1$ and $F(z)$ satisfy
$$\delta(\infty, A_{10}) = 1 > 0,$$
$$\sigma(F) = 3 > 2 = \sigma(A_{10}) > 1 = \max\{\sigma(A_{ij}), (i,j) \neq (1,0)\},$$
where $n = 2$, $m = l = 1$. Then $g(z)$ satisfies $\sigma(g) = \sigma(F) = 3$.

We may use a similar method for constructing an example for Theorem 1.11; we omit it here. Example 4.1 also illustrates Theorem 1.11, since Theorem 1.7 can be seen as a special case of Theorem 1.11.

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