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NONLINEAR PARABOLIC PROBLEMS WITH VARIABLE EXPONENT AND L^1-DATA

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Abstract. In this article, we prove the existence and uniqueness of entropy solutions to nonlinear parabolic equation with variable exponent and L^1-data. The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

1. Introduction

The purpose of this article is to study the existence and uniqueness of entropy solutions to the nonlinear parabolic problem involving the $p(x)$-Laplacian type operator

$$\begin{gathered} u_t - \operatorname{div} a(x, \nabla u) = f \quad \text{in } Q = (0,T) \times \Omega \\ u = 0 \quad \text{on } \Sigma_T = (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded open domain with smooth boundary and T is a positive fixed final time.

The study of various mathematical problems with variable exponent has received considerable attention in recent years. These problems concern applications (see [2, 10, 11, 21, 22]) and raise many difficult mathematical problems.

The operator $-\operatorname{div} a(x, \nabla u)$ is called $p(x)$-Laplacian type operator and is a generalization of the $p(x)$-Laplace operator $-\Delta_{p(x)}(u) := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ and the generalized mean curvature operator $-\operatorname{div}\big((1 + |\nabla u|^2)^{(p(x)-2)/2}\nabla u\big)$. Therefore, the problem (1.1) can be viewed as a generalization of the $p(x)$-Laplace problem

$$\begin{gathered} u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f \quad \text{in } Q = (0,T) \times \Omega \\ u = 0 \quad \text{on } \Sigma_T = (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega. \end{gathered} \tag{1.2}$$

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and the generalized mean curvature problem

$$
\begin{gathered}
u_t - \operatorname{div}\Big(\big(1+|\nabla u|^2\big)^{(p(x)-2)/2}\nabla u\Big) = f \quad \text{in } Q=(0,T)\times\Omega \\
u = 0 \quad \text{on } \Sigma_T = (0,T)\times\partial\Omega \\
u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega.
\end{gathered} \tag{1.3}
$$

The existence and uniqueness of renormalized solutions to problems (1.2) and (1.3) are nowadays well-known (see [3, 24]).

We recall that the notion of renormalized solutions was introduced for the first time by Diperna and Lions [13] in their study of the Boltzmann equation. An equivalent notion of solutions, called entropy solutions, was introduced independently by Bénilan and al. in [4]. Following [4] and using the same notion of solution, Ouaro and Traoré (see [19]) studied the problem

$$
\begin{gathered}
u - \operatorname{div} a(x,\nabla u) = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.4}
$$

where they proved the existence and uniqueness of entropy solution for a data $f \in L^1(\Omega)$. Relying on these results and applying nonlinear semigroup theory, it is easy to deduce the existence of a unique mild solution for the abstract Cauchy problem corresponding to (1.1) and arbitrary L^1-data (cf. section 4). In this paper, we use the abstract semigroup theory to prove the existence and uniqueness of entropy solution to (1.1) for arbitrary L^1-data.

We recall that Wittbold and Zimmermann in [23] studied and proved the existence and uniqueness of a renormalized solution to the stationary problem

$$
\begin{gathered}
\beta(u) - \operatorname{div} a(x,Du) - \operatorname{div} F(u) \ni f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.5}
$$

where $f \in L^1(\Omega), \Omega$ a bounded domain of $\mathbb{R}^N (N \geq 1)$ with Lipschitz boundary $\partial\Omega$ (if $N \geq 2$), $F : \mathbb{R} \to \mathbb{R}^N$ locally Lipschitz continuous, $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0 \in \beta(0)$, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ a Carathéodory function and $p(\cdot) : \overline{\Omega} \to (1,\infty)$ a continuous variable exponent such that $1 < \min_{x\in\overline{\Omega}} p(x) < N$. Relying on these above results and applying nonlinear semigroup theory (see [6]), Bendahmane, Wittbold and Zimmermann proved (see [3]) the existence and uniqueness of a renormalized solution to the problem (1.2).

Apart from the work by Bendahmane and al [3], Zhang and Zhou [24] studied the problem (1.2) by using other methods, where they proved the existence and uniqueness of entropy solutions. They also proved the equivalence between entropy and renormalized solutions of (1.2). The method used in [24] was the following: They employed first the difference and variation methods to prove the existence and uniqueness of a weak solution for the approximate problem of (1.2) under appropriate assumptions. Then they constructed an approximate solution sequence and established some *a priori* estimates. Next, they drew a subsequence to obtain a limit function and proved that this function is a renormalized solution. Based on the strong convergence on the truncations of approximate solutions, they obtained that the renormalized solution to problem (1.2) is also an entropy solution, which leads to an equality in the entropy formulation. Finally, by choosing suitable test functions, they proved the uniqueness of renormalized solutions and entropy solutions and thus, the equivalence of renormalized solutions and entropy solutions. The main operator in problem (1.1) is more general than the $p(\cdot)$-Laplace operator of (1.2) as we will see later.

The aim of our paper is to extend the results in [19], to the case of parabolic equations. Inspired by [3] and [24], we first define two notions of solutions of problem (1.1): The notion of entropy solution and the notion of renormalized solution. Next, we show that the two notions are equivalent which will permit us to use both notion when convenient. After that, according to the results in [19], we prove some properties of the entropy solutions of problem (1.1), by using nonlinear semigroup theory. Next, we prove the existence and uniqueness of entropy solutions to problem (1.1).

This article is organized as follows: In section 2 we recall some results of [19], the assumptions of problem (1.1) and some basic notations and properties of Lebesgue and Sobolev spaces with variable exponents. In section 3, we give the definition of entropy and renormalized solutions to problem (1.1) and prove that the two notions are equivalent. In section 4, using the results of [19], we prove some properties of entropy solutions to problem (1.1). Finally, in section 5 we prove the existence and uniqueness of entropy solutions of (1.1).

2. Preliminaries

In this article, we study problem (1.1) with the following assumptions on the data:

$$p(\cdot):\Omega\to\mathbb{R} \text{ is a measurable function such that } 1<p_-\le p_+<+\infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

For the vector field $a(\cdot,\cdot)$, we assume that $a(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is Carathéodory and is the continuous derivative with respect to ξ of the mapping $A:\Omega\times\mathbb{R}^N\to\mathbb{R}$, i.e. $a(x,\xi)=\nabla_\xi A(x,\xi)$ such that:

$$A(x,0)=0 \quad \text{for almost every } x\in\Omega\,. \tag{2.2}$$

There exists a positive constant C_1 such that

$$|a(x,\xi)|\le C_1(j(x)+|\xi|^{p(x)-1}) \tag{2.3}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$ where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$.

The following inequalities hold

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)>0, \tag{2.4}$$

for almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}^N$, with $\xi\neq\eta$ and

$$\frac{1}{C}|\xi|^{p(x)}\le a(x,\xi).\xi\le Cp(x)A(x,\xi) \tag{2.5}$$

for almost every $x\in\Omega$, $C>0$ and for every $\xi\in\mathbb{R}^N$.

Assumption (2.4) is imposed to obtain uniqueness of the solution to problem (1.1).

Remark 2.1. (1) Strict monotonicity (see assumption (2.4)) of the vector field is certainly not needed to prove uniqueness of the entropy solution. It was assumed it here only just for simplicity.

(2) $a(x,0)=0$ for a.e. $x\in\Omega$. Indeed for a.e. $x\in\Omega$ fixed, denote $z=a(x,0)\in\mathbb{R}^N$. By the continuity of $a(x,\cdot)$, we have $\lim_{\xi\to0}a(x,\xi)=z$. Suppose now that

$z \neq 0$ (if $z = 0$, there is no need to make a proof; this is the case for example when $a(x,\xi) = |\xi|^{p-2}\xi$) and choose $\xi_0 = -sz$ with $s > 0$ used to tend toward 0; then $a(x,\xi_0)\cdot\xi_0 = -s(z+\epsilon(s))\cdot z = -s|z|^2 - sz\epsilon(s) \leq -s|z|^2 + s|z||\epsilon(s)|$, where $\lim_{s\to 0}|\epsilon(s)| = 0$. Therefore, for s sufficiently small, $-s|z|^2 + s|z||\epsilon(s)| < 0$, which is a contradiction by assumption (2.5). Thus, $z = 0$.

(3) As examples of models with respect to assumptions (2.2)-(2.5) for problem (1.1), we can give the following.
(i) Set $A(x,\xi) = (1/p(x))|\xi|^{p(x)}, a(x,\xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \geq 2$. Then we obtain the $p(x)$-Laplace operator

$$\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

(ii) Set $A(x,\xi) = (1/p(x))[\Big(1+|\xi|^2\Big)^{p(x)/2} - 1]$, $a(x,\xi) = \big(1+|\xi|^2\big)^{(p(x)-2)/2}\xi$, where $p(x) \geq 2$. Then we obtain the generalized mean curvature operator

$$\operatorname{div}\Big(\big(1+|\nabla u|^2\big)^{(p(x)-2)/2}\nabla u\Big).$$

As the exponent $p(x)$ appearing in (2.3) and (2.5) depends on the variable x, we must work with Lebesgue and Sobolev spaces with variable exponents. We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,dx$$

is finite. If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$|u|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxembourg norm. The space $(L^{p(\cdot)}(\Omega), |.|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p_- \leq p_+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality.

$$\Big|\int_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p_-} + \frac{1}{p_+}\Big)|u|_{p(\cdot)}|v|_{p'(\cdot)}, \tag{2.6}$$

for all $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$. Now, let

$$W^{1,p(\cdot)}(\Omega) := \Big\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\Big\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |(|\nabla u|)|_{p(\cdot)}.$$

The space $\big(W^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)}\big)$ is a separable and reflexive Banach space. Next, we define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ under the norm

$$\|u\| := |(|\nabla u|)|_{p(\cdot)}.$$

The space $\big(W_0^{1,p(\cdot)}(\Omega), \|u\|\big)$ is a separable and reflexive Banach space. For the interested reader, more details about Lebesgue and Sobolev spaces with variable exponent can be found in [12] (see also [16]).

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result (cf. [15]).

Lemma 2.2. *If $u_n, u \in L^{p(\cdot)}$ and $p_+ < +\infty$, then the following properties hold:*

(1) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p_-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p_+}$;
(2) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p_+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p_-}$;
(3) $|u|_{p(\cdot)} < 1$ *(respectively* $= 1; > 1$*)* $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ *(respectively* $= 1; > 1$*);*
(4) $|u_n|_{p(\cdot)} \to 0$ *(respectively* $\to +\infty$*)* $\Longleftrightarrow \rho_{p(\cdot)}(u_n) \to 0$ *(respectively* $\to +\infty$*);*
(5) $\rho_{p(\cdot)}\big(u/|u|_{p(\cdot)}\big) = 1$.

Following [3], we extend a variable exponent $p : \overline{\Omega} \to [1, +\infty)$ to $\overline{Q} = [0, T] \times \overline{\Omega}$ by setting $p(t, x) := p(x)$ for all $(t, x) \in \overline{Q}$. We also consider the generalized Lebesgue space

$$L^{p(\cdot)}(Q) = \Big\{u : Q \to \mathbb{R} \text{ measurable such that } \iint Q|u(t,x)|^{p(x)}\, d(x,t) < \infty\Big\},$$

endowed with the norm

$$\|u\|_{L^{p(\cdot)}(Q)} := \inf\Big\{\lambda > 0 : \iint_Q \big|\frac{u(t,x)}{\lambda}\big|^{p(x)}\, d(x,t) < 1\Big\},$$

which shares the same properties as $L^{p(\cdot)}(\Omega)$.

We now recall the main result of [19] for the study of (1.4). We first recall the definition of the weak and entropy solutions of (1.4).

Definition 2.3. A weak solution of (1.4) is a function $u \in W_0^{1,1}(\Omega)$ such that $a(\cdot, \nabla u) \in \big(L^1_{\rm loc}(\Omega)\big)^N$ and

$$\int_\Omega a(x, \nabla u) \cdot \nabla\varphi\, dx + \int_\Omega u\varphi\, dx = \int_\Omega f(x)\varphi\, dx, \tag{2.7}$$

for all $\varphi \in C_0^\infty(\Omega)$. A weak energy solution is a weak solution such that $u \in W_0^{1,p(\cdot)}(\Omega)$.

Definition 2.4. A measurable function u is an entropy solution to problem (1.4) if, for every $t > 0$, $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega uT_t(u - \varphi)\, dx + \int_\Omega a(x, \nabla u) \cdot \nabla T_t(u - \varphi)\, dx \leq \int_\Omega f(x)T_t(u - \varphi)\, dx, \tag{2.8}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Now, we recall the two main results in [19].

Theorem 2.5 ([19, Theorem 3.2])**.** *Assume that* (2.1)-(2.5) *hold and $f \in L^\infty(\Omega)$. Then there exists a unique weak energy solution of* (1.4).

Theorem 2.6 ([19, Theorem 4.3])**.** *Assume that* (2.1)-(2.5) *hold and $f \in L^1(\Omega)$. Then there exists a unique entropy solution to problem* (1.4).

Remark 2.7. Theorems 2.5 and 2.6 were generalized by Bonzi and Ouaro (see [8, Theorem 3.2 and 4.3]). According to [8, Theorem 3.2], [19, Theorem 3.2] hold for $f \in L^{(p_-)'}(\Omega)$.

3. Equivalence between entropy and renormalized solutions

Let T_k denote the truncation function at height k, that is

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\,\mathrm{sign}(s) & \text{if } |s| > k. \end{cases}$$

For the notion of entropy solution to problem (1.1), we will use the primitive of the truncation function at height $k \geq 0$ denoted by $\Theta_k : \mathbb{R} \to \mathbb{R}^+$ such that

$$\Theta_k(r) = \int_0^r T_k(s)\,ds = \begin{cases} r^2/2 & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| \geq k. \end{cases}$$

It is obvious that $\Theta_k(r) \geq 0$ and $\Theta_k(r) \leq k|r|$. We denote

$$\begin{aligned} \mathcal{T}_0^{1,p(\cdot)}(Q) = \Big\{ & u : \Omega \times (0,T] \to \mathbb{R} \text{ measurable };T_k(u) \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)), \\ & \text{with } \nabla T_k(u) \in \big(L^{p(\cdot)}(Q)\big)^N, \text{ for every } k > 0 \Big\}. \end{aligned}$$

Next, we define the weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q)$. The proof follows from [4, Lemma 2.1] due to the fact that $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p_-}(\Omega)$.

Proposition 3.1. *For every measurable function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q)$, there exists a unique measurable function $\nu : Q \to \mathbb{R}^N$, which we call the weak gradient of u and denote $\nu = \nabla u$, such that*

$$\nabla T_k(u) = \nu \chi_{\{|u|<k\}}, \text{ almost everywhere in } Q \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $L^1(0,T;W_0^{1,1}(\Omega))$, then ν coincides with the gradient of u.

The notion of the weak gradient allows us to give the following definitions of entropy and renormalized solutions to problem (1.1). We define the spaces:

$$\begin{gathered} V = \big\{ f \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) : |\nabla f| \in L^{p(\cdot)}(Q) \big\}, \\ E = \big\{ \varphi \in V \cap L^\infty(Q) : \varphi_t \in V^* + L^1(Q) \big\}. \end{gathered}$$

According to [20], we have $E \subset \mathcal{C}([0,T];L^1(\Omega))$.

Definition 3.2. An entropy solution to problem (1.1) is a function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q) \cap L^\infty(Q)$ such that the mapping

$$[0,T] \ni t \mapsto \int_\Omega \Theta_k(u-\phi)(t,x)\,dx$$

is a.e. equal to a continuous function for all $k > 0$ and all $\phi \in E$, and

$$\begin{aligned} & \int_\Omega \Theta_k(u-\phi)(T)\,dx - \int_\Omega \Theta_k(u_0 - \phi(0))\,dx \\ & + \int_Q \phi_t T_k(u-\phi)\,dx\,dt + \int_Q a(x,\nabla u) \cdot \nabla T_k(u-\phi)\,dx\,dt \\ & = \int_Q f T_k(u-\phi)\,dt\,dx, \end{aligned} \tag{3.1}$$

for all $k > 0$ and $\phi \in E$.

Definition 3.3. A function $u \in \mathcal{T}_0^{1,p(\cdot)}(Q) \cap L^\infty(Q)$ is a renormalized solution to problem (1.1) if the following conditions are satisfied: (i)

$$\lim_{n\to+\infty} \int_{\{(t,x)\in Q: n\le |u(t,x)|\le n+1\}} |\nabla u|^{p(x)}\,dt\,dx = 0; \tag{3.2}$$

(ii) for all S in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support,

$$\frac{\partial}{\partial t} S(u) - \operatorname{div}\big[S'(u)a(x,\nabla u) + S"(u)a(x,\nabla u)\cdot\nabla u\big] = fS'(u) \quad \text{in } \mathcal{D}'(Q), \tag{3.3}$$

$$S(u)(0) = S(u_0) \quad \text{in } L^1(\Omega)\,. \tag{3.4}$$

Remark 3.4. Using the fact that for any function $\varphi \in V \cap L^\infty(Q)$, there exists functions $\varphi_n \in \mathcal{D}(Q)$ that converge strongly to φ in V and weak-* in $L^\infty(Q)$, we see that in (3.1) and (3.3) we cannot only use the test functions in $\mathcal{D}(Q)$, but also functions in $V \cap L^\infty(Q)$. In fact, we can replace (3.3) by

$$\begin{aligned} &\big\langle \frac{\partial S(u)}{\partial t}, \varphi\big\rangle + \int_0^T \int_\Omega \big[S'(u)a(x,\nabla u)\cdot\nabla\varphi + S"(u)a(x,\nabla u)\cdot\nabla(u)\varphi\big]\,dx\,dt \\ &= \int_0^T \int_\Omega fS'(u)\varphi\,dx\,dt, \end{aligned} \tag{3.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $V^* + L^1(Q)$ and $V \cap L^\infty(Q)$.

To find more estimates for entropy solutions and also to get useful a priori estimates of approximate solutions to the equation (5.2) below, the following integration by parts formula plays a crucial role.

Lemma 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is zero outside a compact set of $\mathbb{R}$. Let us denote $F(s) = \int_0^s f(r)\,dr$. If $u \in V$ is such that $u_t \in V^* + L^1(Q)$ and if $\psi \in \mathcal{C}^\infty(\overline{Q})$, then we have*

$$\langle u_t, f(u)\psi\rangle = \int_\Omega F(u(T))\psi(T)\,dx - \int_\Omega F(u(0))\psi(0)\,dx - \int_Q \psi_t F(u)\,dx\,dt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $V^* + L^1(Q)$ and $V \cap L^\infty(Q)$.

The proof of the above lemma follows the same lines as the proof of [14, Lemma 7.1]; we omit it.

Next, we have a result showing the equivalence between entropy and renormalized solutions of (1.1).

Theorem 3.6. *A function u is an entropy solution of* (1.1) *if and only if it is a renormalized solution.*

The proof of the above theorem is the same as in constant exponent case; see [14].

4. Properties of entropy solutions

In this section, we prove the existence of mild solutions of (1.1) satisfying an L^1-comparison principle. A classical method to prove that consists in approximating (1.1) for $\epsilon > 0$, by an implicit time-discretization

$$\begin{gathered} \frac{u_i^\epsilon - u_{i-1}^\epsilon}{t_i^\epsilon - t_{i-1}^\epsilon} = \operatorname{div} a(x,\nabla u_i^\epsilon) + f_i^\epsilon \quad \text{in } D'(\Omega), \text{ for } i = 1,\dots,n, \\ u_i^\epsilon \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \end{gathered} \tag{4.1}$$

where $n \in \mathbb{N}$, $0 = t_0^\epsilon < t_1^\epsilon < \dots < t_n^\epsilon \leq T$ and $f_i^\epsilon \in L^\infty(\Omega)$, $i = 1, \dots, n$ such that

$$\sum_{i=1}^{n} \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \|f(t) - f_i^\epsilon\|_{L^1(\Omega)} dt \to 0, \quad \max_{i=1,\dots,n} (t_i^\epsilon - t_{i-1}^\epsilon) \to 0,$$
$$T - t_n^\epsilon \to 0, \quad \|u_0 - u_0^\epsilon\|_{L^1(\Omega)} \to 0 \quad \text{as } \epsilon \to 0.$$

The function u_ϵ is piecewise constant, defined by

$$u_\epsilon = u_i^\epsilon \quad \text{on } (t_{i-1}^\epsilon, t_i^\epsilon],\ i = 1, \dots, n; \quad u_\epsilon(0) = u_0^\epsilon.$$

This method is actually the method of nonlinear semigroup theory. Naturally, we are led to give the following concept.

Definition 4.1. A mild solution of (1.1) is a function $u \in C([0,T]; L^1(\Omega))$ which is the uniform limit of the piecewise constant function u_ϵ.

The main result of this section is the following.

Theorem 4.2. *For any $(u_0, f) \in L^1(\Omega) \times L^1(Q)$, there exists a unique mild solution u of* (1.1)*. Moreover, the following contraction principle holds: for any $0 \leq t \leq T$, if u (resp. $\hat{u}$) is a mild solution of* (1.1) *with respect to $(u_0, f) \in L^1(\Omega) \times L^1(Q)$ (resp. $(\hat{u}_0, \hat{f}) \in L^1(\Omega) \times L^1(Q)$), then*

$$\|u(t) - \hat{u}(t)\|_{L^1(\Omega)} \leq \|u_0 - \hat{u}_0\|_{L^1(\Omega)} + \int_0^t \big|\big|f(s) - \hat{f}(s)\big|\big|_{L^1(\Omega)} ds.$$

According to the nonlinear semigroups theory (see [6]), the preceding result is, essentially, a consequence of the result of Proposition 4.3 below. Before stating the proposition, we need to recall some definitions. Let A be a (possibly) multivalued nonlinear operator in $L^1(\Omega)$ that is $A : L^1(\Omega) \to \mathcal{P}(L^1(\Omega))$; as usual, A is identified with its graph $\{(u, v) \in L^1(\Omega) \times L^1(\Omega); v \in Au\}$. The operator A is called accretive if

$$\|u - \hat{u}\|_1 \leq \|u - \hat{u} + \sigma(v - \hat{v})\|_1, \quad \text{for any } (u, v), (\hat{u}, \hat{v}) \in A, \sigma > 0;$$

i.e., for any $\sigma > 0$, the resolvent of $A, (I + \sigma A)^{-1}$, is a single-valued operator and a contraction in the L^1-norm.

The operator A is called T-accretive if $\|(u - \hat{u})^+\|_1 \leq \|(u - \hat{u})^+ + \sigma(v - \hat{v})^+\|_1$, for any $(u, v), (\hat{u}, \hat{v}) \in A$, $\sigma > 0$; equivalently, if

$$\int_{\{u > \hat{u}\}} (v - \hat{v})^+ + \int_{\{u = \hat{u}\}} (v - \hat{v})^+ \geq 0,$$

for any $(u, v), (\hat{u}, \hat{v}) \in A$. Finally, the operator A is called m-accretive (resp. $m-T$-accretive) if A is accretive (resp. T-accretive) and, moreover, $R(I + \sigma A) = L^1(\Omega)$, for any $\sigma > 0$ (cf. [6]).

Proposition 4.3. *There exists an operator*

$$A = \{(u, f) \in L^1(\Omega) \times L^1(\Omega); u \textit{ is an entropy solution of } (1.4)\}$$

such that

(i) *A is T-accretive (and even completely accretive, cf.* [5]*);*
(ii) *$R(I + \sigma A) = L^1(\Omega)$, for any $\sigma > 0$;*
(iii) *$\overline{D(A)} = L^1(\Omega)$.*

Proof. (i) Let u (resp. $\hat{u}$) be a weak solution of (1.4) for f (resp. $\hat{f}$) $\in L^\infty(\Omega)$. We use $\frac{1}{k}T_k(u-\hat{u})^+$ as test function in (2.7) for $k>0$ to get upon addition

$$\begin{aligned}&\int_\Omega (u-\hat{u})\frac{1}{k}T_k(u-\hat{u})^+\,dx + \int_{\{|u-\hat{u}|<k\}}(a(x,\nabla u)-a(x,\nabla\hat{u}))\cdot\nabla(u-\hat{u})^+\,dx\\&= \int_\Omega (f-\hat{f})\frac{1}{k}T_k(u-\hat{u})^+\,dx.\end{aligned}$$

Letting k tend to 0 and using assumption (2.4), we obtain

$$\begin{aligned}\int_\Omega (u-\hat{u})^+\,dx &\le \int_\Omega (f-\hat{f})\operatorname{sign}_0^+(u-\hat{u})\,dx\\&\le \int_{\{u=\hat{u}\}}(f-\hat{f})^+\,dx + \int_\Omega (f-\hat{f})\operatorname{sign}_0^+(u-\hat{u})\,dx\\&= \big[(u-\hat{u})^+,(f-\hat{f})^+\big],\end{aligned}\tag{4.2}$$

where for $g,h\in L^1(\Omega)$, the bracket $[g,h]$ denotes the right-hand side Gâteaux derivative of the L^1-norm at g in the direction of h, i.e.,

$$[g,h] = \lim_{\lambda\to 0}\frac{\|g+\lambda h\|_1-\|g\|_1}{\lambda} = \int_{\{g=0\}}|h|\,dx + \int_\Omega h\operatorname{sign}_0(g)\,dx,$$

with $r\in\mathbb{R}\mapsto \operatorname{sign}_0(r)$, the usual sign-function which is equal to -1 on $(-\infty,0)$, equal to 1 on $(0,+\infty)$ and equal to 0 for $r=0$.

Now, let $f,\hat{f}\in L^1(\Omega)$ and $u,\hat{u}$ be two entropy solutions of (1.4) with f and $\hat{f}$ as data respectively. Then [19], there exist (u_n,f_n) and $(\hat{u}_n,\hat{f}_n)$ such that $u_n,\hat{u}_n$ are weak solutions of (1.4) with $f_n,\hat{f}_n\in L^\infty(\Omega)$ as data and such that: $u_n\to u$ and $\hat{u}_n\to\hat{u}$ in measure, $f_n\to f$ and $\hat{f}_n\to\hat{f}$ in $L^1(\Omega)$, as n approaches ∞.

According to [19], $u_n\to u$ a.e. in Ω, $\hat{u}_n\to\hat{u}$ a.e. in Ω. By setting $f_n=T_n(f)$, $\hat{f}_n=T_n(\hat{f})$ and using (4.2), we have

$$\begin{aligned}\int_\Omega (u_n-\hat{u}_n)^+\,dx &\le \int_\Omega (f_n-\hat{f}_n)\operatorname{sign}_0^+(u-\hat{u})\,dx\\&\le \int_\Omega T_n(f)\operatorname{sign}_0^+(u-\hat{u})\,dx + \int_\Omega T_n(\hat{f})\operatorname{sign}_0^+(u-\hat{u})\,dx\\&\le \|f\|_1+\|\hat{f}\|_1<+\infty.\end{aligned}$$

Therefore, by Fatou's lemma, we deduce that

$$\int_\Omega (u-\hat{u})^+\,dx \le \liminf_{n\to+\infty}\int_\Omega (u_n-\hat{u}_n)^+\,dx.\tag{4.3}$$

From (4.2), we have

$$\int_\Omega (u_n-\hat{u}_n)^+\,dx \le \big[(u_n-\hat{u}_n)^+,(f_n-\hat{f}_n)^+\big].$$

Note also that $[\cdot,\cdot]$ is upper semi-continuous which gives

$$\limsup_{n\to+\infty}\big[(u_n-\hat{u}_n)^+,(f_n-\hat{f}_n)^+\big] \le \big[(u-\hat{u})^+,(f-\hat{f})^+\big].\tag{4.4}$$

Finally, we use (4.3) and (4.4) to obtain

$$\int_\Omega (u-\hat{u})^+\,dx \le \big[(u-\hat{u})^+,(f-\hat{f})^+\big].$$

Assertion (ii) is a direct consequence of [19, Theorem 4.3].

(iii) As $\overline{L^\infty(\Omega)} = L^1(\Omega)$, we will prove that $L^\infty(\Omega) \subset \overline{D(A)}^{\|\cdot\|_1}$. Let $\alpha > 0$, and $f \in L^\infty(\Omega)$. We denote $u_\alpha := (I + \alpha A)^{-1} f$. Then $(u_\alpha, \frac{1}{\alpha}(f - u_\alpha)) \in A$. As $f \in L^\infty(\Omega)$ then, according to Theorem 2.5, u_α is a weak energy solution of (1.4). Let's take $\phi \in D(\Omega)$ as a test function in (2.7) to obtain

$$\alpha \int_\Omega a(x, \nabla u_\alpha) \cdot \nabla \phi \, dx + \int_\Omega u_\alpha \phi \, dx = \int_\Omega f(x) \phi \, dx. \tag{4.5}$$

The following Lemma provides L^∞-a priori estimates of a solution u and is crucial for the next of the proof.

Lemma 4.4. *Let u be a weak energy solution of* (1.4)*, then*

$$\|u\|_s \le C\|f\|_s, \quad \textit{for } 1 \le s \le +\infty.$$

Proof. The proof is rather classical (see. [19]). For the sake of completeness, let us recall the arguments. For $p \in P_0 = \{p \in C^\infty(\mathbb{R}); 0 \le p' \le 1, \operatorname{supp} p' \text{ is compact}, 0 \notin \operatorname{supp} p\}$, we take $p(u_\alpha)$ as a test function in (4.5) to obtain

$$\begin{aligned}
&\int_\Omega p(u_\alpha) f(x) \, dx \\
&= \alpha \int_\Omega a(x, \nabla u_\alpha) \cdot \nabla p(u_\alpha) \, dx + \int_\Omega p(u_\alpha) u_\alpha \, dx \\
&= \alpha \int_\Omega \big[a(x, \nabla u_\alpha) - a(x, 0)\big] \cdot \nabla u_\alpha p'(u_\alpha) \, dx + \alpha \int_\Omega a(x, 0) \cdot \nabla u_\alpha p'(u_\alpha) \, dx \\
&\quad + \int_\Omega p(u_\alpha) u_\alpha \, dx \\
&\ge \alpha \int_\Omega a(x, 0) \cdot \nabla u_\alpha p'(u_\alpha) \, dx + \int_\Omega p(u_\alpha) u_\alpha \, dx \\
&= \int_\Omega p(u_\alpha) u_\alpha \, dx \quad \text{(by the divergence formula).}
\end{aligned} \tag{4.6}$$

Next, we choose p such that $p(k) = |k|^{s-2}k$ for $1 \le s < +\infty$ in (4.6) to obtain

$$\int_\Omega |u_\alpha|^{s-2} u_\alpha f \, dx \ge \int_\Omega |u_\alpha|^s \, dx. \tag{4.7}$$

By Hölder inequality, from (4.7) we obtain

$$\int_\Omega |u_\alpha|^s \, dx \le \|f\|_s \Big(\int_\Omega \big(|u_\alpha|^{s-1}\big)^{s'} \, dx \Big)^{1/s'},$$

which gives

$$\|u_\alpha\|_s \le \|f\|_s. \tag{4.8}$$

As $f \in L^\infty(\Omega)$, then (4.8) implies $\|u_\alpha\|_\infty \le \|f\|_\infty$. □

Now, let us come back to the proof of Proposition 4.3. We take u_α as a test function in (4.5) to obtain

$$\begin{aligned}
\alpha \int_\Omega a(x, \nabla u_\alpha) \cdot \nabla u_\alpha \, dx &= - \int_\Omega u_\alpha^2 \, dx + \int_\Omega f(x) u_\alpha \, dx \\
&\le \|u_\alpha\|_q \|f\|_p.
\end{aligned} \tag{4.9}$$

Then, by Lemma 4.4 and (2.5), from (4.9) we deduce that

$$\alpha \int_\Omega |\nabla u_\alpha|^{p(x)} dx \le C\|f\|_p \|f\|_q < \infty \quad (\text{because } f \in L^\infty(\Omega)). \tag{4.10}$$

Now, by using the Hölder type inequality, for all $\phi \in D(\Omega)$, we have

$$\begin{aligned} &\big|\alpha \int_\Omega a(x, \nabla u_\alpha) \cdot \nabla \phi dx\big| \\ &\le C_1 \alpha \int_\Omega \big(j(x) + |\nabla u_\alpha|^{p(x)-1}\big)|\nabla \phi| dx \\ &\le C'\alpha \|j\|_{p'(\cdot)} \|\nabla \phi\|_{p(\cdot)} + C'\alpha \big|\big(|\nabla u_\alpha|^{p(x)-1}\big)\big|_{p'(\cdot)} \|\nabla \phi\|_{p(\cdot)} \\ &\le C \max\Big(\alpha^{1-\frac{1}{p_-}} (\alpha \rho_{p(\cdot)}(\nabla u_\alpha))^{\frac{1}{p_-}}, \alpha^{1-\frac{1}{p_+}} (\alpha \rho_{p(\cdot)}(\nabla u_\alpha))^{\frac{1}{p_+}}\Big) + C\alpha. \end{aligned} \tag{4.11}$$

According to (4.10), from (4.11) we deduce that

$$\big|\alpha \int_\Omega a(x, \nabla u_\alpha) \cdot \nabla \phi dx\big| \to 0 \quad \text{as } \alpha \to 0. \tag{4.12}$$

From (4.5) by using (4.12) we obtain

$$u_\alpha \to f \quad \text{as } \alpha \to 0, \text{ in } \mathcal{D}'(\Omega). \tag{4.13}$$

Note also that $(u_\alpha)_{\alpha>0}$ is uniformly bounded by Lemma 4.4, then up to a subsequence, $u_\alpha \to f$ in $L^p(\Omega)$, for all $1 < p < +\infty$ and a.e. in Ω.

Now, $\|u_\alpha\|_p \le \|f\|_p$ for all $1 < p < +\infty$ by Lemma 4.4, then by the Lebesgue dominated convergence theorem, we deduce that

$$u_\alpha \to f \quad \text{as } \alpha \to 0, \text{ in } L^p(\Omega),\ \forall 1 < p < +\infty. \tag{4.14}$$

As Ω is bounded, (4.14) implies

$$u_\alpha \to f \quad \text{in } L^1(\Omega) \text{ as } \alpha \to 0. \tag{4.15}$$

Therefore, by (4.15), we deduce that $\overline{D(A)} = L^1(\Omega)$. $\square$

By Proposition 4.3, the nonlinear operator A is m-accretive in $L^1(\Omega)$. Then, by the general theory of nonlinear semigroups (see [6]) we conclude that the abstract evolution problem corresponding to (1.1) admits a unique mild solution $u \in C([0,T]; L^1(\Omega))$ for any initial datum $u_0 \in \overline{D(A)}^{\|\cdot\|_{L^1(\Omega)}}$ and any right-hand side $f \in L^1(0,T; L^1(\Omega))$.

Lemma 4.5. *Let u be an entropy solution to problem* (1.1)*, then*

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}, \tag{4.16}$$

$$\begin{aligned} &\|\nabla T_k(u)\|_{L^{p(\cdot)}(Q)} \\ &\le k \max\Big\{\big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big)^{1/p_-}, \big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big)^{1/p_+}\Big\}. \end{aligned} \tag{4.17}$$

Proof. **Step 1:** Proof of (4.17). Taking $\phi = 0$ as a test function in (3.1), we obtain

$$\begin{aligned} &\int_\Omega \Theta_k(u)(T)\, dx - \int_\Omega \Theta_k(u_0)\, dx + \int_Q a(x, \nabla u) \cdot \nabla T_k(u)\, dx\, dt \\ &= \int_Q f T_k(u)\, dx\, dt. \end{aligned} \tag{4.18}$$

By the definition of Θ_k, we have $\Theta_k(u) \geq 0$. Using hypothesis (2.5), inequality (4.18) becomes

$$\begin{aligned} \frac{1}{C}\int_Q |\nabla T_k(u)|^{p(x)}\,dx\,dt &\leq \int_\Omega \Theta_k(u_0)\,dx + \int_Q fT_k(u)\,dx\,dt \\ &\leq \int_\Omega k|u_0|dx + \int_Q fT_k(u)\,dx\,dt \\ &\leq k\Big(\int_\Omega |u_0|dx + \int_Q |f|\,dx\,dt\Big), \end{aligned}$$

then, according to Lemma 2.2, we deduce that

$$\begin{aligned} &\|\nabla T_k(u)\|_{L^{p(\cdot)}(Q)} \\ &\leq k\max\Big\{\big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big)^{1/p_-}, \big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big)^{1/p_+}\Big\}. \end{aligned}$$

Step 2: Proof of (4.16). In the following, we use the function $S_n : \mathbb{R} \to \mathbb{R}$ defined by:

$$S_n(s) = \int_0^s h_n(r)dr, \quad \text{where } h_n(s) = 1 - |T_1(s - T_n(s))|. \tag{4.19}$$

Note that S_n satisfies

$$\begin{gathered} S_n(r) = S_n(T_{n+1}(r)), \|S_n'\|_{L^\infty(\mathbb{R})} \leq 1, \\ \operatorname{supp} S_n' \subset [-(n+1), n+1], S_n'' = 1_{[-n-1,-n]} - 1_{[n,n+1]}. \end{gathered} \tag{4.20}$$

Let $t_1 \in (0,T)$ and $\theta_\epsilon(t) = \big(1 - \frac{(t-t_1)^+}{\epsilon}\big)^+$. Then θ_ϵ is continuous on $[0,+\infty)$, $\theta_\epsilon = 1$ on $[0,t_1], \theta_\epsilon = 0$ on $[t_1+\epsilon,+\infty)$ and θ_ϵ is linear on $[t_1, t_1+\epsilon]$. Using $\varphi = \frac{1}{k}T_k(u)\theta_\epsilon$ as a test function in (3.3)(since entropy and renormalized solutions are equivalent) and taking $S = S_n$, we obtain

$$\begin{aligned} &\frac{1}{k}\int_0^T\int_\Omega \theta_\epsilon(S_n(u))_t T_k(u)\,dx\,dt \\ &+ \frac{1}{k}\int_0^T\int_\Omega S_n'(u)a(x,\nabla u)\cdot\nabla(T_k(u)\theta_\epsilon)dx\,dt \\ &= \frac{1}{k}\int_0^T\int_\Omega fS_n'(u)T_k(u)\theta_\epsilon\,dx\,dt \\ &\quad - \frac{1}{k}\int_0^T\int_\Omega S_n''(u)a(x,\nabla u)\cdot\nabla u T_k(u)\theta_\epsilon\,dx\,dt. \end{aligned} \tag{4.21}$$

Since $S_n''(s) = 0$ for $|s| \notin [n, n+1]$, we can write

$$S_n''(u)a(x,\nabla u)\cdot\nabla u T_k(u) = S_n''(u)a(x,\nabla T_{n+1}(u))\cdot\nabla(T_{n+1}(u))T_k(u) \in L^1(Q).$$

Since $\theta_\epsilon \to 1_{[0,t_1]}$ and is bounded by 1 as $\epsilon \to 0$, using Lebesgue dominated convergence theorem in equality (4.21), we obtain

$$\begin{aligned} &\frac{1}{k}\int_0^{t_1}\int_\Omega (S_n(u))_t T_k(u)\,dx\,dt + \frac{1}{k}\int_0^{t_1}\int_\Omega S_n'(u)a(x,\nabla u)\cdot\nabla(T_k(u))\,dx\,dt \\ &+ \frac{1}{k}\int_0^{t_1}\int_\Omega S_n''(u)a(x,\nabla u)\cdot\nabla u T_k(u)\,dx\,dt \\ &= \frac{1}{k}\int_0^{t_1}\int_\Omega fS_n'(u)T_k(u)\,dx\,dt. \end{aligned} \tag{4.22}$$

Let $n \geq M$. We have $T_k(u) = T_k(S_n(u))$ (since $S_n(s) = s$ on $[-M, M], |S_n(s)| \geq M$ and $\operatorname{sign}(S_n(s)) = \operatorname{sign}(s)$ outside $[-M, M]$), $(S_n(u))(t_1) \to u(t_1, \cdot)$ in $L^1(\Omega)$, $S_n(u_0) \to u_0$ in $L^1(\Omega)$ and $S_n'(u) \to 1$ a.e. in Q as $n \to +\infty$. Since $|S_n''(s)| \leq 1$ and $S_n''(s) \neq 0$ only if $|s| \in [n, n+1]$, using (2.3) we can write

$$
\begin{aligned}
&\Big| \int_0^{t_1} \int_\Omega S_n''(u) a(x, \nabla u) \cdot \nabla u T_k(u) \, dx \, dt \Big| \\
&\leq k \int_{\{n \leq |u| \leq n+1\}} \Big| a(x, \nabla u) \cdot \nabla u \Big| \, dx \, dt \\
&\leq k \int_{\{n \leq |u| \leq n+1\}} C_1 \Big(j(x) + |\nabla u|^{p(x)-1} \Big) |\nabla u| \, dx \, dt \\
&\leq k \int_\Omega C_1 \Big(j(x) + |\nabla u|^{p(x)-1} \Big) |\nabla u| 1_{\{n \leq |u| \leq n+1\}} \, dx \to 0 \quad \text{as } n \to +\infty.
\end{aligned}
$$

Passing to the limit in (4.22) as $n \to +\infty$, we obtain

$$
\begin{aligned}
&\frac{1}{k} \int_0^{t_1} \int_\Omega u_t T_k(u) \, dx \, dt + \frac{1}{k} \int_0^{t_1} \int_\Omega a(x, \nabla T_k(u)) \cdot \nabla (T_k(u)) \, dx \, dt \\
&= \frac{1}{k} \int_0^{t_1} \int_\Omega f T_k(u) \, dx \, dt,
\end{aligned} \tag{4.23}
$$

for all $t_1 \in (0, T)$. By (2.5), from (4.23), we obtain

$$
\frac{1}{k} \int_0^{t_1} \int_\Omega u_t T_k(u) \, dx \, dt \leq \frac{1}{k} \int_0^{t_1} \int_\Omega f T_k(u) \, dx \, dt.
$$

Letting $k \to 0$ in the inequality above, we obtain

$$
\int_0^{t_1} \int_\Omega u_t \operatorname{sign}_0(u) \, dx \, dt \leq \int_0^{t_1} \int_\Omega f \operatorname{sign}_0(u) \, dx \, dt.
$$

which implies

$$
\|u(t_1, \cdot)\|_{L^1(\Omega)} \leq \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}, \quad \text{for all } t_1 \in (0, T)
$$

i.e.

$$
\|u\|_{L^\infty(0,T;L^1(\Omega))} \leq \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} .
$$

This completes the proof. $\square$

Now, for any continuous and monotonic function ψ, we define the proper lower semi-continuous and convex or upper semi-continuous and concave function

$$
B_\psi(s) = \int_0^s \psi(r) \, dr.
$$

To prove the existence of weak solutions, we need an energy estimate similar to the one given in [1, Lemma 1.5].

Lemma 4.6. *Let $\psi \in C^{0,1}(\mathbb{R})$ be monotone, let u be a measurable function such that $u \in L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$. Then $B_\psi(u) \in L^\infty(0, T; L^1(\Omega))$ and, for almost every $t \in [0, T]$,*

$$
\begin{aligned}
&\int_\Omega B_\psi(u(t)) \xi(t) \, dx - \int_\Omega B_\psi(u_0) \xi(0) \, dx \\
&= \int_0^t \int_\Omega u_t \psi(u) \xi \, dx \, dt + \int_0^t \int_\Omega B_\psi(u) \xi_t \, dx \, dt
\end{aligned} \tag{4.24}
$$

for any $\xi \in C^{0,1}(\overline{Q})$ *such that* $\psi(u)\xi \in L^2(0,T;W_0^{1,1}(\Omega))$.

For the proof of the above lemma, see the proof of [9, Lemma 4].

By a weak solution of (1.1) we understand a solution in the sense of distributions that belongs to the energy space, i.e.,

$$\begin{gathered} u \in V = \big\{ f \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)); |\nabla f| \in L^{p(\cdot)}(Q) \big\}, \\ \frac{\partial u}{\partial t} - \operatorname{div} a(x,\nabla u) = f \text{ in } \mathcal{D}'(Q), \quad u(0,\cdot) = u_0\,. \end{gathered} \tag{4.25}$$

To complete this section we prove the following proposition.

Proposition 4.7. *Assume that* (2.1)-(2.5) *hold,* $u_0 \in L^\infty(\Omega)$*,* $f \in L^\infty(Q)$ *and* u *is the unique mild solution of* (1.1)*. Then* u *is a weak solution of* (1.1)*.*

Proof. For $i = 0, 1, \dots, n$, let u_i^ϵ be the unique weak energy solution of

$$\epsilon f_i^\epsilon + u_{i-1}^\epsilon \in (I + \epsilon A) u_i^\epsilon.$$

We have

$$\int_\Omega a(x,\nabla u_i^\epsilon)\cdot\nabla\varphi dx + \int_\Omega \frac{u_i^\epsilon - u_{i-1}^\epsilon}{\epsilon}\varphi dx = \int_\Omega f_i^\epsilon \varphi dx, \tag{4.26}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$. Taking $\varphi = u_i^\epsilon$ as test function in (4.26), integrating over $(t_{i-1}^\epsilon, t_i^\epsilon)$ and summing up the inequalities over $i = 1, \dots, n$, we obtain

$$\begin{aligned} &\sum_{i=1}^n \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \int_\Omega \frac{u_i^\epsilon - u_{i-1}^\epsilon}{\epsilon} u_i^\epsilon \,dx\,dt + \sum_{i=1}^n \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \int_\Omega a(x,\nabla u_i^\epsilon)\cdot\nabla u_i^\epsilon \,dx\,dt \\ &= \sum_{i=1}^n \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \int_\Omega f_i^\epsilon u_i^\epsilon \,dx\,dt. \end{aligned} \tag{4.27}$$

By (2.5) and as B_{Id} is convex, from (4.27) we deduce that

$$\begin{aligned} &\sum_{i=1}^n \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \int_\Omega \frac{B_{Id}(u_i^\epsilon) - B_{Id}(u_{i-1}^\epsilon)}{\epsilon} \,dx\,dt + \sum_{i=1}^n \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \frac{1}{C} \int_\Omega |\nabla u_i^\epsilon|^{p(x)} \,dx\,dt \\ &\le \sum_{i=1}^n \int_{t_{i-1}^\epsilon}^{t_i^\epsilon} \int_\Omega f_i^\epsilon u_i^\epsilon \,dx\,dt. \end{aligned}$$

Consequently, if we set $\epsilon = t_i^\epsilon - t_{i-1}^\epsilon$, then $f_\epsilon(t) = f_i^\epsilon$ and $u_\epsilon(t) = u_i^\epsilon$ for $t \in (t_{i-1}^\epsilon, t_i^\epsilon]$, $i = 1, \dots, n$; $u_\epsilon(0) = u_0^\epsilon$. It follows that

$$\begin{aligned} &\int_\Omega \Big[B_{Id}(u_\epsilon(T)) - B_{Id}(u_\epsilon(0)) \Big] dx + \frac{1}{C} \int_0^T \int_\Omega |\nabla u_\epsilon|^{p(x)} \,dx\,dt \\ &\le \int_0^T \int_\Omega f_\epsilon u_\epsilon \,dx\,dt. \end{aligned}$$

As $B_{Id}(u_\epsilon(T)) - B_{Id}(u_\epsilon(0)), u_\epsilon, f_\epsilon \in L^\infty(\Omega)$, we obtain

$$\int_0^T \int_\Omega |\nabla u_\epsilon|^{p(x)} \,dx\,dt \le C \Rightarrow \int_0^T \int_\Omega |\nabla u_\epsilon|^{p_-} \,dx\,dt \le C. \tag{4.28}$$

Using the Poincaré inequality with constant exponent, we deduce that $(u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$. So, there exists a subsequence still denoted $(u_\epsilon)_{\epsilon>0}$, such that

$$u_\epsilon \rightharpoonup u \quad \text{in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \text{ as } \epsilon \to 0, \tag{4.29}$$

$$\nabla u_\epsilon \rightharpoonup \nabla u \quad \text{in } \big(L^{p(\cdot)}(Q)\big)^N \text{ as } \epsilon \to 0. \tag{4.30}$$

Since $(\nabla u_\epsilon)_{\epsilon>0}$ is uniformly bounded in $\big(L^{p(\cdot)}(Q)\big)^N$, by (2.3) we deduce that $\big(a(x,\nabla u_\epsilon)\big)_{\epsilon>0}$ is uniformly bounded in $\big(L^{p'(\cdot)}(Q)\big)^N$ and then we can assume that

$$a(x,\nabla u_\epsilon) \rightharpoonup \Phi \quad \text{in } \big(L^{p'(\cdot)}(Q)\big)^N \text{ as } \epsilon \to 0. \tag{4.31}$$

From (4.26), we have

$$\int_\Omega a(x,\nabla(u_\epsilon))\cdot\nabla\varphi\,dx + \int_\Omega \frac{u_\epsilon(t)-u_\epsilon(t-\epsilon)}{\epsilon}\varphi\,dx = \int_\Omega f_\epsilon(t)\varphi dx, \tag{4.32}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega)$. Then, taking $\psi \in W^{1,1}(0,T;W^{1,1}(\Omega)\cap L^\infty(\Omega))\cap E$, $\psi(T)=0$ as a test function in (4.32), we obtain

$$\begin{aligned} &\int_0^T\int_\Omega a(x,\nabla u_\epsilon(t))\cdot\nabla\psi(t)\,dx\,dt + \int_0^T\int_\Omega \frac{u_\epsilon(t)-u_\epsilon(t-\epsilon)}{\epsilon}\psi(t)\,dx\,dt \\ &= \int_0^T\int_\Omega f_\epsilon(t)\psi(t)\,dx\,dt. \end{aligned} \tag{4.33}$$

We have

$$\begin{aligned} &\int_0^T\int_\Omega \frac{u_\epsilon(t)-u_\epsilon(t-\epsilon)}{\epsilon}\psi(t)\,dx\,dt \\ &= \int_0^T\int_\Omega \frac{u_\epsilon(t)}{\epsilon}\psi(t)\,dx\,dt - \int_0^T\int_\Omega \frac{u_\epsilon(t-\epsilon)}{\epsilon}\psi(t)\,dx\,dt \\ &= \int_0^T\int_\Omega \frac{u_\epsilon(t)}{\epsilon}\psi(t)\,dx\,dt - \int_{-\epsilon}^{T-\epsilon}\int_\Omega \frac{u_\epsilon(s)}{\epsilon}\psi(s+\epsilon)\,dx\,ds \quad (\text{where } s=t-\epsilon) \\ &= \int_0^{T-\epsilon}\int_\Omega \frac{u_\epsilon(t)}{\epsilon}\psi(t)\,dx\,dt + \int_{T-\epsilon}^T\int_\Omega \frac{u_\epsilon(t)}{\epsilon}\psi(t)\,dx\,dt \\ &\quad - \int_{-\epsilon}^0\int_\Omega \frac{u_\epsilon(s)}{\epsilon}\psi(s+\epsilon)\,dx\,ds - \int_0^{T-\epsilon}\int_\Omega \frac{u_\epsilon(s)}{\epsilon}\psi(s+\epsilon)\,dx\,ds \\ &= -\int_0^{T-\epsilon}\int_\Omega u_\epsilon(t)\frac{\psi(t+\epsilon)-\psi(t)}{\epsilon}\,dx\,dt + \int_{T-\epsilon}^T\int_\Omega \frac{u_\epsilon(t)\psi(t)}{\epsilon}dx\,dt \\ &\quad - \int_0^\epsilon\int_\Omega \frac{u_{0,\epsilon}(t)\psi(t)}{\epsilon}\,dx\,dt, \\ &\to -\int_0^T\int_\Omega u(t)\psi_t\,dx\,dt - \int_\Omega u_0\psi(0)\,dx\,dt \quad \text{as } \epsilon\to 0, \end{aligned}$$

where $u_\epsilon(t)=u_0$ for $t\le 0$. Therefore, taking limit in (4.33) as $\epsilon\to 0$, we obtain

$$\begin{aligned} &\int_0^T\int_\Omega \Phi\cdot\nabla\psi\,dx\,dt - \int_0^T\int_\Omega u\psi_t\,dx\,dt - \int_\Omega u_0\psi(0)\,dx\,dt \\ &= \int_0^T\int_\Omega f(t)\psi\,dx\,dt. \end{aligned} \tag{4.34}$$

Thus, to complete the proof of Proposition 4.7, we only need to show that $\Phi = a(x,\nabla u)$. To do so, we apply the Minty-Browder's method. Firstly, we prove that

$$\limsup_{\epsilon\to 0}\iint Qa(x,\nabla u_\epsilon).\nabla u_\epsilon\,dx\,dt \le \int_Q \Phi\cdot\nabla u\,dx\,dt. \tag{4.35}$$

Using (4.27), we have

$$\begin{aligned}&\int_0^T \int_\Omega a(x,\nabla u_\epsilon)\cdot \nabla u_\epsilon \,dx\,dt\\ &\leq -\int_\Omega \Big[B_{Id}(u_\epsilon(T)) - B_{Id}(u_0)\Big]\,dx + \int_0^T\int_\Omega f_\epsilon u_\epsilon \,dx\,dt.\end{aligned} \tag{4.36}$$

Since $B_{Id}(u_\epsilon(T)) \geq 0$, then by Fatou's lemma, we have

$$\int_\Omega \liminf_{\epsilon\to 0} B_{Id}(u_\epsilon(T))\,dx \leq \liminf_{\epsilon\to 0}\int_\Omega B_{Id}(u_\epsilon(T))\,dx. \tag{4.37}$$

Because of the lower semi-continuity of B_{Id}, we have

$$\int_\Omega B_{Id}(u(T))\,dx \leq \int_\Omega \liminf_{\epsilon\to 0} B_{Id}(u_\epsilon(T))\,dx. \tag{4.38}$$

Inequalities (4.37) and (4.38) imply

$$\int_\Omega B_{Id}(u(T))\,dx \leq \liminf_{\epsilon\to 0}\int_\Omega B_{Id}(u_\epsilon(T))\,dx,$$

i.e.

$$-\liminf_{\epsilon\to 0}\int_\Omega B_{Id}(u_\epsilon(T))\,dx \leq -\int_\Omega B_{Id}(u(T))\,dx.$$

Then, passing to the limit in (4.36) as $\epsilon\to 0$ and according to Lemma 4.6 we have

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\,dx\,dt\\ &\leq -\int_\Omega \Big[B_{Id}(u(T)) - B_{Id}(u(0))\Big]\,dx + \int_0^T\int_\Omega fu\,dx\,dt\\ &= \langle f-u_t, u\rangle.\end{aligned} \tag{4.39}$$

Now, we prove that

$$\iint Qa(x,\nabla u).\nabla\xi\,dx\,dt = \iint Q\Phi\cdot\nabla\xi\,dx\,dt, \tag{4.40}$$

for any $\xi\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$.
By the monotonicity of a, for any $\rho\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$,

$$\iint Qa(x,\nabla\rho).\nabla(u_\epsilon-\rho)\,dx\,dt \leq \iint Qa(x,\nabla u_\epsilon).\nabla(u_\epsilon-\rho)\,dx\,dt. \tag{4.41}$$

Since u_ϵ is a weak energy solution of $\epsilon f_i^\epsilon + u_{i-1}^\epsilon \in (I+\epsilon A)u_i^\epsilon$ then, by [19, Proposition 4.11], ∇u_ϵ converges in measure to ∇u. We can then extract a subsequence such that $\nabla u_\epsilon\to\nabla u$ a.e. in Q. Then according to (2.3), we may apply Lebesgue dominated convergence theorem and pass to the limit in (4.41) as $\epsilon\to 0$ to obtain

$$\liminf_{\epsilon\to 0}\int\int_Q a(x,\nabla u_\epsilon).\nabla(u_\epsilon-\rho)\,dx\,dt \geq \int\int_Q a(x,\nabla\rho).\nabla(u-\rho)\,dx\,dt. \tag{4.42}$$

Combining (4.39) and (4.42), we have

$$\langle f-u_t, u-\rho\rangle \geq \int\int_Q a(x,\nabla\rho).\nabla(u-\rho)\,dx\,dt,$$

for all $\rho\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$.

Choosing $\rho = u + \sigma\xi, \sigma \in \mathbb{R}, \xi \in L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$, we obtain
$$\langle f - u_t, \sigma\xi\rangle \le \sigma \int\int_Q a(x, \nabla(u + \sigma\xi)).\nabla\xi \, dx \, dt. \tag{4.43}$$
Dividing inequality (4.43) by $\sigma > 0$, resp. $\sigma < 0$ and passing to the limit with $\sigma \downarrow 0$, resp. $\sigma \uparrow 0$, we obtain
$$\langle f - u_t, \xi\rangle = \int\int_Q a(x, \nabla u).\nabla\xi \, dx \, dt, \tag{4.44}$$
for any $\xi \in L^{p_-}(0, T; W_0^{1,p(\cdot)}(\Omega))$. By (4.34), we have
$$\begin{aligned} &\int_0^T \int_\Omega \Phi \cdot \nabla\psi \, dx \, dt \\ &= \int_0^T \int_\Omega u\psi_t \, dx \, dt + \int_\Omega u_0\psi(0) \, dx + \int_0^T \int_\Omega f\psi \, dx \, dt \\ &= \langle f - u_t, \psi\rangle. \end{aligned} \tag{4.45}$$
Combining (4.44) and (4.45) yields (4.40). To conclude, we pass to the limit in (4.33) as $\epsilon \to 0$ to obtain
$$\begin{aligned} \int_0^T \int_\Omega f\phi \, dx \, dt = -\int_0^T \int_\Omega u\phi_t \, dx \, dt - \int_\Omega (u\phi)(0) dx \\ + \int_0^T \int_\Omega a(x, \nabla u) \cdot \nabla\phi \, dx \, dt, \end{aligned} \tag{4.46}$$
for all $\phi \in E \cap L^\infty(Q)$. Hence u is a weak solution of (1.1). □

Our aim is to prove that this weak solution is also an entropy solution of (1.1). The proof of this result consists of two main steps. Firstly, we prove $\epsilon-$uniform a-priori-estimates in certain Bochner spaces as well as in appropriate variable exponent Lebesgue spaces for u_ϵ and ∇u_ϵ. Secondly, we pass to the limit in the entropy relation as $\epsilon \to 0$.

5. Existence and uniqueness of an entropy solution

Theorem 5.1. *Let* (2.1)-(2.5) *hold. Let $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$. There exists a unique entropy solution for* (1.1).

The proof of the above theorem is done in several steps.

5.1. A priori estimates.

As $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$ and L^∞ is dense in L^1, then we can find two sequences of functions $\big(f_\epsilon\big)_{\epsilon>0} \subset L^\infty(Q)$ and $\big(u_{0,\epsilon}\big)_{\epsilon>0} \subset L^\infty(\Omega)$ strongly converging respectively to f and u_0 such that
$$\|f_\epsilon\|_{L^1(Q)} \le \|f\|_{L^1(Q)}, \quad \|u_{0,\epsilon}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}. \tag{5.1}$$
Now, let u_ϵ be a weak solution to problem (1.1) with f_ϵ and $u_{0,\epsilon}$ as data, i.e.
$$\begin{aligned} \int_0^T \int_\Omega f_\epsilon\phi \, dx \, dt = -\int_0^T \int_\Omega u_\epsilon\phi_t \, dx \, dt - \int_\Omega u_{0,\epsilon}\phi(0, \cdot) dx \\ + \int_0^T \int_\Omega a(x, \nabla u_\epsilon) \cdot \nabla\phi \, dx \, dt, \end{aligned} \tag{5.2}$$
for all $\phi \in E \cap L^\infty(Q)$.

Lemma 5.2. *The estimates in Lemma 4.5 hold with u replaced by u_ϵ, and all the constants are independent of ϵ, i.e.*

$$\|u_\epsilon\|_{L^\infty(0,T;L^1(\Omega))} \le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}, \tag{5.3}$$

$$\begin{aligned} &\|\nabla T_k(u_\epsilon)\|_{L^{p(\cdot)}(Q)} \\ &\le k \max\Big\{ \big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big)^{1/p_-}, \big(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\big)^{1/p_+} \Big\}. \end{aligned} \tag{5.4}$$

The proof of the above lemma is similar to that of Lemma 4.5.

5.2. **Basic convergence results.** The a priori estimates in lemmas 4.5 and 5.2, together with the $C([0,T];L^1(\Omega))$-convergence guaranteed by nonlinear semigroup theory, imply the following basic convergence results.

Lemma 5.3. *For a subsequence $\big(u_\epsilon\big)_{\epsilon>0}$ as $\epsilon \to 0$:*

$$u_\epsilon \to u \quad a.e. \text{ in } Q, \tag{5.5}$$

$$\nabla T_k(u_\epsilon) \rightharpoonup \nabla T_k(u) \quad \text{in } (L^{p(\cdot)}(Q))^N, \tag{5.6}$$

$$T_k(u_\epsilon) \to T_k(u) \text{in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \tag{5.7}$$

for all $k > 0$.

Proof. Proof of (5.5). Let u_{ϵ_1} and u_{ϵ_2} be two weak solutions of problem (1.1). Choosing $\theta_\epsilon T_1(u_{\epsilon_1} - u_{\epsilon_2})$ as a test function corresponding to u_{ϵ_1} and $\theta_\epsilon T_1(u_{\epsilon_2} - u_{\epsilon_1})$ as a test function corresponding to u_{ϵ_2}, we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \theta_\epsilon (u_{\epsilon_1})_t T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt \\ &= \int_0^T \int_\Omega \theta_\epsilon a(x, \nabla u_{\epsilon_1}) \cdot \nabla T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt \\ &\quad + \int_0^T \int_\Omega \theta_\epsilon f_{\epsilon_1} T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\int_0^T \int_\Omega \theta_\epsilon (u_{\epsilon_2})_t T_1(u_{\epsilon_2} - u_{\epsilon_1})\,dx\,dt \\ &= \int_0^T \int_\Omega \theta_\epsilon a(x, \nabla u_{\epsilon_2}) \cdot \nabla T_1(u_{\epsilon_2} - u_{\epsilon_1})\,dx\,dt \\ &\quad + \int_0^T \int_\Omega \theta_\epsilon f_{\epsilon_2} T_1(u_{\epsilon_2} - u_{\epsilon_1})\,dx\,dt. \end{aligned} \tag{5.9}$$

Adding (5.8) and (5.9), then by using (2.4) and letting ϵ approach zero we have

$$\begin{aligned} &\int_0^{t_1} \int_\Omega (u_{\epsilon_1} - u_{\epsilon_2})_t T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt \\ &= \int_0^{t_1} \int_\Omega \big(a(x, \nabla u_{\epsilon_1}) - a(x, \nabla u_{\epsilon_2})\big) \cdot \nabla T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt \\ &\quad + \int_0^{t_1} \int_\Omega \big(f_{\epsilon_2} - f_{\epsilon_1}\big) T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt \\ &\le \int_0^{t_1} \int_\Omega \big(f_{\epsilon_2} - f_{\epsilon_1}\big) T_1(u_{\epsilon_1} - u_{\epsilon_2})\,dx\,dt. \end{aligned} \tag{5.10}$$

From (5.10) we deduce that

$$
\begin{aligned}
&\int_\Omega \Theta_1(u_{\epsilon_1}-u_{\epsilon_2})(t_1)\,dx \\
&\le \int_\Omega \Theta_1(u_{0,\epsilon_1}-u_{0,\epsilon_2})\,dx + \|f_{\epsilon_2}-f_{\epsilon_1}\|_{L^1(Q)} \\
&\le \|u_{0,\epsilon_1}-u_{0,\epsilon_2}\|_{L^1(\Omega)} + \|f_{\epsilon_2}-f_{\epsilon_1}\|_{L^1(Q)} := a_{\epsilon_1\epsilon_2}.
\end{aligned}
\tag{5.11}
$$

By the definition of Θ_1, we have

$$
\Theta_1(u_{\epsilon_1}-u_{\epsilon_2})(t_1) = \begin{cases} \frac{[(u_{\epsilon_1}-u_{\epsilon_2})(t_1)]^2}{2} & \text{if } |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|<1 \\ |(u_{\epsilon_1}-u_{\epsilon_2})(t_1)| & \text{if } |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|\ge 1. \end{cases}
$$

On the set $\{|u_{\epsilon_1}-u_{\epsilon_2}|\ge 1\}$, we have $\frac{|(u_{\epsilon_1}-u_{\epsilon_2})(t_1)|}{2}\le |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|$. Then, from (5.11) we deduce

$$
\begin{aligned}
&\int_{\{|u_{\epsilon_1}-u_{\epsilon_2}|<1\}} \frac{(u_{\epsilon_1}-u_{\epsilon_2})^2(t_1)}{2}\,dx + \int_{\{|u_{\epsilon_1}-u_{\epsilon_2}|\ge 1\}} \frac{|u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|}{2}\,dx \\
&\le \int_\Omega \Theta_1(u_{\epsilon_1}-u_{\epsilon_2})(t_1)\,dx \le a_{\epsilon_1\epsilon_2}.
\end{aligned}
$$

Using Hölder inequality,

$$
\begin{aligned}
&\int_\Omega |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|\,dx \\
&= \int_{\{|u_{\epsilon_1}-u_{\epsilon_2}|<1\}} |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|\,dx + \int_{\{|u_{\epsilon_1}-u_{\epsilon_2}|\ge 1\}} |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|\,dx \\
&\le \Big(\int_{\{|u_{\epsilon_1}-u_{\epsilon_2}|<1\}} |u_{\epsilon_1}(t_1)-u_{\epsilon_2}(t_1)|^2\,dx\Big)^{1/2} \operatorname{meas}(\Omega)^{1/2} + 2a_{\epsilon_1\epsilon_2} \\
&\le (2\operatorname{meas}(\Omega))^{1/2} a_{\epsilon_1\epsilon_2}^{1/2} + 2a_{\epsilon_1\epsilon_2}.
\end{aligned}
\tag{5.12}
$$

Since $\big(f_\epsilon\big)_{\epsilon>0}$ and $\big(u_{0,\epsilon}\big)_{\epsilon>0}$ are convergent respectively in $L^1(Q)$ and $L^1(\Omega)$, we have $a_{\epsilon_1\epsilon_2}\to 0$ for $\epsilon_1,\epsilon_2\to 0$. Thus from (5.12) we deduce that $\big(u_\epsilon\big)_{\epsilon>0}$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$ and u_ϵ converges to u in $C([0,T];L^1(\Omega))$. Then we find an a.e. convergent subsequence (still denoted by $\big(u_\epsilon\big)_{\epsilon>0}$) in Q such that $u_\epsilon\to u$ a.e. in Q. The proof of (5.5) is complete.

Proof of (5.6) and (5.7). By (5.4), the sequence $\big(\nabla T_k(u_\epsilon)\big)_{\epsilon>0}$ is bounded in $\big(L^{p(\cdot)}(Q)\big)^N$; hence the sequence $\big(T_k(u_\epsilon)\big)_{\epsilon>0}$ is bounded in $W_0^{1,p(\cdot)}(Q)$. Then, up to a subsequence we can assume that for any $k>0$, $\big(T_k(u_\epsilon)\big)_{\epsilon>0}$ converges weakly to σ_k in $W_0^{1,p(\cdot)}(Q)$ and so $\big(T_k(u_\epsilon)\big)_{\epsilon>0}$ converges strongly to σ_k in $L^{p_-}(Q)$. By (5.5), we have $u_\epsilon\to u$ a.e. in Q. As for $k>0, T_k$ is continuous, then $T_k(u_\epsilon)\to T_k(u)$ a.e. in Q and $\sigma_k=T_k(u)$ a.e. in Q, which yields (5.7). Using also the boundedness of $\big(\nabla T_k(u_\epsilon)\big)_{\epsilon>0}$ in $\big(L^{p(\cdot)}(Q)\big)^N$, we can find a subsequence (still denoted by $(u_\epsilon)_{\epsilon>0}$) from $(u_\epsilon)_{\epsilon>0}$ such that $\nabla T_k(u_\epsilon)$ converges weakly to $\nabla T_k(u)$ in $\big(L^{p(\cdot)}(Q)\big)^N$, i.e. (5.6) holds. $\square$

5.3. **Strong convergence.** We start by recalling a suitable time-regularization procedure, which was first introduced by Landes (see [17]) and employed by several authors to solve nonlinear time dependent problems with L^1 or measure data (see e.g. [7]). We denote this time regularized function to $T_n(u)$ by $(T_n(u))_\mu$, with $\mu > 0$. It is defined as the unique solution $(T_n(u))_\mu \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q)$, with $\nabla(T_n(u))_\mu \in \big(L^{p(\cdot)}(Q)\big)^N$, of the equation

$$\partial_t(T_n(u))_\mu + \mu((T_n(u))_\mu - T_n(u)) = 0 \quad \text{in } \mathcal{D}'(Q), \tag{5.13}$$

with the initial condition

$$(T_n(u))_\mu|_{t=0} = w_0^\mu \quad \text{in } \Omega, \tag{5.14}$$

where w_0^μ is a sequence of functions such that

$$\begin{gathered} w_0^\mu \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad \|w_0^\mu\|_{L^\infty(\Omega)} \leq n \\ w_0^\mu \to T_n(u_0) \quad \text{a.e. in } \Omega \text{ as } \mu \to \infty, \\ \frac{1}{\mu}\|w_0^\mu\|_{W_0^{1,p(\cdot)}(\Omega)} \to 0 \quad \text{as } \mu \to \infty. \end{gathered} \tag{5.15}$$

Following [17] we can prove that

$$\begin{gathered} \frac{\partial(T_n(u))_\mu}{\partial t} \in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)) \cap L^\infty(Q), \quad \|(T_n(u))_\mu\|_{L^\infty(Q)} \leq n, \\ (T_n(u))_\mu \to T_n(u) \text{ a.e. in } Q, \text{ weak-* in } L^\infty(Q) \\ \text{and strongly in } L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega)). \end{gathered} \tag{5.16}$$

To continue our proof of Theorem 5.1, we need the following result.

Proposition 5.4. *For all $k > 0$ we have:*

- (i) $a(x,\nabla T_k(u_\epsilon)) \rightharpoonup a(x,\nabla T_k(u))$ *in* $\big(L^{p'(\cdot)}(Q)\big)^N$,
- (ii) $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ *a.e. in Q,*
- (iii) $a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u))\cdot\nabla T_k(u)$ *strongly in $L^1(Q)$ and a.e. in Q,*
- (iv) $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ *in* $\big(L^{p(\cdot)}(Q)\big)^N$.

Proof. (i) The sequence $\big(a(x,\nabla T_k(u_\epsilon))\big)_{\epsilon>0}$ is bounded in $\big(L^{p'(\cdot)}(Q)\big)^N$ according to (2.3). We can extract a subsequence such that $a(x,\nabla T_k(u_\epsilon)) \to \zeta_k$ in $\big(L^{p'(\cdot)}(Q)\big)^N$. We have to show that $\zeta_k = a(x,\nabla T_k(u))$ a.e. in Q. To this end, we take a subsequence $(u_\epsilon)_{\epsilon>0}$ such that $u_\epsilon \to u$ almost everywhere in Q. For $h > 2k$, we introduce the function

$$w_\epsilon = T_{2k}\Big(u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - \big(T_k(u)\big)_\mu\Big),$$

where $\big(T_k(u)\big)_\mu$ is the approximation of $T_k(u)$ defined in (5.13). The use of w_ϵ as a test function to prove the strong convergence of truncations was first introduced in the stationary case in [18], then adapted to parabolic equations in [20]. The advantage in working with w_ϵ is that since

$$\nabla w_\epsilon = \nabla\Big(u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - \big(T_k(u)\big)_\mu\Big)\chi_{E_\epsilon},$$

with $E_\epsilon = \big\{\big|u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - \big(T_k(u)\big)_\mu\big| \leq 2k\big\}$, in particular we have $\nabla w_\epsilon = 0$ if $|u_\epsilon| > h + 4k$. Thus the estimate on $T_k(u_\epsilon)$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ appearing in Lemma 5.3 implies that w_ϵ is bounded in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$. Then by the almost everywhere convergence of u_ϵ to u as $\epsilon\to 0$, we deduce that

$$w_\epsilon \rightharpoonup T_{2k}\Big(u - T_h(u) + T_k(u) - \big(T_k(u)\big)_\mu\Big) \tag{5.17}$$

in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and a.e. in Q.

In the following, we set $M = h+4k$, moreover we will denote by $w(\epsilon,\mu,h)$ all quantities (possibly different) such that

$$\lim_{h\to+\infty}\lim_{\mu\to+\infty}\limsup_{\epsilon\to 0}|w(\epsilon,\mu,h)| = 0. \tag{5.18}$$

Similarly we will write only $w(\epsilon)$ or $w(\epsilon,\mu)$, to mean that the limits are made only on the specified parameters. Choosing w_ϵ as a test function in (5.2) we have

$$\int_0^T\int_\Omega (u_\epsilon)_t w_\epsilon\,dx\,dt + \int_0^T\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla w_\epsilon\,dx\,dt = \int_0^T\int_\Omega f_\epsilon w_\epsilon\,dx\,dt. \tag{5.19}$$

Notice that

$$\begin{aligned}
&|\int_0^T\int_\Omega f_\epsilon w_\epsilon\,dx\,dt| \\
&\le \int_0^T\int_\Omega |f_\epsilon - f||T_{2k}(u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - (T_k(u))_\mu)|\,dx\,dt \\
&\quad + \int_0^T\int_\Omega |fT_{2k}(u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - (T_k(u))_\mu)|\,dx\,dt \\
&\le 2k\int_0^T\int_\Omega |f_\epsilon - f|\,dx\,dt + \int_0^T\int_\Omega |fT_{2k}(u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - (T_k(u))_\mu)|\,dx\,dt.
\end{aligned}$$

Since f_ϵ is strongly compact in $L^1(Q)$, using (5.5), the definition of $\big(T_k(u)\big)_\mu$ and the Lebesgue dominated convergence theorem, we have

$$\lim_{h\to+\infty}\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\Big|\int_0^T\int_\Omega f_\epsilon w_\epsilon\,dx\,dt\Big| \le \lim_{h\to+\infty}\int_0^T\int_\Omega |fT_{2k}(u - T_h(u)|\,dx\,dt = 0.$$

Thus, recalling the notation introduced in (5.18), we have proven that

$$\int_0^T\int_\Omega f_\epsilon w_\epsilon\,dx\,dt = w(\epsilon,\mu,h). \tag{5.20}$$

Let us estimate the second term in (5.19). Since $\nabla w_\epsilon = 0$ if $|u_\epsilon| > M = h+4k$, we have

$$\int_0^T\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla w_\epsilon\,dx\,dx = \int_0^T\int_\Omega a(x,\nabla T_M(u_\epsilon))\cdot\nabla w_\epsilon\,dt\,dt.$$

Next we split the integral in the sets $\{|u_\epsilon| \le k\}$ and $\{|u_\epsilon| > k\}$, so that we have, recalling that $h > 2k$,

$$
\begin{aligned}
&\int_0^T \int_\Omega a(x, \nabla T_M(u_\epsilon)) \cdot \nabla T_{2k}(u_\epsilon - T_h(u_\epsilon) + T_k(u_\epsilon) - (T_k(u))_\mu)\,dx\,dt \\
&= \iint_{\{|u_\epsilon| \le k\}} a(x, \nabla u_\epsilon) \cdot \nabla(u_\epsilon - (T_k(u))_\mu)\,dx\,dt \\
&\quad + \iint_{\{|u_\epsilon| > k\}} a(x, \nabla T_M(u_\epsilon)) \cdot \nabla(u_\epsilon - T_h(u_\epsilon))\,dx\,dt \\
&\quad - \iint_{\{|u_\epsilon| > k\}} a(x, \nabla T_M(u_\epsilon)) \cdot \nabla(T_k(u))_\mu\,dx\,dt := I_1 + I_2 + I_3.
\end{aligned}
\tag{5.21}
$$

Let us estimate I_2. Since $u_\epsilon - T_h(u_\epsilon) = 0$ if $|u_\epsilon| \le h$, using (2.3), Remark 2.1 and Young inequality, we obtain

$$
\begin{aligned}
&|I_2| \\
&= \Big| \iint_{\{|u_\epsilon| > k\}} a(x, \nabla T_M(u_\epsilon)) \cdot \nabla(u_\epsilon - T_h(u_\epsilon))\,dx\,dt \Big| \\
&\le \iint_{\{h \le |u_\epsilon| \le M\}} |a(x, \nabla u_\epsilon)| |\nabla u_\epsilon|\,dx\,dt \\
&\le \iint_{\{h \le |u_\epsilon| \le M\}} C_1(j(x) + |\nabla u_\epsilon|^{p(x)-1}) |\nabla u_\epsilon|\,dx\,dt \\
&\le \iint_{\{h \le |u_\epsilon| \le M\}} C_1 j(x) |\nabla u_\epsilon|\,dx\,dt + \iint_{\{h \le |u_\epsilon| \le M\}} C_1 |\nabla u_\epsilon|^{p(x)}\,dx\,dt \\
&\le \iint_{\{h \le |u_\epsilon| \le M\}} \frac{C_1}{p'_-} |j(x)|^{p'(x)}\,dx\,dt + \iint_{\{h \le |u_\epsilon| \le M\}} \frac{C_1}{p_-} |\nabla u_\epsilon|^{p(x)}\,dx\,dt \\
&\quad + \iint_{\{h \le |u_\epsilon| \le M\}} C_1 |\nabla u_\epsilon|^{p(x)}\,dx\,dt \\
&\le C \iint_{\{h \le |u_\epsilon| \le M\}} C_1 |\nabla u_\epsilon|^{p(x)}\,dx\,dt \\
&\quad + C' \iint_{\{h \le |u_\epsilon| \le M\}} \frac{C_1}{p'_-} |j(x)|^{p'(x)}\,dx\,dt.
\end{aligned}
\tag{5.22}
$$

The functions $j(t,x)$ and $\big(\nabla u_\epsilon\big)_{\epsilon > 0}$ are bounded in $L^{p'_-}(0,T; W_0^{1,p(\cdot)}(\Omega))$ and in $L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega))$ respectively, and meas$\{h \le |u_\epsilon| \le h+4k\}$ converges uniformly to zero as h tends to infinity with respect to ϵ. Then, passing to the limit in (5.22) as $\epsilon \to 0$ and $h \to +\infty$ respectively, and using Lebesgue dominated convergence theorem, we obtain

$$
I_2 = w(\epsilon, h).
$$

For I_3, let us remark that, since $\big(\nabla u_\epsilon\big)_{\epsilon > 0}$ is bounded in $L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega))$, (2.3) implies that $\big(a(x, \nabla T_M(u_\epsilon))\big)_{\epsilon > 0}$ is bounded in $\big(L^{p'(\cdot)}(Q)\big)^N$. The almost everywhere convergence of u_ϵ to u, as $\epsilon \to 0$, implies that $|\nabla T_k(u)| \chi_{\{|u_\epsilon| > k\}}$ strongly converges to zero in $L^{p_-}(0,T; W_0^{1,p(\cdot)}(\Omega))$. So that, by the Lebesgue dominated convergence theorem, we have

$$\limsup_{\epsilon\to 0}\iint_{\{|u_\epsilon|>k\}} a(x,\nabla T_M(u_\epsilon))\cdot\nabla T_k(u)\,dx\,dt=0.$$

Thus, we obtain

$$\begin{aligned}
I_3&=\iint_{\{|u_\epsilon|>k\}} a(x,\nabla T_M(u_\epsilon))\cdot\nabla (T_k(u))_\mu\,dx\,dt\\
&=\iint_{\{|u_\epsilon|>k\}} a(x,\nabla T_M(u_\epsilon))\cdot\nabla T_k(u)\,dx\,dt\\
&\quad+\iint_{\{|u_\epsilon|>k\}} a(x,\nabla T_M(u_\epsilon))\cdot\nabla((T_k(u))_\mu-T_k(u))\,dx\,dt\\
&=w(\epsilon)+\iint_{\{|u_\epsilon|>k\}} a(x,\nabla T_M(u_\epsilon))\cdot\nabla((T_k(u))_\mu-T_k(u))\,dx\,dt.
\end{aligned}$$

Using the fact that $\big(a(x,\nabla T_M(u_\epsilon))\big)_{\epsilon>0}$ is bounded in $\big(L^{p'(\cdot)}(Q)\big)^N$ and thanks to (5.16), we can apply the Lebesgue dominated convergence theorem to obtain

$$\iint_{\{|u_\epsilon|>k\}} a(x,\nabla T_M(u_\epsilon))\cdot\nabla((T_k(u))_\mu-T_k(u))\,dx\,dt=w(\epsilon,\mu),$$

therefore we conclude that $I_3=w(\epsilon,\mu)$.

Then from (5.21), according to the fact that I_2 and I_3 converge to zero, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla w_\epsilon\,dx\,dt\\
&=\iint_{\{|u_\epsilon|\le k\}} a(x,\nabla u_\epsilon)\cdot\nabla(u_\epsilon-(T_k(u))_\mu)\,dx\,dt+w(\epsilon,\mu,h).
\end{aligned}\tag{5.23}$$

Putting together (5.19), (5.20) and (5.23) we have

$$\begin{aligned}
&\int_0^T\int_\Omega (u_\epsilon)_t w_\epsilon\,dx\,dt+\iint_{\{|u_\epsilon|\le k\}} a(x,\nabla u_\epsilon)\cdot\nabla(u_\epsilon-(T_k(u))_\mu)\,dx\,dt\\
&=w(\epsilon,\mu,h).
\end{aligned}\tag{5.24}$$

For the first term of (5.24), we can apply [20, Lemma 2.1] to obtain

$$\int_0^T\int_\Omega (u_\epsilon)_t w_\epsilon\,dx\,dt\ge w(\epsilon,\mu,h).$$

Hence (5.24) becomes

$$\iint_{\{|u_\epsilon|\le k\}} a(x,\nabla u_\epsilon)\cdot\nabla(u_\epsilon-(T_k(u))_\mu)\,dx\,dt\le w(\epsilon,\mu,h).\tag{5.25}$$

Since $\nabla(T_k(u))_\mu\to\nabla T_k(u)$ strongly in $\big(L^{p(\cdot)}(Q)\big)^N$ as $\mu\to+\infty$, we deduce from (5.25) that

$$\int_0^T\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla(T_k(u_\epsilon)-(T_k(u)))\,dx\,dt\le w(\epsilon,\mu,h).\tag{5.26}$$

Therefore, passing to the limit in (5.26) as ϵ tends to zero, μ and h tend to infinity respectively, we deduce that

$$\limsup_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla(T_k(u_\epsilon)-(T_k(u)))\,dx\,dt\le 0. \tag{5.27}$$

Now, let $\varphi\in\mathcal{D}(Q)$ and $\lambda\in\mathbb{R}^*$. Using (5.27) and (2.4), we obtain

$$\begin{aligned}
&\lambda\lim_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla\varphi\,dx\,dt\\
&\ge\limsup_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)+\lambda\varphi\big]\,dx\,dt\\
&\ge\limsup_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)+\lambda\varphi\big]\,dx\,dt\\
&\ge\lambda\int_0^T\int_\Omega a(x,\nabla(T_k(u)-\lambda\varphi))\cdot\nabla\varphi\,dx\,dt.
\end{aligned}\tag{5.28}$$

Dividing (5.28) by $\lambda>0$ and by $\lambda<0$ respectively, passing to the limit with $\lambda\to 0$ it follows that

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla T_k(u_\epsilon))\cdot\nabla\varphi\,dx\,dt=\int_0^T\int_\Omega a(x,\nabla T_k(u)\cdot\nabla\varphi\,dx\,dt.$$

This means that for all $k>0$,

$$\int_0^T\int_\Omega \zeta_k\nabla\varphi\,dx=\int_0^T\int_\Omega a(x,\nabla T_k(u)\cdot\nabla\varphi\,dx\,dt.$$

Hence $\zeta_k=a(x,\nabla T_k(u))$ a.e. in Q and we have

$$a(x,\nabla T_k(u_\epsilon))\rightharpoonup a(x,\nabla T_k(u))\quad\text{in } \big(L^{p'(\cdot)}(Q)\big)^N.$$

(ii) From (5.26), we have

$$\begin{aligned}
&\int_0^T\int_\Omega(a(x,\nabla T_k(u_\epsilon))-a(x,\nabla T_k(u)))\cdot\nabla(T_k(u_\epsilon)-(T_k(u)))\,dx\,dt\\
&\le-\int_0^T\int_\Omega a(x,\nabla T_k(u))\cdot\nabla(T_k(u_\epsilon)-(T_k(u)))\,dx\,dt+w(\epsilon,\mu,h).
\end{aligned}\tag{5.29}$$

The weak convergence of $\nabla T_k(u_\epsilon)$ to $\nabla T_k(u)$ in $\big(L^{p(\cdot)}(Q)\big)^N$ allows to conclude that

$$\limsup_{\epsilon\to 0}\int_0^T\int_\Omega a(x,\nabla T_k(u))\cdot\nabla(T_k(u_\epsilon)-(T_k(u)))\,dx\,dt=0.$$

Therefore, passing to the limit in (5.29) as ϵ tends to zero, μ and h tend to infinity respectively, we deduce that

$$\limsup_{\epsilon\to 0}\int_0^T\int_\Omega(a(x,\nabla T_k(u_\epsilon))-a(x,\nabla T_k(u)))\cdot\nabla(T_k(u_\epsilon)-(T_k(u)))\,dx\,dt=0. \tag{5.30}$$

Now, set

$$g_\epsilon(t,x)=\big[a(x,\nabla u_\epsilon)-a(x,\nabla u)\big]\cdot\nabla\big[T_k(u_\epsilon)-T_k(u)\big]\ge 0.$$

$g_\epsilon(t,x)\to 0$ strongly in $L^1(Q)$ as $\epsilon\to 0$. Up to a subsequence, $g_\epsilon(t,x)\to 0$ a.e. in Q, which means that there exists $\omega\subset Q$ such that meas $(\omega)=0$ and $g_\epsilon(t,x)\to 0$ in $Q\backslash\omega$.

Let $(t,x) \in Q\backslash\omega$. Using assumptions (2.5) and(2.3), it follows that the sequence $\big(\nabla T_k(u_\epsilon(t,x))\big)_{\epsilon>0}$ is bounded in $\mathbb{R}\times\mathbb{R}^N$ and so we can extract a subsequence which converges to some θ in $\mathbb{R}\times\mathbb{R}^N$. Passing to the limit in the expression of $g_\epsilon(t,x)$, it follows that

$$0 = \big[a(x,\theta) - a(x,\nabla T_k(u))\big]\cdot\big[\theta - T_k(u)\big]$$

and it yields $\theta = \nabla T_k(u)$ for all $(t,x) \in Q\backslash\omega$. As the limit does not depend on the subsequence, the whole sequence $\big(\nabla T_k(u_\epsilon(t,x))\big)_{\epsilon>0}$ converges to θ in $\mathbb{R}\times\mathbb{R}^N$. This means that $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ a.e. in Q.

(iii) The continuity of $a(x,\xi)$ with respect to $\xi \in \mathbb{R}\times\mathbb{R}^N$ gives us

$$a(x,\nabla T_k(u_\epsilon)) \to a(x,\nabla T_k(u)) \quad \text{a.e. in } Q.$$

Therefore,

$$a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u))\cdot\nabla T_k(u) \quad \text{a.e. in } Q.$$

Setting $z_\epsilon = a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)$ and $z = a(x,\nabla T_k(u))\cdot\nabla T_k(u)$, we have

$$z_\epsilon > 0, \quad z_\epsilon \to z \text{ a.e. in } Q,\ z \in L^1(\Omega),$$
$$\iint_Q z_\epsilon\,dx\,dt \to \iint_Q z\,dx\,dt$$

and as

$$\iint_Q |z_\epsilon - z|\,dx\,dt = 2\iint_Q (z-z_\epsilon)^+\,dx\,dt + \iint_Q (z_\epsilon - z)\,dx\,dt$$

and $(z-z_\epsilon)^+ \le z$, it follows by using the Lebesgue dominated convergence theorem that

$$\lim_{\epsilon\to 0}\iint_Q |z_\epsilon - z|\,dx\,dt = 0,$$

which implies

$$a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon) \to a(x,\nabla T_k(u))\cdot\nabla T_k(u) \quad \text{strongly in } L^1(Q) \text{ and a.e. in } Q.$$

To prove (iv), we need the following lemmas.

Lemma 5.5 ([15]). *Let $u, u_n \in L^{p(\cdot)}(Q)$, $n = 1,2,\dots$. Then the following statements are equivalent to each other:*

1. $\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0$*;*
2. $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$*;*
3. *u_n converges to u in Q in measure and* $\lim_{n\to\infty}\rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

Next we have a Lebesgue generalized convergence theorem.

Lemma 5.6. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions and f a measurable function such that $f_n \to f$ a.e. in Q. Let $(g_n)_{n\in\mathbb{N}} \subset L^1(Q)$ such that for all $n \in \mathbb{N}$, $|f_n| \le g_n$ a.e. in Q and $g_n \to g$ in $L^1(Q)$. Then*

$$\iint_Q f_n\,dx \to \iint_Q f\,dx.$$

Now, set $f_\epsilon = |\nabla T_k(u_\epsilon)|^{p(x)}$, $f = |\nabla T_k(u)|^{p(x)}$, $g_\epsilon = a(x,\nabla T_k(u_\epsilon))\cdot\nabla T_k(u_\epsilon)$ and $g = a(x,\nabla T_k(u))\cdot\nabla T_k(u)$. We have:

- f_ϵ is a sequence of measurable functions, f is a measurable function and according to (ii), $f_\epsilon \to f$ a.e. in Q.

- Using (iii), we have $(g_\epsilon)_{\epsilon>0} \subset L^1(Q)$, $g_\epsilon \to g$ a.e. in Q, $g_\epsilon \to g$ in $L^1(Q)$ and using (2.5), we have $|f_\epsilon| \leq C g_\epsilon$.

Then, by Lemma 5.6, we have $\iint_Q f_\epsilon \,dx\,dt \to \iint_Q f\,dx\,dt$, which is equivalent to say

$$\iint_Q |\nabla T_k(u_\epsilon)|^{p(x)}\,dx\,dt \to \iint_Q |\nabla T_k(u)|^{p(x)}\,dx\,dt.$$

We deduce from (ii) that the sequence $\big(\nabla T_k(u_\epsilon)\big)_{\epsilon>0}$ converges to $\nabla T_k(u)$ in Q in measure. Then, by Lemma 5.5 we deduce that

$$\lim_{\epsilon\to 0}\iint_Q |\nabla T_k(u_\epsilon) - \nabla T_k(u)|^{p(x)}\,dx\,dt = 0,$$

which is equivalent to saying that $\nabla T_k(u_\epsilon) \to \nabla T_k(u)$ in $\big(L^{p(\cdot)}(Q)\big)^N$. $\square$

5.4. **Existence of entropy solutions.** For a given $a, k > 0$ defines the function $T_{k,a}(s) = T_a(s - T_k(s))$.

$$T_{k,a}(s) = \begin{cases} s - k\ \mathrm{sign}(s) & \text{if } k \leq |s| < k + a,\\ a\ \mathrm{sign}(s) & \text{if } |s| \geq k + a,\\ 0 & \text{if } |s| \leq k. \end{cases}$$

Let u_ϵ be a weak solution of (1.1). Using $\theta_\epsilon T_{k,a}(u_\epsilon)$ as a test function in (5.2) and letting ϵ goes to zero, we find

$$\begin{aligned} &\int_0^{t_1}\int_\Omega (u_\epsilon)_t T_{k,a}(u_\epsilon)\,dx\,dt + \int_0^{t_1}\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla T_{k,a}(u_\epsilon)\,dx\,dt \\ &= \int_0^{t_1}\int_\Omega f_\epsilon T_{k,a}(u_\epsilon)\,dx\,dt. \end{aligned} \tag{5.31}$$

We have

$$\begin{aligned} &\int_0^{t_1}\int_\Omega (u_\epsilon)_t T_{k,a}(u_\epsilon)\,dx\,dt \\ &= \int_0^{t_1}\int_\Omega (u_\epsilon)_t T_a(u_\epsilon - T_k(u_\epsilon))\,dx\,dt \\ &= \int_0^{t_1}\int_{\{|u_\epsilon|>k\}} (u_\epsilon)_t T_a(u_\epsilon \mp k)\,dx\,dt \\ &= \int_0^{t_1}\int_{\{|u_\epsilon|>k\}} (u_\epsilon \mp k)_t T_a(u_\epsilon \mp k)\,dx\,dt \\ &= \int_{\{|u_\epsilon|>k\}} \Theta_a(u_\epsilon \mp k)(t_1)\,dx - \int_{\{|u_{0,\epsilon}|>k\}} \Theta_a(u_{0,\epsilon} \mp k)\,dx. \end{aligned} \tag{5.32}$$

Using (2.5) and (5.32), from (5.31) we obtain

$$\begin{aligned} &\int_{\{|u_\epsilon|>k\}} \Theta_a(u_\epsilon \mp k)(t_1)\,dx - \int_{\{|u_{0,\epsilon}|>k\}} \Theta_a(u_{0,\epsilon} \mp k)\,dx \\ &+ \frac{1}{C}\iint_{\{k\leq |u_\epsilon| \leq k+a\}} |\nabla u_\epsilon|^{p(x)}\,dx\,dt \\ &\leq \int_0^{t_1}\int_\Omega f_\epsilon T_{k,a}(u_\epsilon)\,dx\,dt, \end{aligned}$$

which yields

$$\begin{aligned}&\iint_{\{k\le|u_\epsilon|\le k+a\}}|\nabla u_\epsilon|^{p(x)}\,dx\,dt\\&\le C'\Big(\iint_{\{|u_\epsilon|>k\}}|f_\epsilon|\,dx\,dt+\int_{\{|u_{0,\epsilon}|>k\}}|u_{0,\epsilon}|\,dx\Big).\end{aligned}\tag{5.33}$$

Recalling that $u_\epsilon\to u$ a.e. in Q, we have

$$\lim_{k\to+\infty}\operatorname{meas}\{(t,x)\in Q:|u_\epsilon|>k\}=0\quad\text{uniformly with respect to }\epsilon.$$

Therefore, passing to the limit in (5.33) with ϵ and k tending to zero and infinity respectively, we conclude that

$$\lim_{k\to+\infty}\iint_{\{(t,x)\in Q:k\le|u_\epsilon|\le k+a\}}|\nabla u|^{p(x)}\,dx\,dt=0.$$

Choosing $a=1$, we obtain the renormalized condition (3.2), i.e.,

$$\lim_{k\to+\infty}\iint_{\{(t,x)\in Q:k\le|u_\epsilon|\le k+1\}}|\nabla u|^{p(x)}\,dx\,dt=0.$$

Now, let $\varphi\in\mathcal{D}(Q)$ with $\varphi(.,T)=0$ and S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 satisfying that $\operatorname{supp}S'\subset[-M,M]$ for some $M>0$. Taking $S'(u_\epsilon)\varphi$ as a test function in (5.2), it yields

$$\begin{aligned}&\int_0^T\int_\Omega(u_\epsilon)_tS'(u_\epsilon)\varphi\,dx\,dt+\int_0^T\int_\Omega a(x,\nabla u_\epsilon)\cdot\nabla(S'(u_\epsilon)\varphi)\,dx\,dt\\&=\int_0^T\int_\Omega f_\epsilon S'(u_\epsilon)\varphi\,dx\,dt.\end{aligned}\tag{5.34}$$

We have $(u_\epsilon)_tS'(u_\epsilon)\varphi=\big(S(u_\epsilon)\big)_t\varphi$ and $\nabla(S'(u_\epsilon)\varphi)=S'(u_\epsilon)\nabla\varphi+S"(u_\epsilon)\nabla u_\epsilon\varphi$. Then, equality (5.34) becomes

$$\begin{aligned}&\int_0^T\int_\Omega\big(S(u_\epsilon)\big)_t\varphi\,dx\,dt+\int_0^T\int_\Omega S'(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla\varphi\,dx\,dt\\&\quad+\int_0^T\int_\Omega S"(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon\varphi\,dx\,dt\\&=\int_0^T\int_\Omega f_\epsilon S'(u_\epsilon)\,\varphi\,dx\,dt.\end{aligned}\tag{5.35}$$

We consider the first term on the left-hand side of (5.35). Since S is continuous, (5.5) implies that $S(u_\epsilon)$ converges to $S(u)$ a.e. in Q and weakly$-*$ in $L^\infty(Q)$. Then $(S(u_\epsilon))_t$ converges to $(S(u))_t$ in $\mathcal{D}'(Q)$ as $\epsilon\to0$, that is

$$\int_0^T\int_\Omega\big(S(u_\epsilon)\big)_t\varphi\,dx\,dt\to\int_0^T\int_\Omega(S(u))_t\varphi\,dx\,dt.$$

For the other terms on the left-hand side of (5.35), as $\operatorname{supp}S'\subset[-M,M]$, we have

$$\begin{gathered}S'(u_\epsilon)a(x,\nabla u_\epsilon)=S'(u_\epsilon)a(x,\nabla T_M(u_\epsilon)),\\S"(u_\epsilon)a(x,\nabla u_\epsilon)\cdot\nabla u_\epsilon=S"(u_\epsilon)a(x,\nabla T_M(u_\epsilon))\cdot\nabla T_M(u_\epsilon).\end{gathered}$$

Using (5.5) and Proposition 5.4, we have

$$S'(u_\epsilon)a(x,\nabla T_M(u_\epsilon))\to S'(u)a(x,\nabla T_M(u))\quad\text{in }\big(L^{p'(\cdot)}(Q)\big)^N,$$

$$S"(u_\epsilon)a(x,\nabla T_M(u_\epsilon))\cdot\nabla T_M(u_\epsilon)\to S"(u)a(x,\nabla T_M(u))\cdot\nabla T_M(u)\quad\text{in } L^1(Q).$$

For the right-hand side of (5.35), thanks to the strong convergence of f_ϵ, we have

$$f_\epsilon S'(u_\epsilon)\to fS'(u)\quad\text{in } L^1(Q).$$

Therefore, we can pass to the limit in (5.35) as $\epsilon\to 0$ to obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \big(S(u)\big)_t\varphi\,dx\,dt+\int_0^T\int_\Omega S'(u)a(x,\nabla u)\cdot\nabla\varphi\,dx\,dt\\
&+\int_0^T\int_\Omega S"(u)a(x,\nabla u)\cdot\nabla u\varphi\,dx\,dt\\
&=\int_0^T\int_\Omega fS'(u)\,\varphi\,dx\,dt.
\end{aligned}\tag{5.36}$$

Employing the integration by parts formula for the evolution term, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \big(S(u)\big)_t\varphi\,dx\,dt\\
&=\int_\Omega S(u(T,x))\varphi(T,x)\,dx-\int_\Omega S(u_0)\varphi(0,x)\,dx-\int_0^T\int_\Omega S(u)(\varphi)_t\,dx\,dt\\
&=-\int_\Omega S(u_0)\varphi(0,x)\,dx-\int_0^T\int_\Omega S(u)(\varphi)_t\,dx\,dt\quad(\text{since }\varphi(T,x)=0).
\end{aligned}$$

Therefore, we deduce from (5.36) that

$$\begin{aligned}
&-\int_\Omega S(u_0)\varphi(0,x)\,dx-\int_0^T\int_\Omega S(u)(\varphi)_t\,dx\,dt\\
&+\int_0^T\int_\Omega\big[S'(u)a(x,\nabla u)\cdot\nabla\varphi+S"(u)a(x,\nabla u)\cdot\nabla u\varphi\big]\,dx\,dt\\
&=\int_0^T\int_\Omega fS'(u)\varphi\,dx\,dt.
\end{aligned}\tag{5.37}$$

This complete the proof of the existence of a renormalized solution, and then of the entropy solution (cf. Theorem 3.6).

5.5. **Uniqueness of the entropy solution.** Now, we prove the uniqueness of the entropy solution. By Theorem 3.6, it is enough to prove the uniqueness of the renormalized solution. Let u and v be two renormalized solutions for problem (1.1). Let S_n be defined as in (4.19). We choose $T_k\big(S_n(u)-S_n(v)\big)$ as a test function in both the equations solved by u and v. Subtracting the equations, we have

$$\begin{aligned}
&\int_0^T\int_\Omega\big(S_n(u)-S_n(v)\big)_tT_k\big(S_n(u)-S_n(v)\big)\,dx\,dt\\
&+\int_0^T\int_\Omega\big(S_n'(u)a(x,\nabla u)-S_n'(v)a(x,\nabla v)\big)\cdot\nabla T_k\big(S_n(u)-S_n(v)\big)\,dx\,dt\\
&+\int_0^T\int_\Omega\big(S_n''(u)a(x,\nabla u)\cdot\nabla u-S_n''(v)a(x,\nabla v)\cdot\nabla v\big)T_k\big(S_n(u)-S_n(v)\big)\,dx\,dt\\
&=\int_0^T\int_\Omega f\big(S_n'(u)-S_n'(v)\big)T_k\big(S_n(u)-S_n(v)\big)\,dx\,dt.
\end{aligned}\tag{5.38}$$

We set

$$J_0 = \int_0^T \int_\Omega \big(S_n(u) - S_n(v)\big)_t T_k\big(S_n(u) - S_n(v)\big)\,dx\,dt$$

$$J_1 = \int_0^T \int_\Omega \big(S_n'(u)a(x,\nabla u) - S_n'(v)a(x,\nabla v)\big)\cdot\nabla T_k\big(S_n(u) - S_n(v)\big)\,dx\,dt$$

$$J_2 = \int_0^T \int_\Omega \big(S_n''(u)a(x,\nabla u)\cdot\nabla u - S_n''(v)a(x,\nabla v)\cdot\nabla v\big)T_k\big(S_n(u) - S_n(v)\big)\,dx\,dt$$

$$J_3 = \int_0^T \int_\Omega f\big(S_n'(u) - S_n'(v)\big)T_k\big(S_n(u) - S_n(v)\big)\,dx\,dt.$$

We estimate J_0, J_1, J_2 and J_3 one by one. Recalling the definition of $\Theta_k(r)$, J_0 can be written as

$$J_0 = \int_\Omega \Theta_k\big(S_n(u) - S_n(v)\big)(T)\,dx - \int_\Omega \Theta_k\big(S_n(u) - S_n(v)\big)(0)\,dx.$$

Because u and v have the same initial condition, and by the properties of Θ_k, we obtain

$$J_0 = \int_\Omega \Theta_k\big(S_n(u) - S_n(v)\big)(T)\,dx \ge 0. \tag{5.39}$$

We deal with J_1 splitting it as below

$$\begin{aligned}
J_1 = &\iint_{\{|S_n(u)-S_n(v)|\le k\}\cap\{|u|\le n,|v|\le n\}} \big(a(x,\nabla u) - a(x,\nabla v)\big)\cdot\nabla(u-v)\,dx\,dt\\
&+ \iint_{\{|S_n(u)-S_n(v)|\le k\}\cap\{|u|\le n,|v|>n\}} \big(a(x,\nabla u) - S_n'(v)a(x,\nabla v)\big)\\
&\cdot\nabla(u - S_n(v)\,dx\,dt\\
&+ \iint_{\{|S_n(u)-S_n(v)|\le k\}\cap\{|u|>n\}} \big(S_n'(u)a(x,\nabla u) - S_n'(v)a(x,\nabla v)\big)\\
&\cdot\nabla(S_n(u) - S_n(v)\,dx\,dt := J_1^1 + J_1^2 + J_1^3.
\end{aligned}$$

Since $\{|S_n(u) - S_n(v)| \le k, |u| > n\} \subset \{|u| > n, |v| > n - k\}$, we have, using the fact that $S_n'(t) = 0$ if $|t| > n+1$ and $|S_n'(t)| \le 1$:

$$\begin{aligned}
|J_1^3| \le &\iint_{\{n\le|u|\le n+1\}} |a(x,\nabla u)||\nabla u|\,dx\,dt\\
&+ \iint_{\{n\le|u|\le n+1\}\cap\{n-k\le|v|\le n+1\}} |a(x,\nabla u)||\nabla v|\,dx\,dt\\
&+ \iint_{\{n\le|u|\le n+1\}\cap\{n-k\le|v|\le n+1\}} |a(x,\nabla v)||\nabla u|\,dx\,dt\\
&+ \iint_{\{n-k\le|v|\le n+1\}} |a(x,\nabla v)||\nabla v|\,dx\,dt.
\end{aligned} \tag{5.40}$$

Using assumption (2.3) and Young's inequality, from the first integral in the right-hand side of (5.40), we obtain

$$\begin{aligned}
&\iint_{\{n\le|u|\le n+1\}} |a(x,\nabla u)||\nabla u|\,dx\,dt\\
&\le \iint_{\{n\le|u|\le n+1\}} C_1(j(t,x) + |\nabla u|^{p(x)-1})|\nabla u|\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\le \iint_{\{n\le|u|\le n+1\}} C_1 j(t,x)|\nabla u|\,dx\,dt + \iint_{\{n\le|u|\le n+1\}} C_1|\nabla u|^{p(x)}\,dx\,dt\\
&\le \iint_{\{n\le|u_\epsilon|\le n+1\}} \frac{C_1}{p'_-}|j(x)|^{p'(x)}\,dx\,dt + \iint_{\{n\le|u|\le n+1\}} \frac{C_1}{p_-}|\nabla u|^{p(x)}\,dx\,dt\\
&\quad + \iint_{\{n\le|u|\le n+1\}} C_1|\nabla u|^{p(x)}\,dx\,dt\\
&\le C\iint_{\{n\le|u|\le n+1\}} |\nabla u|^{p(x)}\,dx\,dt + C'\iint_{\{n\le|u|\le n+1\}} |j(x)|^{p'(x)}\,dx\,dt.
\end{aligned}
$$

Function $j(x)$ is bounded in $L^{p'_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and meas$\{n\le|u_\epsilon|\le n+1\}$ converges uniformly to zero as n tends to infinity. Using the condition (3.2), we can conclude that

$$
\lim_{n\to+\infty}\iint_{\{n\le|u|\le n+1\}} |a(x,\nabla u)||\nabla u|\,dx\,dt = 0.
$$

Similarly, we prove that all the other integrals in the right-hand side of (5.40) converge to zero as $n\to+\infty$. Thus J_1^3 converges to zero. Changing the roles of u and v, the same arguments prove that J_1^2 also converges to zero. We use Fatou's lemma to obtain

$$
\liminf_{n\to+\infty} J_1 \ge \iint_{\{|u-v|\le k\}} \big(a(x,\nabla u)-a(x,\nabla v)\big)\cdot\nabla(u-v)\,dx\,dt. \tag{5.41}
$$

Let us study the limit of J_2 now. We have

$$
\begin{aligned}
J_2 &= \int_0^T\int_\Omega S''_n(u)a(x,\nabla u)\cdot\nabla u T_k\big(S_n(u)-S_n(v)\big)\,dx\,dt\\
&\quad + \int_0^T\int_\Omega S''_n(v)a(x,\nabla v)\cdot\nabla v T_k\big(S_n(v)-S_n(u)\big)\,dx\,dt := J_2^1 + J_2^2.
\end{aligned}
$$

By symmetry between J_2^1 and J_2^2, it is sufficient to prove that J_2^1 tends to zero.

Since $|S''_n(s)|\le 1$ and $S''_n(s)\ne 0$ only if $|s|\in[n,n+1]$, using (2.3) we can write

$$
\begin{aligned}
\big|J_2^1\big| &\le k\iint_{\{n\le|u|\le n+1\}} |a(x,\nabla u)\cdot\nabla u|\,dx\,dt\\
&\le k\int_{\{n\le|u|\le n+1\}} C_1\big(j(x)+|\nabla u|^{p(x)-1}\big)|\nabla u|\,dx\,dt\\
&\le k\int_\Omega C_1\Big(j(x)+|\nabla u|^{p(x)-1}\Big)|\nabla u|1_{\{n\le|u|\le n+1\}}\,dx\,dt \to 0 \quad \text{as } n\to+\infty.
\end{aligned}
$$

We conclude that

$$
\lim_{n\to+\infty} J_2 = 0. \tag{5.42}
$$

Let us recall that by definition of S_n we have that S'_n converges to 1 for every s in $\mathbb{R}$. Then

$$
f(S'_n(u)-S'_n(v))\to 0 \quad \text{strongly in } L^1(Q) \text{ as } n\to+\infty.
$$

Using the Lebesgue dominated convergence theorem, we deduce that

$$
\lim_{n\to+\infty} J_3 = 0. \tag{5.43}
$$

Putting together (5.39), (5.41), (5.42) and (5.43), from (5.38), we obtain that as n tends to infinity,

$$\iint_{\{|u-v|\le k\}} \big(a(x,\nabla u)-a(x,\nabla v)\big)\cdot\nabla(u-v)\,dx\,dt\le 0$$

and then letting k gœs to infinity (recall that u and v are finite a.e. in Q), we deduce that

$$\iint_Q \big(a(x,\nabla u)-a(x,\nabla v)\big)\cdot\nabla(u-v)\,dx\,dt\le 0.$$

The strict monotonicity assumption (2.4) then implies that $\nabla u=\nabla v$ a.e. in Q. Then, let $\xi_n=T_1\big(T_{n+1}(u)-T_{n+1}(v)\big)$. We have $\xi_n\in L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$ and, since $\nabla u=\nabla v$ a.e. in Q,

$$\nabla\xi_n=\begin{cases}0 & \text{on } \{|u|\le n+1,|v|\le n+1\}\\ & \quad\cup\{|u|>n+1,|v|>n+1\}\\ 1_{\{|u-T_{n+1}(v)|\le 1\}}\nabla u & \text{on } \{|u|\le n+1,|v|>n+1\}\\ -1_{\{|v-T_{n+1}(u)|\le 1\}}\nabla v & \text{on } \{|u|>n+1,|v|\le n+1\}.\end{cases}$$

But, if $|s|>n+1,|t|\le n+1$ and $|t-T_{n+1}(s)|\le 1$, then $n\le|t|\le n+1$, which implies

$$\begin{aligned}\int_Q|\nabla\xi_n|^{p(x)}\,dx\,dt&\le\int_{\{n\le|u|\le n+1\}}|\nabla u|^{p(x)}\,dx\,dt+\int_{\{n\le|v|\le n+1\}}|\nabla v|^{p(x)}\,dx\,dt\\&\to 0\quad\text{as } n\to+\infty.\end{aligned}$$

Then, $\xi_n\to 0$ in $L^{p_-}(0,T;W_0^{1,p(\cdot)}(\Omega))$, and thus in $\mathcal{D}'(Q)$ as $n\to+\infty$. Since $\xi_n\to T_1(u-v)$ a.e. as $n\to+\infty$ and remains bounded by 1, we also have $\xi_n\to T_1(u-v)$ in $\mathcal{D}'(Q)$. Hence, $T_1(u-v)=0$, i.e. $u=v$ on Q. Therefore we obtain the uniqueness of the renormalized solution to (1.1), and then the uniqueness of the entropy solution.

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References

[1] H. W. Alt, S. Luckhaus; *Quasi-linear elliptic-parabolic differential equations*, Math. Z., **183** (1983), 311-341.

[2] S. N. Antontsev, J. F. Rodrigues; *On stationary thermo-rheological viscous flows*, Annal. del. Univ. de Ferrara. 52 (2006), 19-36.

[3] M. Bendahmane, P. Wittbold, A. Zimmermann; *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1−data*, J. Diff. Equ., 249 (2010), 1483-1515.

[4] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez; *An L^1 theory of existence and uniqueness of nonlinear elliptic equations*, Ann Scuola Norm. Sup. Pisa., **22**, No. 2 (1995), 240-273.

[5] Ph. Bénilan, M. G. Crandal; *Completely accretive operators, Semigroup theory and evolution equations*, Proc. 2nd Int. Conf., Delft / Neth. 1989, Lect. Notes Pure Appl. Math. 135, (1991), 41-75.

[6] Ph. Bénilan, M. G. Crandal, A. Pazy; *Evolution equations governed by accretive operators*, forthcoming book.

[7] D. Blanchard, F. Murat; *Renormalized solutions of nonlinear parabolic problems with L^1 data: Existence and Uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A, **127** (6) (1997), 1137-1152.

[8] B. K. Bonzi, S. Ouaro; *Entropy solutions for doubly nonlinear elliptic problems with variable exponent*, J. Math. Anal. Appl. **370**, no.2 (2010), 392-405.
[9] J. Carillo; *Entropy solutions for nonlinear degenerate problems*, Arch. Ration. Mech. Anal. **147** No.4 (1999), 269-361.
[10] Y. Chen, S. Levine, M. Rao; *Variable exponent, linear growth functionals in image restoration*, SIAM. J. Appl. Math., **66** (2006), 1383-1406.
[11] L. Diening; *Theoritical and numerical results for electrorheological fluids*, Ph. D. thesis, University of Freiburg, Germany, 2002.
[12] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka; *Lebesgue and Sobolev spaces with variable exponents*, Springer, (2010).
[13] R. J. DiPerna, P.L. Lions; *On the Cauchy problem for Boltzmann equations: Global existence and weak stability*, Ann. of Math.(2) **130** (1989), 321-366.
[14] J. Droniou, A. Prignet; *Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data*, No DEA 14 (2007), no. 1-2, 181-205.
[15] X. Fan, D. Zhao; *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl., **263** (2001), 424-446.
[16] O. Kovacik, J. Rakosnik; *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, czech. Math. J., **41** (1991), 592-618.
[17] R. Landes; *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A, **89** (3-4)(1981), 217-237.
[18] C. Leone, A. Porretta; *Entropy solutions for nonlinear elliptic equations in L^1*, Nonlinear Anal., **32** (3) (1998), 325-334.
[19] S. Ouaro, S. Traoré; *Existence and uniqueness of entropy solutions to nonlinear elliptic problems with variable growth*, Int. J. Evol. Equ., **4** (2009), no. 4, 451-471.
[20] A. Porretta; *Existence results for nonlinear parabolic equations via strong convergence of truncations*, Ann.Mat.Pura Appl., **177** (4)(1999), 143-172.
[21] K. R. Rajagopal, M. Ruzicka; *Mathematical Modeling of Electrorheological Materials*, Contin. Mech. Thermodyn., **13** (2001), 59-78.
[22] M. Ruzicka; *Electrorheological fluids: modelling and mathematical theory*, Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2002.
[23] P. Wittbold, A. Zimmermann; *Existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponents and L^1-data*, Nonlinear Analysis TMA, **72** (2010), 2990-3008.
[24] C. Zhang, S. Zhou; *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and L^1-data*, J. Differ. Equations, **248** (2010), 1376-1400.

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