Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 304, pp. 1–24.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NON-HOMOGENEOUS PROBLEM FOR FRACTIONAL LAPLACIAN INVOLVING CRITICAL SOBOLEV EXPONENT

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Communicated by Binlin Zhang

Abstract. In this article, we study the existence of positive solutions for the nonhomogeneous fractional equation involving critical Sobolev exponent

$$(-\Delta)^s u + \lambda u = u^p + \mu f(x), \quad u > 0 \quad \text{in } \Omega,$$
$$u = 0, \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $0 < 2s < \min\{N, 2\}$, λ and $\mu > 0$ are two parameters, $p = \frac{N+2s}{N-2s}$ and $f \in C^{0,\alpha}(\bar{\Omega})$, where $\alpha \in (0, 1)$. $f \geq 0$ and $f \not\equiv 0$ in Ω. For some λ and N, by the barrier method and mountain pass lemma, we prove that there exists $0 < \bar{\mu} := \bar{\mu}(s, \mu, N) < +\infty$ such that there are exactly two positive solutions if $\mu \in (0, \bar{\mu})$ and no positive solutions for $\mu > \bar{\mu}$. Moreover, if $\mu = \bar{\mu}$, there is a unique solution $(\bar{\mu}; u_{\bar{\mu}})$, which means that $(\bar{\mu}; u_{\bar{\mu}})$ is a turning point for the above problem. Furthermore, in case $\lambda > 0$ and $N \geq 6s$ if Ω is a ball in $\mathbb{R}^N$ and f satisfies some additional conditions, then a uniqueness existence result is obtained for $\mu > 0$ small enough.

1. Introduction and main results

In this article, we focus our attention on the non-homogeneous fractional problems. To be more precise, we consider the existence of multiple positive solutions for the following nonlinear elliptic equations involving the fractional Laplacian

$$\begin{gathered} (-\Delta)^s u + \lambda u = u^p + \mu f(x), \quad u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.1}$$

where $s \in (0, 1)$ is fixed, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, λ and $\mu > 0$ are two parameters, $p = 2_s^* - 1$ where $2_s^* = \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent. Moreover, $f(x)$ is a non-homogeneous perturbation satisfying following assumption:

(A1) $f \in C^{0,\alpha}(\bar{\Omega})$, where $\alpha \in (0, 1)$. $f \geq 0$ and $f \not\equiv 0$ in Ω.

The fractional Laplacian $(-\Delta)^s$ is a classical linear integro-differential operator of order $2s$ which gives the standard Laplacian when $s = 1$.

2010 *Mathematics Subject Classification.* 35A15,35J60, 46E35.
Key words and phrases. Non-homogeneous; fractional Laplacian; critical Sobolev exponent; variational method.

Submitted September 23, 2017. Published December 11, 2017.

A range of powers of particular interest is $s \in (0,1)$ and we can write the operator as

$$(-\Delta)^s u(x) = C_{N,s}\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy, \quad x \in \mathbb{R}^N, \quad u \in \mathcal{S}(\mathbb{R}^N), \tag{1.2}$$

where P.V. is the principal value, $C_{N,s}$ is a normalization constant and $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying $\mathcal{C}^\infty$ functions in $\mathbb{R}^N$. For an elementary introduction to the fractional Laplacian and fractional Sobolev spaces we refer the readers to [15, 20].

The motivation to study problem (1.1) comes from the nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

Solutions of (1.3) are standing wave solutions of the fractional Schrödinger equation of the form

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - f(x,|\psi|), \quad x \in \mathbb{R}^N. \tag{1.4}$$

that is solutions of the form $\psi(x,t) = e^{-iEt}u(x)$, where E is a constant, $u(x)$ is a solution of (1.3). The fractional Schrödinger equation is a fundamental equation in fractional quantum mechanics. It was discovered by Laskin ([18, 19]) as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths, where the Feynman path integral leads to the classical Schrödinger equation, and the path integral Lévy trajectories leads to the fractional Schrdinger equation. Different to the classical Laplacian operator, the usual analysis tools for elliptic PDEs can not be directly applied to (1.3) since $(-\Delta)^s$ is a nonlocal operator. In the remarkable work of Caffarelli-Silvestre [7], the authors expressed the nonlocal operator $(-\Delta)^s$ as a Dirichlet-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space.

Since then, problems with the fractional Laplacian have been extensively studied, especially on the existence and nonexistence of positive solutions, multiple solutions, ground states and regularity, see for example, [1, 4, 6, 7, 10, 12, 22, 26, 27, 28, 29, 30, 32, 36] and the references therein. In particular, by using definition (1.2), the Brézis-Nirenberg type problem was discussed in [1, 28]. On the other hand, by adapting the s-harmonic extension introduced by Caffarelli and Silvestre [7], Cabré and Tan [5] and Tan [32] investigated the Brézis-Nirenberg type problem for the special case $s = \frac{1}{2}$. For the general case $0 < s < 1$, Colorado et al. in [1] studied the concave-convex elliptic problem involving the fractional Laplacian. For the related results about the nonhomogeneous fractional Laplacian equations, for example, we refer to [25, 24, 35, 34] and the references therein.

In the local case that $s = 1$, (1.1) reduce to the equation

$$\begin{gathered} -\Delta u + \lambda u = u^{2^*-1} + \mu f(x), \quad u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

By using variational methods, the existence of multiple positive solutions and nonexistence results for classical non-homogeneous elliptic equation like (1.5) have been studied, see [8, 9, 14, 21] and the references therein. Naito and Sato [21] considered the problem (1.5) on the bounded domains. By using variational methods and Pohozaev identity, the authors investigate the multiplicity of positive solutions to the problem and find the phenomenon depending on the space dimension N. Precisely, they showed that the situation is drastically different between the cases $N = 3, 4, 5$ and $N \geq 6$ if $\mu > 0$.

It is nature to ask whether we can find multiple positive solutions of (1.5) if we replace the Laplacian operator $-\Delta$ by the fractional Laplacian operator $(-\Delta)^s$? As far as we know such a problem was not considered before. Firstly, Since (1.1) has no trivial solutions, it presents specific mathematical difficulties. Secondly, as we mention above, the fractional Laplacian operator $(-\Delta)^s$ is nonlocal, and this brings some essential difference with the elliptic equations with the classical Laplacian operator, such as regularity, maximum principle, Pohozaev identity and so on.

Before presenting our main results, we first give some notation. Let λ_1 be the first eigenvalue of the non-local operator $(-\Delta)^s$ with homogeneous Dirichlet condition on Ω (see [28]). We denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the so-called *Gagliardo norm*

$$\|g\|_{H^s(\mathbb{R}^N)} = \|g\|_{L^2(\mathbb{R}^N)} + \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|g(x)-g(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}, \tag{1.6}$$

and $X_0^s(\Omega)$ is the function space defined as

$$X_0^s(\Omega) = \Big\{u \in H^s(\mathbb{R}^N) : u = 0 \quad \text{a.e. in } \mathbb{R}^N\backslash\Omega\Big\}. \tag{1.7}$$

We refer to [28, 29] for a general definition of $X_0^s(\Omega)$ and its properties and to [15] for an account of the properties of $H^s(\mathbb{R}^N)$. In $X_0^s(\Omega)$ we can consider the norm

$$\|v\|_{X_0^s(\Omega)} = \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}.$$

The pair $(X_0^s(\Omega), \|\cdot\|_{X_0^s(\Omega)})$ yields a Hilbert space (see for instance [15]) with scalar product

$$\langle u,v\rangle_{X_0^s(\Omega)} = \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}. \tag{1.8}$$

We also consider another norm in $X_0^s(\Omega)$,

$$\|u\|_\lambda = \Big(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy + \lambda\int_\Omega |u|^2dx\Big)^{1/2}. \tag{1.9}$$

If $\lambda > -\lambda_1$, $\|\cdot\|_\lambda$ is equivalent with $\|\cdot\|_{X_0^s(\Omega)}$, see [15] for more details.

Observe that by [15], if $u, v \in X_0^s(\Omega)$, then

$$\int_\Omega v(-\Delta)^s u dx = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v dx = \langle u,v\rangle_{X_0^s(\Omega)}. \tag{1.10}$$

This leads us to define the solutions to our problem (1.1) in a variational framework. In this paper, we also suppose that $f \in (X_0^s(\Omega))'$, where $(X_0^s(\Omega))'$ denote the dual space of $X_0^s(\Omega)$.

Definition 1.1. We say that $u \in X_0^s(\Omega)$ is a positive solution of (1.1) if $u > 0$ a.e. in Ω such that

$$\begin{aligned} &\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,dx\,dy + \lambda\int_\Omega u\varphi dx \\ &= \int_\Omega u^p\varphi dx + \mu\int_\Omega f\varphi dx \end{aligned} \tag{1.11}$$

for every $\varphi \in X_0^s(\Omega)$.

Definition 1.2. For any fixed $\mu > 0$, we say that $\underline{u}_\mu$ is a positive *minimal* solution of (1.1) if $\underline{u}_\mu$ satisfies $0 < \underline{u}_\mu \leq u_\mu$ in Ω for any positive solution u_μ of (1.1).

In our context, the fractional Sobolev constant is given by

$$S(N,s) := \inf_{v \in H^s(\mathbb{R}^N)\setminus\{0\}} Q_{N,s}(v) > 0, \tag{1.12}$$

where

$$Q_{N,s}(v) := \frac{\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,dx\,dy}{(\int_{\mathbb{R}^N} |v(x)|^{2_s^*}\,dx)^{2/2_s^*}}, \quad v \in H^s(\mathbb{R}^N)$$

is the associated Rayleigh quotient. The constant $S(N,s)$ is well defined and independent of the domain (see for instance [1]). By [13], $S(N,s)$ is attained by a family of functions

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2s)/2}}{(|x|^2+\varepsilon^2)^{(N-2s)/2}}, \quad \varepsilon > 0, \tag{1.13}$$

that is

$$\big\|(-\Delta)^{s/2}u_\varepsilon\big\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = S(N,s)\|u_\varepsilon\|^2_{L^2(\mathbb{R}^N)}. \tag{1.14}$$

The main goal of this paper is to exhibit the existence and nonexistence results for (1.1) with more general nonlinear term f under some weaker assumptions. Our main results are as follows:

Theorem 1.3. *Let* (A1) *hold and $\lambda > -\lambda_1$. Then, there exists $\bar{\mu} \in (0,+\infty)$ such that*

- (i) *if $0 < \mu < \bar{\mu}$, the problem* (1.1) *has a positive minimal solution $\underline{u}_\mu \in X_0^s(\Omega)$. Furthermore, $\underline{u}_\mu$ is increasing in μ for $\mu \in (0,\bar{\mu})$, and $\underline{u}_\mu \to 0$ in $X_0^s(\Omega)$ as $\mu \to 0$;*
- (ii) *if $\mu = \bar{\mu}$, the problem* (1.1) *has a unique positive solution in $X_0^s(\Omega)$;*
- (iii) *if $\mu > \bar{\mu}$, the problem* (1.1) *has no positive solution in $X_0^s(\Omega)$.*

Remark 1.4. There is no positive solution of (1.1) if $\lambda \leq -\lambda_1$. In fact, assume to the contrary that there exists a positive solution u of (1.1) with $\lambda \leq -\lambda_1$. Let φ_1 be the eigenfunction corresponding to the first eigenvalue λ_1 with $\varphi_1 > 0$ in Ω. Then, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi_1 dx - \lambda_1 \int_\Omega u\varphi_1 dx \\ &\geq \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi_1 dx + \lambda \int_\Omega u\varphi_1 dx \\ &= \int_\Omega (u^p\varphi_1 + \mu f\varphi_1)dx > 0. \end{aligned}$$

This is a contradiction.

Theorem 1.3 indicates that equation (1.1) has a minimal solution $\underline{u}_\mu \in X_0^s(\Omega)$ for $0 < \mu \leq \bar{\mu}$, unique positive solution for $\mu = \bar{\mu}$, and has no solution for $\mu > \bar{\mu}$. A natural questions is whether there are more solutions for some $0 < \mu \leq \bar{\mu}$, or analogous to Theorem 1.3, the uniqueness result hold for some special μ. Our main results in this direction can be stated as follows.

Theorem 1.5. *Assume* (A1) *holds. Then*

(i) *if* $0 < \mu < \bar{\mu}$, (1.1) *has a second positive solution* $\bar{u}_\mu \in X_0^s(\Omega)$ *satisfies* $\bar{u}_\mu > \underline{u}_\mu$ *in* Ω *for* $\lambda \in (-\lambda_1, 0]$ *and* $N > 2s$*; or* $\lambda > 0$ *and* $2s < N < 6s$*. Moreover,* $(\bar{\mu}; u_{\bar{\mu}})$ *is a bifurcation point for problem* (1.1)*;*

(ii) *there exists* $\mu^* = \mu^*(\lambda) \in (0, \bar{\mu})$ *such that* (1.1) *has a second positive solution* $\bar{u}_\mu \in X_0^s(\Omega)$ *satisfies* $\bar{u}_\mu > \underline{u}_\mu$ *for every* $\mu^* \le \mu < \bar{\mu}$ *if* $\lambda > 0$, $N \ge 6s$.

Theorem 1.6. *Assume* (A1) *holds,* $\lambda > 0$*, and* $N \ge 6s$. $\Omega = \{x \in \mathbb{R}^N : |x| < R\}$ *with some* $R > 0$*, and let* $f = f(|x|)$ *be radially symmetric about the origin and* $f(r)$ *is decreasing in* $r \in [0, R]$*. Then, there exists* $\mu_* \in (0, \mu^*)$ *such that* (1.1) *has a unique positive solution* $\underline{u}_\mu$ *for* $\mu \in (0, \mu_*]$.

By Theorems 1.5 and 1.6, it is obviously that the existence of the second solution depend on λ and the space dimension N. To prove Theorems 1.5 and 1.6, we consider the auxiliary equation

$$(-\Delta)^s v + \lambda v = (v + \underline{u}_\mu)^p - \underline{u}_\mu^p \quad \text{in } \Omega, \quad v \in X_0^s(\Omega) \tag{1.15}$$

by classical Mountain-Pass Lemma and variational methods.

The rest of this article is organized as follows. In Section 2, we first present variational framework to deal with problem (1.1), Then we show the existence of positive minimal solutions to (1.1) and prove Theorem 1.3. In Section 3, by studying the auxiliary equation (1.15), we give the proof of Theorem 1.5. At last, in Section 4, we prove Theorem 1.6.

2. Existence and properties of minimal solutions

In this section, we show the existence of positive minimal solutions to (1.1) and present some properties of the solutions which will be used in the sequel. Now we give a Maximum Principle which will be used frequently in our text.

Proposition 2.1 (Maximum principle)**.** *If* (A1) *holds,* $u \ge 0$ *is a solution of* (1.1)*, then either* $u \equiv 0$ *in* Ω *or* u *is strictly positive in* Ω.

Proof. Let $k(x,u) = -\lambda u + u^{2_s^*-1} + \mu f$, then there exists $C > 0$ which is independent with u such that $|k(x,u)| \le C(1 + |u^{2_s^*-1}|)$. Then by [1, Proposition 2.2], we have $u \in L^\infty(\Omega)$. Moreover, similar as the proof of [30, Proposition 2.1.9], we deduce that $u \in C^{0,\gamma}(\Omega)$ for any $0 < \gamma < 2s$ if $2s \le 1$, or $u \in C^{1,\gamma}(\Omega)$ for any $0 < \gamma < 2s-1$ if $2s > 1$.

We will discuss this problem into following two cases.

Case 1: $-\lambda_1 < \lambda \le 0$. In this case, we will get $(-\Delta)^s u \ge 0$. Then by [30, Proposition 2.1.7], we have $u > 0$.

Case 2: $\lambda > 0$. Since $u \ge 0$ is a solution of (1.1), for any $\varphi \ge 0$ and $\varphi \in X_0^s(\Omega)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dx \, dy + \lambda \int_\Omega u\varphi \\
&= \int_\Omega u^{2_s^*-1} \varphi dx + \mu \int_\Omega f\varphi dx \ge 0.
\end{aligned}$$

Then u is a super-solution of

$$(-\Delta)^s u = -\lambda u + u^{2_s^*-1} + \mu f \quad \text{in } \Omega,$$

$$u \in X_0^s(\Omega).$$

Thus by [23, Theorem 1.2], we conclude that $u > 0$ in Ω. $\square$

Taking into account that we are looking for positive solutions for problem (1.1), we will consider the Dirichlet problem

$$\begin{gathered} (-\Delta)^s u + \lambda u = (u_+)^p + \mu f(x) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{2.1}$$

where $u_+ := \max\{u, 0\}$. The crucial observation here is that, by Proposition 2.1, if u is a solution of (2.1) then u is strictly positive in Ω and, therefore, it is also a solution of (1.1).

The energy functional related to the problem (2.1) is given by

$$\begin{aligned} I_{\lambda,\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy + \frac{\lambda}{2} \int_\Omega u^2 dx - \frac{1}{p+1} \int_\Omega (u_+)^{p+1} dx \\ - \mu \int_\Omega f u dx. \end{aligned}$$

The functional $I_{\lambda,\mu}$ is well-defined for every $u \in X_0^s(\Omega)$ and belongs to $\mathcal{C}^1(X_0^s(\Omega), \mathbb{R})$. Moreover, for any $u, \varphi \in X_0^s(\Omega)$, we have

$$\begin{aligned} \langle I'_{\lambda,\mu}(u), \varphi \rangle = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\big(u(x) - u(y)\big)\big(\varphi(x) - \varphi(y)\big)}{|x - y|^{N+2s}}\, dx\, dy \\ + \lambda \int_\Omega u\varphi dx - \int_\Omega (u_+)^p \varphi dx - \mu \int_\Omega f\varphi dx. \end{aligned} \tag{2.2}$$

Clearly, critical points of $I_{\lambda,\mu}$ are the weak solutions for the problem (1.1).

Lemma 2.2. *Assume that* (A1) *holds. There exists $\mu_0 > 0$ such that, for $\mu \in (0, \mu_0]$, the* (1.1) *has a positive solution $u_\mu \in X_0^s(\Omega)$ satisfying $\|u_\mu\|_{X_0^s(\Omega)} \to 0$ as $\mu \to 0$. Furthermore,* (1.1) *has a unique positive solution u_μ in a neighborhood of the origin in $X_0^s(\Omega)$ for $\mu > 0$ small enough.*

Proof. Define $\Phi : [0, +\infty) \times X_0^s(\Omega) \to (X_0^s(\Omega))'$ by

$$\Phi(\mu, u) = (-\Delta)^s u + \lambda u - (u_+)^p - \mu f. \tag{2.3}$$

Then Φ is a continuous operator, and for $w \in X_0^s(\Omega)$, we have

$$\Phi_u(\mu, u) w = (-\Delta)^s w + \lambda w - p(u_+)^{p-1} w. \tag{2.4}$$

In particular, $\Phi_u(0, 0)w = (-\Delta)^s w + \lambda w$. It is clear that $\Phi_u(0, 0) : X_0^s(\Omega) \to (X_0^s(\Omega))'$ is invertible for $\lambda > -\lambda_1$. Then, by the implicit function theorem, there exists a function $u_\mu \in X_0^s(\Omega)$ for $\mu \in (0, \mu_0]$ with some $\mu_0 > 0$ such that $\Phi(\mu, u_\mu) = 0$ and $\|u_\mu\|_{X_0^s(\Omega)} \to 0$ as $\mu \to 0$. Furthermore, there is no other solution of $\Phi(\mu, u) = 0$ in a neighborhood of the origin in $X_0^s(\Omega)$ for $\mu > 0$ sufficiently small. Then, u_μ solves the problem

$$(-\Delta)^s u + \lambda u = (u_+)^p + \mu f \quad \text{in } \Omega$$

for each $\mu \in (0, \mu_0]$ and the local uniqueness of the solution holds. By Proposition 2.1, we obtain $u_\mu > 0$ in Ω. Thus, (1.1) has a unique positive solution u_μ in a neighborhood of the origin in $X_0^s(\Omega)$ for $\mu > 0$ sufficiently small. $\square$

Lemma 2.3. *Assume that there exists a positive function $\hat{u} \in X_0^s(\Omega)$ satisfying*

$$(-\Delta)^s \hat{u} + \lambda \hat{u} \geq \hat{u}^p + \hat{\mu} f \quad \text{in } \Omega \tag{2.5}$$

for some $\hat{\mu} > 0$. Then, for any $\mu \in (0, \hat{\mu}]$, there exists a positive solution $u \in X_0^s(\Omega)$ of (1.1) *satisfying $0 < u(x) \leq \hat{u}(x)$ for $x \in \Omega$. Furthermore, for any positive solution $\tilde{u} \in X_0^s(\Omega)$ of* (1.1)*, the solution u satisfies $u(x) \leq \tilde{u}(x)$ for $x \in \Omega$.*

Proof. Let $\mu \in (0, \hat{\mu}]$, and put $u_0 \equiv 0$. Inductively, we can define $\{u_n\}$, by a solution of the problem

$$(-\Delta)^s u_n + \lambda u_n = (u_{n-1})^p + \mu f, \quad u_n \in X_0^s(\Omega)$$

for $n = 1, 2, \dots$. Furthermore, $\{u_n\}$ satisfies

$$0 < u_1(x) < u_2(x) < \dots < \hat{u}(x), \quad \text{for } x \in \Omega. \tag{2.6}$$

In fact, it is clear that $u_1 \in X_0^s(\Omega)$ and $0 < u_1 < \hat{u}$ in Ω by Proposition 2.1. Then, it follows that $u_1^p \in L^{(p+1)'}(\Omega) \subset (X_0^s(\Omega))'$, where $(p+1)' = 2N/(N+2s)$ is the conjugate exponent of $p + 1 = 2N/(N - 2s)$. By induction, we obtain $u_n \in X_0^s(\Omega)$ and $u_{n-1} < u_n < \hat{u}$ in Ω for each $n = 1, 2, \dots$. Thus, (2.6) holds.

By the definition of u_n, it follows that

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2} \psi dx + \lambda \int_\Omega u_n \psi dx = \int_\Omega u_{n-1}^p \psi dx + \mu \int_\Omega f \psi dx \tag{2.7}$$

for any $\psi \in X_0^s(\Omega)$. Putting $\psi = u_n$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 dx + \lambda \int_\Omega u_n^2 dx &= \int_\Omega u_{n-1}^p u_n dx + \mu \int_\Omega f u_n dx \\ &\leq \int_\Omega \hat{u}^{p+1} dx + \mu \int_\Omega f \hat{u} dx \end{aligned}$$

Thus, $\{u_n\}$ is bounded in $X_0^s(\Omega)$. Hence, there exist a subsequence, denoted again $\{u_n\}$, and $u \in X_0^s(\Omega)$ satisfying, as $n \to \infty$, $u_n \rightharpoonup u$ in $X_0^s(\Omega)$ weakly, $u_n \to u$ in $L^2(\Omega)$ strongly, and $u_n \to u$ a.e. in Ω. By the monotone convergence theorem, we have

$$\int_\Omega u_{n-1}^p \psi dx \to \int_\Omega u^p \psi dx \quad \text{as } n \to \infty.$$

Then, letting $n \to \infty$ in (2.7), we obtain

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \psi dx + \lambda \int_\Omega u \psi dx = \int_\Omega u^p \psi dx + \mu \int_\Omega f \psi dx \tag{2.8}$$

for any $\psi \in X_0^s(\Omega)$. This implies that $u \in X_0^s(\Omega)$ is a solution to (1.1). From (2.6), we have $0 < u \leq \hat{u}$ in Ω.

Let $\tilde{u} \in X_0^s(\Omega)$ be a positive solution of (1.1). Then, $\tilde{u} > u_0 \equiv 0$, and $\tilde{u} > u_n$ for each $n = 1, 2, \dots$, by induction. Thus, we obtain $\tilde{u} \geq u$ in Ω. □

For each $\mu > 0$, define the solution set S_μ by

$$S_\mu = \{u \in X_0^s(\Omega) : u \text{ is a positive solution of } (1.1)\}.$$

Lemma 2.2 implies that $S_\mu \neq \emptyset$ for sufficient small $\mu > 0$.

Lemma 2.4. *Let* (A1) *hold.*

(i) *Assume that $S_{\mu_0} \neq \emptyset$ for some $\mu_0 > 0$. Then, $S_\mu \neq \emptyset$ for all $\mu \in (0, \mu_0)$.*
(ii) *If $S_\mu \neq \emptyset$, then there exists a minimal solution $\underline{u}_\mu \in S_\mu$.*

(iii) *Assume that $u_\mu \in S_\mu$ and $u_{\hat\mu} \in S_{\hat\mu}$ are minimal solutions with $0 < \mu < \hat\mu$. Then, $u_\mu < u_{\hat\mu}$ in Ω.*

(iv) *Let u_μ be the solution of* (1.1) *obtained in Lemma 2.2, and let $\underline{u}_\mu \in S_\mu$ be the minimal solution. Then, $u_\mu \equiv \underline{u}_\mu$ for $\mu > 0$ sufficiently small.*

Proof. (i) Let $\mu \in (0, \mu_0)$ and $u_0 \in S_{\mu_0}$. Applying Lemma 2.2 and Lemma 2.3 with $\hat u = u_0$ and $\hat\mu = \mu_0$, we obtain a positive solution $u \in X_0^s(\Omega)$ of (1.1). This implies that $S_\mu \neq \emptyset$ for all $\mu \in (0, \mu_0)$.

(ii) Assume that $u \in S_\mu$. Applying Lemma 2.3 with $\hat u = u$ and $\hat\mu = \mu$, there exists $\underline{u}_\mu \in S_\mu$ such that $\underline{u}_\mu \le u$ in Ω. By the latter part of Lemma 2.3, $\underline{u}_\mu$ is the minimal solution of S_μ.

(iii) Applying Lemma 2.3 with $\hat u = u_{\hat\mu}$, we deduce that $\underline{u}_\mu \le \underline{u}_{\hat\mu}$ in Ω. Put $z = \underline{u}_{\hat\mu} - \underline{u}_\mu \ge 0$. Then, z satisfies $(-\Delta)^s z + \lambda z \ge (\hat\mu - \mu) f \ge 0$, $\not\equiv 0$ in Ω. By Proposition 2.1, we obtain $z > 0$ in Ω, that is, $\underline{u}_{\hat\mu} > \underline{u}_\mu$ in Ω.

(iv) Since $\underline{u}_\mu \in S_\mu$ is the minimal solution, we have $\underline{u}_\mu \le u_\mu$ in Ω. Note that the solution of (1.1) satisfies (2.8) for any $\psi \in X_0^s(\Omega)$. Putting $u = \psi = \underline{u}_\mu$ in (2.8), we have

$$\|\underline{u}_\mu\|_\lambda^2 = \|\underline{u}_\mu\|_{L^{p+1}}^{p+1} + \mu \int_\Omega f \underline{u}_\mu dx \le \|u_\mu\|_{L^{p+1}}^{p+1} + \mu \int_\Omega f u_\mu dx.$$

By the Sobolev inequality, we obtain

$$\|\underline{u}_\mu\|_\lambda^2 \le C \|u_\mu\|_{X_0^s(\Omega)}^{p+1} + \mu \|f\|_{(X_0^s(\Omega))'} \|u_\mu\|_{X_0^s(\Omega)},$$

with some constant $C > 0$. Since $\|\cdot\|_\lambda$ is equivalent with $\|\cdot\|_{X_0^s(\Omega)}$, Lemma 2.2 implies that $\|\underline{u}_\mu\|_{X_0^s(\Omega)} \to 0$ as $\mu \to 0$. By the local uniqueness of the solution u_μ in a neighborhood of the origin in $X_0^s(\Omega)$, we obtain $u_\mu \equiv \underline{u}_\mu$ for μ sufficiently small. $\square$

Next, let us consider the eigenvalue problem

$$(-\Delta)^s \phi + \lambda \phi = \kappa a(x) \phi, \quad \phi \in X_0^s(\Omega), \tag{2.9}$$

where $\lambda \in \mathbb{R}$, $a(x) \in L^{N/2s}(\Omega)$, and $a(x) > 0$ in Ω. We assume that $\lambda > -\lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta)^s$ with zero Dirichlet boundary condition on Ω. In order to find the first eigenvalue of (2.9), we consider the following minimization problem

$$\kappa_1 = \inf_{\psi \in X_0^s(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} \psi|^2 dx + \lambda \int_\Omega \psi^2 dx}{\int_\Omega a(x) \psi^2 dx}. \tag{2.10}$$

Lemma 2.5. *Let* (A1) *hold. The infimum κ_1 in* (2.10) *is positive and achieved by some $\phi_1 \in X_0^s(\Omega)$ with $\phi_1 > 0$ in Ω. In particular, (κ_1, ϕ_1) is the first eigenvalue and the first eigenfunction to the problem* (2.9).

Proof. Let $\{\psi_n\} \subset X_0^s(\Omega)$ be a minimizing sequence of (2.10) satisfying

$$\int_\Omega a(x) \psi_n^2 dx = 1, \quad \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \psi_n|^2 dx + \lambda \int_\Omega \psi_n^2 dx \to \kappa_1 \quad \text{as } n \to \infty.$$

Since $\{\psi_n\}$ is bounded in $X_0^s(\Omega)$, there exists a subsequence, still denoted by $\{\psi_n\}$, and a function $\phi_1 \in X_0^s(\Omega)$ such that, as $n \to \infty$, $\psi_n \to \phi_1$ weakly in $X_0^s(\Omega)$, $\psi_n \to \phi_1$ strongly in $L^2(\Omega)$, $\psi_n \to \phi_1$ a.e. in Ω. Then, it follows that

$$\kappa_1 = \liminf_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \psi_n|^2 dx + \lambda \int_\Omega \psi_n^2 dx \ge \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \phi_1|^2 dx + \lambda \int_\Omega \phi_1^2 dx.$$

Since $a(x) \in L^{N/2s}(\Omega)$, $\{\psi_n^2\}$ is bounded in $L^{N/(N-2s)}(\Omega)$, we obtain

$$\int_\Omega a(x)\psi_n^2 dx \to \int_\Omega a(x)\phi_1^2 dx = 1 \quad \text{as } n \to \infty.$$

Hence, $\phi_1 \not\equiv 0$ achieves the infimum $\kappa_1 > 0$. Clearly, $|\phi_1|$ also achieves κ_1, since

$$\begin{aligned}
\|\phi_1\|^2_{X_0^s(\Omega)} &= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(\phi_1^+(x) - \phi_1^+(y)) - (\phi_1^-(x) - \phi_1^-(y))|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&\geq \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(\phi_1^+(x) - \phi_1^+(y)) + (\phi_1^-(x) - \phi_1^-(y))|^2}{|x-y|^{N+2s}} \, dx \, dy \\
&= \||\phi_1|\|^2_{X_0^s(\Omega)}.
\end{aligned}$$

Then, we assume that $\phi_1 \geq 0$ a.e. in Ω. Note that ϕ_1 satisfies

$$(-\Delta)^s \phi_1 + \lambda \phi_1 = \kappa_1 a(x) \phi_1 \quad \text{in } \Omega.$$

Thus, $\phi_1 > 0$ in Ω by Proposition 2.1. □

Define g_0 by the unique solution of the problem

$$(-\Delta)^s g_0 + \lambda g_0 = f \quad \text{in } Omega,\ g_0 \in X_0^s(\Omega). \tag{2.11}$$

By Proposition 2.1, we find that $g_0 > 0$ in Ω. Let us consider the eigenvalue problem

$$(-\Delta)^s \phi + \lambda \phi = \kappa (g_0)^{p-1} \phi \quad \text{in } \Omega,\ \phi \in X_0^s(\Omega). \tag{2.12}$$

since $g_0 \in X_0^s(\Omega) \subset L^{2N/N-2s}(\Omega)$, we have $(g_0)^{p-1} \in L^{N/2s}(\Omega)$. By Lemma 2.5, there exist the first eigenvalue $\kappa_1 > 0$ and the corresponding eigenfunction $\phi_1 > 0$ in Ω.

Proof of Theorem 1.3 (i) and (iii). Put $\bar{\mu} = \sup\{\mu > 0 : S_\mu \neq \emptyset\}$. By Lemma 2.2 implies that $\bar{\mu} > 0$. Now we show that $\bar{\mu} < \infty$. Let $\mu > 0$ such that $S_\mu \neq \emptyset$, and let $u \in S_\mu$. Put $v = u - \mu g_0$, where g_0 is the solution of (2.11). then, v satisfies $(-\Delta)^s v + \lambda v = u^p > 0$ in Ω. By Proposition 2.1, we have $v > 0$ in Ω, and hence, $u > \mu g_0$. Then, it follows that

$$(-\Delta)^s u + \lambda u > \mu^{p-1} (g_0)^{p-1} u \quad \text{in } \Omega. \tag{2.13}$$

Let $\phi_1 > 0$ be the eigenfunction corresponding to the first eigenvalue κ_1 to the problem (2.12); that is,

$$(-\Delta)^s \phi_1 + \lambda \phi_1 = \kappa_1 (g_0)^{p-1} \phi_1 \quad \text{in } \Omega. \tag{2.14}$$

Multiply (2.13) by ϕ_1 and (2.14) by u, respectively, and integrating them on Ω, we have

$$\begin{aligned}
\mu^{p-1} \int_\Omega (g_0)^{p-1} u \phi_1 dx &< \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \phi_1 dx + \lambda \int_\Omega u \phi_1 dx \\
&= \kappa_1 \int_\Omega (g_0)^{p-1} u \phi_1 dx.
\end{aligned}$$

Then $\mu < \kappa_1^{1/p-1}$ if $S_\mu \neq \emptyset$, and hence $\bar{\mu} \leq \kappa_1^{1/p-1} < +\infty$.

By the definition of $\bar{\mu}$, (1.1) has no positive solution for $\mu > \bar{\mu}$, so (iii) of Theorem 1.3 holds. From Lemma 2.4, we obtain (i) of Theorem 1.3. □

For $\mu \in (0, \bar{\mu})$, let $\underline{u}_\mu$ be the minimal solution of (1.1) obtained in Theorem 1.3. We consider the following linearized eigenvalue problem

$$(-\Delta)^s \phi + \lambda \phi = \kappa p (\underline{u}_\mu)^{p-1} \phi \quad \text{in } \Omega, \ \phi \in X_0^s(\Omega). \tag{2.15}$$

Since $\underline{u}_\mu \in X_0^s(\Omega) \subset L^{2N/N-2s}(\Omega)$, we have $(\underline{u}_\mu)^{p-1} \in L^{N/2s}(\Omega)$. By Lemma 2.5, there exists the first eigenvalue $\kappa_1(\mu) > 0$ of the problem (2.15), and it holds

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} \psi|^2 dx + \lambda \int_\Omega \psi^2 dx \geq \kappa_1(\mu) \int_\Omega p(\underline{u}_\mu)^{p-1} \psi^2 dx \tag{2.16}$$

for any $\psi \in X_0^s(\Omega)$.

To show the existence and uniqueness of solution for (1.1) with $\mu = \bar{\mu}$, we need the following lemmas.

Lemma 2.6. *If $\mu \in (0, \bar{\mu})$, then $\kappa_1(\mu) > 1$.*

Proof. For $0 < \mu < \hat{\mu} < \bar{\mu}$, let $\underline{u}_\mu$ and $\underline{u}_{\hat{\mu}}$ be the minimal solution of S_μ and $S_{\hat{\mu}}$, respectively. Put $z = \underline{u}_{\hat{\mu}} - \underline{u}_\mu$. We find that $z > 0$ from Lemma 2.4(iii), and that z satisfies

$$(-\Delta)^s z + \lambda z > p(\underline{u}_\mu)^{p-1} z \quad \text{in } \Omega. \tag{2.17}$$

Let $\phi_1 > 0$ be the eigenfunction corresponding to the first eigenvalue $\kappa_1(\mu)$ to the problem (2.15), that is,

$$(-\Delta)^s \phi_1 + \lambda \phi_1 = \kappa_1(\mu) p (\underline{u}_\mu)^{p-1} \phi_1 \quad \text{in } \Omega. \tag{2.18}$$

Multiplying (2.17) and (2.18) by ϕ_1 and z, respectively, and integrating them on Ω, we obtain

$$\begin{aligned} \kappa_1(\mu) \int_\Omega p(\underline{u}_\mu)^{p-1} \phi_1 z dx &= \int_{\mathbb{R}^N} (-\Delta)^{s/2} \phi_1 (-\Delta)^{s/2} z dx + \lambda \int_\Omega \phi_1 z dx \\ &> \int_\Omega p(\underline{u}_\mu)^{p-1} \phi_1 z dx. \end{aligned}$$

This implies that $\kappa_1(\mu) > 1$. □

Lemma 2.7. *For $\mu \in (0, \bar{\mu})$, let $\underline{u}_\mu$ be the minimal solution of* (1.1) *obtained in Theorem 1.3. Then, there exists a constant $M > 0$ independent of μ such that $\|\underline{u}_\mu\|_{X_0^s(\Omega)} \leq M$ for all $\mu \in (0, \bar{\mu})$.*

Proof. Put $v_\mu = \underline{u}_\mu - \mu g_0$, where g_0 is the solution of problem (2.11). Then, $v_\mu \in X_0^s(\Omega)$ and satisfies $(-\Delta)^s v_\mu + \lambda v_\mu = (v_\mu + \mu g_0)^p$ in Ω; that is,

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} v_\mu (-\Delta)^{s/2} \psi dx + \lambda \int_\Omega v_\mu \psi dx = \int_\Omega (v_\mu + \mu g_0)^p \psi dx$$

for any $\psi \in X_0^s(\Omega)$. Putting $\psi = v_\mu$, we have

$$\|v_\mu\|_\lambda^2 = \int_\Omega (v_\mu + \mu g_0)^p v_\mu dx.$$

Since $\|\cdot\|_\lambda$ is equivalent with $\|\cdot\|_{X_0^s(\Omega)}$, it suffices to show that there exists a constant $M' > 0$ independent of μ such that $\|v_\mu\|_\lambda \leq M'$ for $\mu \in (0, \bar{\mu})$.

For any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) > 0$ such that

$$(t+s)^p \leq (1+\varepsilon)(t+s)^{p-1} t + C s^p \quad \text{for } t, \ s \geq 0.$$

Then, we have

$$\|v_\mu\|_\lambda^2 \le (1+\varepsilon)\int_\Omega (\underline{u}_\mu)^{p-1} v_\mu^2 dx + C\mu^p \int_\Omega (g_0)^p v_\mu dx.$$

From (2.16) and Lemma 2.6 it follows that

$$\int_\Omega (\underline{u}_\mu)^{p-1} v_\mu^2 dx < \frac{1}{p}\|v_\mu\|_\lambda^2.$$

By using Hölder and Sobolev inequalities, we obtain

$$\int_\Omega (g_0)^p v_\mu dx \le \|g_0\|_{L^{p+1}}^p \|v_\mu\|_{L^{p+1}} \le C\|g_0\|_{L^{p+1}}^p \|v_\mu\|_{X_0^s(\Omega)} \le C'\|g_0\|_{L^{p+1}}^p \|v_\mu\|_\lambda$$

with some constant C, $C' > 0$. Then, it follows that

$$\|v_\mu\|_\lambda^2 \le \frac{(1+\varepsilon)}{p}\|v_\mu\|_\lambda^2 + \bar{\mu}^p C'\|g_0\|_{L^{p+1}}^p \|v_\mu\|_\lambda.$$

This implies that $\|v_\mu\|_\lambda$ is bounded for $\mu \in (0,\bar{\mu})$, and hence $\|v_\mu\|_{X_0^s(\Omega)}$ is bounded for $\mu \in (0,\bar{\mu})$. □

Lemma 2.8. *For $\mu = \bar{\mu}$, the problem* (1.1) *has a positive minimal solution $\underline{u}_{\bar{\mu}} \in X_0^s(\Omega)$, and there hold $\underline{u}_\mu < \underline{u}_{\bar{\mu}}$ in Ω for $\mu < \bar{\mu}$ and $\underline{u}_\mu \to \underline{u}_{\bar{\mu}}$ a.e. in Ω as $\mu \uparrow \bar{\mu}$.*

Proof. Let $\{\mu_n\}$ be sequence such that $\mu_n < \mu_{n+1}$ and $\mu_n \to \bar{\mu}$ as $n \to \infty$. Since $\underline{u}_\mu$ is increasing in $\mu \in (0,\bar{\mu})$ by Lemma 2.4 (iii), we have $\underline{u}_{\mu_n} < \underline{u}_{\mu_{n+1}}$ in Ω. Lemma 2.7 implies that $\{\underline{u}_{\mu_n}\}$ is bounded in $X_0^s(\Omega)$. Then, there exists a positive function $\bar{u} \in X_0^s(\Omega)$ such that, as $n \to \infty$, $\underline{u}_{\mu_n} \rightharpoonup \bar{u}$ weakly in $X_0^s(\Omega)$, $\underline{u}_{\mu_n} \to \bar{u}$ strongly in $L^2(\Omega)$. and $\underline{u}_{\mu_n} \to \bar{u}$ a.e. in Ω. We note here that $\underline{u}_{\mu_n}$ satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}\underline{u}_{\mu_n}(-\Delta)^{s/2}\psi dx + \lambda\int_\Omega \underline{u}_{\mu_n}\psi dx = \int_\Omega \underline{u}_{\mu_n}^p \psi dx + \mu_n\int_\Omega f\psi dx \tag{2.19}$$

for any $\psi \in X_0^s(\Omega)$, and that $\bar{u}$ satisfies

$$\int_\Omega \bar{u}^p\psi dx \le \|\bar{u}\|_{L^{p+1}}^p \|\psi\|_{L^{p+1}} < \infty.$$

Letting $n \to \infty$ in (2.19), by the monotone convergence theorem, we obtain

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}\bar{u}(-\Delta)^{s/2}\psi dx + \lambda\int_\Omega \bar{u}\psi dx = \int_\Omega \bar{u}^p\psi dx + \bar{\mu}\int_\Omega f\psi dx$$

Thus, $\bar{u} \in X_0^s(\Omega)$ is a positive solution of (1.1); i.e., $\bar{u} \in S_{\bar{\mu}}$. From Lemma 2.4 (ii), there exists a minimal solution $\underline{u}_{\bar{\mu}} \in S_{\bar{\mu}}$ Then, $\underline{u}_{\bar{\mu}} \le \bar{u}$. We will verify that $\underline{u}_{\bar{\mu}} \equiv \bar{u}$. In fact, from Lemma 2.4 (iii), we have $\underline{u}_{\mu_n} < \underline{u}_{\bar{\mu}}$ in Ω for $n = 1, 2, \dots$. It follows that $\bar{u} \le \underline{u}_{\bar{\mu}}$, and hence $\underline{u}_{\bar{\mu}} \equiv \bar{u}$. Since $\underline{u}_\mu$ is increasing in $\mu \in (0,\bar{\mu})$, we have $\underline{u}_\mu < \underline{u}_{\bar{\mu}}$ in Ω for $\mu < \bar{\mu}$ and $\underline{u}_\mu \to \underline{u}_{\bar{\mu}}$ a.e. in Ω as $\mu \uparrow \bar{\mu}$. □

Denote by $\kappa_1(\bar{\mu})$, the first eigenvalue of the linearized problem (2.15) with $\mu = \bar{\mu}$. By Lemma 2.5, the first eigenvalue $\kappa_1(\bar{\mu})$ is given by

$$\kappa_1(\bar{\mu}) = \inf_{\psi \in X_0^s(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\psi|^2 dx + \lambda\int_\Omega \psi^2 dx}{\int_\Omega p(\underline{u}_{\bar{\mu}})^{p-1}\psi^2 dx}. \tag{2.20}$$

Since $\underline{u}_\mu < \underline{u}_{\bar{\mu}}$ by Lemma 2.4, we have $\kappa_1(\mu) \ge \kappa_1(\bar{\mu})$ for $\mu \in (0,\bar{\mu})$.

Lemma 2.9. *Assume that the problem* (2.9) *has the first eigenvalue* $\kappa_1 > 1$. *Then, for any* $f \in (X_0^s(\Omega))'$, *the problem*

$$(-\Delta)^s u + \lambda u = a(x)u + f, \quad \text{in } \Omega. \tag{2.21}$$

has a unique solution in $X_0^s(\Omega)$.

Proof. Define $I_f(u)$, for $u \in X_0^s(\Omega)$, by

$$I_f(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{\lambda}{2}\int_\Omega u^2 dx - \frac{1}{2}\int_\Omega a(x)u^2 dx - \int_\Omega fudx.$$

From (2.10), it follows that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\psi|^2 dx + \lambda \int_\Omega \psi^2 dx \geq \kappa_1 \int_\Omega a(x)\psi^2 dx \tag{2.22}$$

for any $\psi \in X_0^s(\Omega)$. Then, we have

$$I_f(u) \geq \big(\frac{1}{2} - \frac{1}{2\kappa_1}\big)\|u\|_\lambda^2 - \|f\|_{(X_0^s(\Omega))'}\|u\|_{X_0^s(\Omega)},$$

Since $\|\cdot\|_\lambda$ is equivalent with $\|\cdot\|_{X_0^s(\Omega)}$, we obtain $I_f(u) \to \infty$ as $\|u\|_\lambda \to \infty$. Thus, I_f is coercive and bounded from below in $X_0^s(\Omega)$. Since I_f is weakly lower semicontinuous on $X_0^s(\Omega)$, there exists $u \in X_0^s(\Omega)$ which attains the infimum, and hence (2.21) has a solution in $X_0^s(\Omega)$. To show the uniqueness of the solution of (2.21), it suffices to show that (2.21) has only trivial solution when $f \equiv 0$. Assume to the contrary that there exists a non-trivial solution $u \in X_0^s(\Omega)$. Then, from (2.21) we have

$$\int_\Omega a(x)u^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \lambda \int_\Omega u^2 dx \geq \kappa_1 \int_\Omega a(x)u^2 dx.$$

This contradicts $\kappa_1 > 1$. Thus, (2.21) has a unique solution in $X_0^s(\Omega)$. □

Lemma 2.10. *We have* $\kappa_1(\mu) \to \kappa_1(\bar{\mu})$ *as* $\mu \uparrow \bar{\mu}$ *and* $\kappa_1(\bar{\mu}) = 1$.

Proof. First, we will show that $\kappa_1(\mu) \to \kappa_1(\bar{\mu})$ as $\mu \uparrow \bar{\mu}$. By Lemma 2.5, we find that

$$\begin{aligned}
\kappa_1(\bar{\mu}) &= \inf_{\psi \in X_0^s(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\psi|^2 dx + \lambda \int_\Omega \psi^2 dx}{\int_\Omega p(\underline{u}_{\bar{\mu}})^{p-1}\psi^2 dx} \\
&= \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi_1|^2 dx + \lambda \int_\Omega \phi_1^2 dx}{\int_\Omega p(\underline{u}_{\bar{\mu}})^{p-1}\phi_1^2 dx},
\end{aligned}$$

where ϕ_1 is the eigenfunction corresponding to the first eigenvalue $\kappa_1(\bar{\mu})$. Let $\{\mu_n\}$ be a sequence such that $\mu_n < \mu_{n+1}$ and $\mu_n \to \bar{\mu}$ as $n \to \infty$. By the monotone convergence theorem, we have

$$\int_\Omega p(\underline{u}_{\mu_n})^{p-1}\phi_1^2 dx \to \int_\Omega p(\underline{u}_{\bar{\mu}})^{p-1}\phi_1^2 dx$$

as $n \to \infty$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $0 < \bar{\mu} - \mu < \delta$ then

$$\frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi_1|^2 dx + \lambda \int_\Omega \phi_1^2 dx}{\int_\Omega p(\underline{u}_\mu)^{p-1}\phi_1^2 dx} - \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi_1|^2 dx + \lambda \int_\Omega \phi_1^2 dx}{\int_\Omega p(\underline{u}_{\bar{\mu}})^{p-1}\phi_1^2 dx} > 0.$$

Put

$$\tilde{\kappa}(\mu) = \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\phi_1|^2 dx + \lambda \int_\Omega \phi_1^2 dx}{\int_\Omega p(\underline{u}_\mu)^{p-1}\phi_1^2 dx}.$$

It follows from the above inequality that $0 < \tilde{\kappa}(\mu) - \kappa_1(\bar{\mu}) < \varepsilon$. Since $\kappa_1(\bar{\mu}) \leq \kappa_1(\mu) \leq \tilde{\kappa}(\mu)$, we have

$$0 \leq \kappa_1(\mu) - \kappa_1(\bar{\mu}) \leq \tilde{\kappa}(\mu) - \kappa_1(\bar{\mu}) < \varepsilon \quad \text{if } 0 < \bar{\mu} - \mu < \delta.$$

This implies that $\kappa_1(\mu) \to \kappa_1(\bar{\mu})$ as $\mu \uparrow \bar{\mu}$. Since $\kappa_1(\mu) > 1$ for $\mu \in (0, \bar{\mu})$ by Lemma 2.5, we have $\kappa_1(\bar{\mu}) \geq 1$. Finally, we will show $\kappa_1(\bar{\mu}) = 1$. Assume to the contrary that $\kappa_1(\bar{\mu}) > 1$. Define $\Phi : (0,\infty) \times X_0^s(\Omega) \to (X_0^s(\Omega))'$ by (2.3). For $u \in X_0^s(\Omega)$, we have (2.4), and, in particular,

$$\Phi_u(\bar{\mu}, \underline{u}_{\bar{\mu}})w = (-\Delta)^s w + \lambda w - p(\underline{u}_{\bar{\mu}})^{p-1} w.$$

By Lemma 2.9, for every $f \in (X_0^s(\Omega))'$, there exists a unique solution $w \in X_0^s(\Omega)$ of $\Phi_u(\bar{\mu}, \underline{u}_{\bar{\mu}})w = f$; that is, $\Phi_u : X_0^s(\Omega) \to (X_0^s(\Omega))'$ is invertible at $(\bar{\mu},\ \underline{u}_{\bar{\mu}})$. Then, by the implicit function theorem, there exist $\varepsilon > 0$ such that $\Phi(\mu, u) = 0$ has a solution $u_\mu \in X_0^s(\Omega)$ for $\mu \in (\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon)$. From Lemma 2.2, we obtain a positive solution u_μ of (1.1) for $\mu \in (\bar{\mu} - \varepsilon, \bar{\mu} + \varepsilon)$. This contradicts the definition of $\bar{\mu}$. Thus, we obtain $\kappa_1(\bar{\mu}) = 1$. □

Proof of Theorem 1.3 (ii). Let $\underline{u}_{\bar{\mu}} \in S_{\bar{\mu}}$ be the minimal solution obtained in Lemma 2.2, we will show the uniqueness of $\underline{u}_{\bar{\mu}} \in S_{\bar{\mu}}$. Assume $u \in S_{\bar{\mu}}$, and put $z = u - \underline{u}_{\bar{\mu}}$. Since $\underline{u}_{\bar{\mu}}$ is the minimal solution, z satisfies $z \geq 0$ and

$$(-\Delta)^s z + \lambda z = u^p - (\underline{u}_{\bar{\mu}})^p \quad \text{in } \Omega. \tag{2.23}$$

Let $\phi_1 \in X_0^s(\Omega)$ be the first eigenfunction of the linearized problem of (2.15) with $\mu = \bar{\mu}$. Since $\kappa_1(\bar{\mu}) = 1$ from Lemma 2.10, we have

$$(-\Delta)^s \phi_1 + \lambda \phi_1 = p(\underline{u}_{\bar{\mu}})^{p-1} \phi_1 \quad \text{in } \Omega. \tag{2.24}$$

Multiplying (2.23) and (2.24) by ϕ_1 and z, respectively, and integrating them on Ω, we have

$$\begin{aligned} \int_\Omega (u^p - \underline{u}_{\bar{\mu}}^p)\phi_1 dx &= \int_{\mathbb{R}^N} (-\Delta)^{s/2}\phi_1 (-\Delta)^{s/2} z dx + \lambda \int_\Omega \phi_1 z dx \\ &= p \int_\Omega (\underline{u}_{\bar{\mu}})^{p-1}(u - \underline{u}_{\bar{\mu}})\phi_1 dx. \end{aligned}$$

Hence, it follows that

$$\int_\Omega F(u, \underline{u}_{\bar{\mu}})\phi_1 dx = 0$$

where $F(\sigma, \tau) = \sigma^p - \tau^p - p\tau^{p-1}(\sigma - \tau)$. We note here that, for $\sigma \geq \tau \geq 0$, $F(\sigma, \tau) \geq 0$ and $F(\sigma, \tau) = 0$ holds if and only if $\sigma = \tau$. Then, from $\phi_1 > 0$, we conclude that $F(u, \underline{u}_\mu) = 0$ a.e. in Ω, and hence $u = \underline{u}_\mu$ a.e. in Ω. Thus, Theorem 1.3(ii) is obtained. □

3. Existence of the second solution: Proof of Theorem 1.5

Let $\underline{u}_\mu$ be the minimal solution of (1.1) for $\mu \in (0, \bar{\mu})$ obtained in Theorem 1.3. To find a second solution of (1.1), we introduce the problem

$$(-\Delta)^s v + \lambda v = (v + \underline{u}_\mu)^p - \underline{u}_\mu^p \quad \text{in } \Omega,\ v \in X_0^s(\Omega). \tag{3.1}$$

Assume that (3.1) has a positive solution v, and put $\bar{u}_\mu = v + \underline{u}_\mu$. Then, $\bar{u}_\mu \in X_0^s(\Omega)$ solves (1.1) and satisfies $\bar{u}_\mu > \underline{u}_\mu$ in Ω. We will show the existence of solutions of (3.1) by using Mountain-Pass Lemma. To this end, we define the corresponding variational functional of (3.1) by

$$\bar{I}_{\lambda,\mu}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}v|^2dx + \frac{\lambda}{2}\int_\Omega v^2dx - \int_\Omega G(v,\underline{u}_\mu)dx, \tag{3.2}$$

for $v \in X_0^s(\Omega)$, where

$$G(\sigma,\tau) := \frac{1}{p+1}(\sigma_+ + \tau)^{p+1} - \frac{1}{p+1}\tau^{p+1} - \tau^p\sigma_+. \tag{3.3}$$

Obviously, $\bar{I}_{\lambda,\mu} : X_0^s(\Omega) \to \mathbb{R}$ is $\mathcal{C}^1$; If $v \in X_0^s(\Omega)$ is a critical point, then v satisfies

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}v(-\Delta)^{s/2}\psi + \int_\Omega \lambda v\psi - \int_\Omega g(v,\underline{u}_\mu)\psi dx = 0 \tag{3.4}$$

for any $\psi \in X_0^s(\Omega)$, where

$$g(\sigma,\tau) := (\sigma_+ + \tau)^p - \tau^p, \tag{3.5}$$

By Proposition 2.1, we have $v > 0$, and hence v is a positive solution to (3.1).

The following lemma gives the properties of the functions $G(\sigma,\tau)$, $g(\sigma,\tau)$ defined in (3.3) and (3.5). For a proof we refer the reader to [21, Appendix B.1], so we omit it here.

Lemma 3.1. (i) *There exists a constant $C = C(p) > 0$ such that*

$$g(\sigma,\tau) \le C(\sigma^p + \tau^{p-1}\sigma) \quad \text{for } \sigma,\tau \ge 0.$$

(ii) *For $\sigma,\tau \ge 0$,*

$$\frac{1}{p+1}\sigma^{p+1} \le G(\sigma,\tau) \le \frac{1}{2}g(\sigma,\tau)\sigma.$$

(iii) *For any $\varepsilon > 0$, there is a constant $C = C(\varepsilon) > 0$ such that*

$$G(\sigma,\tau) - \frac{p}{2}\tau^{p-1}\sigma^2 \le \varepsilon\tau^{p-1}\sigma^2 + C\sigma^{p+1} \quad \text{for } \sigma,\tau \ge 0.$$

(iv) *Put $c_p = \min\{1, p-1\}$. Then,*

$$g(\sigma,\tau)\sigma - (2+c_p)G(\sigma,\tau) \ge -\frac{c_p p}{2}\tau^{p-1}\sigma^2 \quad \text{for } \sigma,\tau \ge 0.$$

(v) *If $N \ge 6s$, that is $1 < p \le 2$, then*

$$(p+1)G(\sigma,\tau) - g(\sigma,\tau)\sigma \le \frac{p(p-1)}{2}\tau^{p-1}\sigma^2 \quad \text{for } \sigma,\tau \ge 0.$$

To use Mountain-Pass Lemma to find critical point of $\bar{I}_{\lambda,\mu}$, we first define

$$H(\sigma,\tau) := G(\sigma,\tau) - \frac{1}{p+1}(\sigma_+)^{p+1} \quad \text{and} \quad h(\sigma,\tau) := g(\sigma,\tau) - (\sigma_+)^p, \tag{3.6}$$

and the following two lemmas show the properties of $H(\sigma,\tau)$ and $h(\sigma,\tau)$, the reader can refer [21, Appendix B.2] for the proof.

Lemma 3.2. *(i) There exists a constant $C > 0$ such that, for $s,t \ge 0$,*

$$H(\sigma,\tau) \le C(\sigma^p\tau + \tau^p\sigma),$$
$$h(\sigma,\tau)\sigma \le C(\sigma^p\tau + \tau^p\sigma).$$

(ii) For $\sigma_0 \ge 0$ and $\tau_0 > 0$, there is a constant $C = C(\sigma_0,\tau_0) > 0$ such that

$$H(\sigma,\tau) \ge C\sigma^p \quad \text{for } \sigma \ge \sigma_0, \quad \tau \ge \tau_0.$$

(iii) Let $N \geq 6s$, that is $1 < p \leq 2$. For $\varepsilon > 0$,

$$\frac{p}{2}\tau^{p-1}\sigma^2 - H(\sigma,\tau) \leq \frac{\varepsilon p}{2}\tau^{p-1}\sigma^2 + \frac{1-\varepsilon}{p+1}\sigma^{p+1} \quad \text{for } \sigma,\tau \geq 0.$$

Lemma 3.3. *Let $\{v_n\}$ be a sequence which is bounded in $X_0^s(\Omega)$. Assume that $v_n \to v$ a.e. in Ω as $n \to \infty$ for some $v \in X_0^s(\Omega)$. Then, as $n \to \infty$, we have*

$$\int_\Omega H(v_n, \underline{u}_\mu)dx \to \int_\Omega H(v, \underline{u}_\mu)dx, \tag{3.7}$$

$$\int_\Omega h(v_n, \underline{u}_\mu)v_n dx \to \int_\Omega h(v, \underline{u}_\mu)v dx, \tag{3.8}$$

and, for any $\psi \in X_0^s(\Omega)$,

$$\int_\Omega g(v_n, \underline{u}_\mu)\psi dx \to \int_\Omega g(v, \underline{u}_\mu)\psi dx. \tag{3.9}$$

Now, we can verify that the functional $\bar{I}_{\lambda,\mu}$ exhibits the Mountain-Pass Geometry.

Lemma 3.4. *Let $\mu \in (0, \bar{\mu})$, then the functional $\bar{I}_{\lambda,\mu}$ exhibits the Mountain-Pass Geometry, i.e. the functional $\bar{I}_{\lambda,\mu}$ satisfies*

- (i) *$\bar{I}_{\lambda,\mu}(0) = 0$;*
- (ii) *There exist some positive constant $\delta = \delta(\mu) > 0$ and $\rho = \rho(\mu) > 0$, such that $\bar{I}_{\lambda,\mu}(u) \geq \rho > 0$ for all $w \in X_0^s(\Omega)$ with $\|u\|_{X_0^s(\Omega)} = \delta$;*
- (iii) *For any $v \in X_0^s(\Omega)$ with $v \geq 0$, $v \not\equiv 0$, we have $\bar{I}_{\lambda,\mu}(tv) \to -\infty$ as $t \to +\infty$.*

Proof. (i) That $\bar{I}_{\lambda,\mu}(0) = 0$ is trivial.

(ii) For any $u \in X_0^s(\Omega)$, we have

$$\begin{aligned}
\bar{I}_{\lambda,\mu}(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega G(u, \underline{u}_\mu)dx \\
&= \Big(\frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx + \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega \frac{p}{2}(\underline{u}_\mu)^{p-1}u^2 dx\Big) \\
&\quad - \int_\Omega (G(u, \underline{u}_\mu) - \frac{p}{2}(\underline{u}_\mu)^{p-1}u^2)dx := I_1 - I_2.
\end{aligned}$$

From Lemma 2.6 and the Sobolev inequality, we have, with $\kappa_1(\mu) > 1$,

$$I_1 \geq \frac{1}{2}(1 - \frac{1}{\kappa_1(\mu)})\|u\|_\lambda^2 \quad \text{and} \quad I_2 \leq \frac{\varepsilon}{p\kappa_1(\mu)}\|u\|_\lambda^2 + C\|u\|_{X_0^s(\Omega)}^{p+1}.$$

From Lemma 3.1 (*iii*), for any $\varepsilon > 0$, there is a constant $C = C(\varepsilon) > 0$ such that

$$I_2 \leq \varepsilon \int_\Omega p(\underline{u}_\mu)^{p-1}u^2 dx + C\|u\|_{L^{p+1}}^{p+1}.$$

Thus, for $\varepsilon > 0$ sufficiently small, we obtain

$$\bar{I}_{\lambda,\mu}(u) \geq C_1\|u\|_\lambda^2 - C\|u\|_{X_0^s(\Omega)}^{p+1} \geq C_1'\|u\|_{X_0^s(\Omega)}^2 - C\|u\|_{X_0^s(\Omega)}^{p+1}$$

with some constants $C_1', C > 0$. This implies that there are positive constants δ and ρ such that $\bar{I}_{\lambda,\mu}(u) \geq \rho$ holds for all for all $u \in X_0^s(\Omega)$ with $\|u\|_{X_0^s(\Omega)} = \delta$.

(iii) By Lemma 3.1 (ii), we have $G(tv, \underline{u}_\mu) \geq t^{p+1}v^{p+1}/(p+1)$. Then, it follows that

$$\bar{I}_{\lambda,\mu}(tw) \leq \frac{t^2}{2}\|u\|_{X_0^s(\Omega)}^2 - \frac{t^{p+1}}{p+1}\int_\Omega v^{p+1}dx.$$

Thus, we obtain $\bar{I}_{\lambda,\mu}(tv) \to -\infty$ as $t \to +\infty$. □

As a consequence of Lemma 3.4 and Mountain-Pass Lemma, for the constant

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \bar{I}_{\lambda,\mu}(\gamma(t)) > 0, \tag{3.10}$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1], X_0^s(\Omega)),\ \gamma(0) = 0,\ \gamma(1) \neq 0 \text{ and } \ \bar{I}_{\lambda,\mu}(\gamma(1)) < 0\}.$$

There exists a $(PS)_c$ sequence $\{u_n\}$ in $X_0^s(\Omega)$ at the level c, that is,

$$\bar{I}_{\lambda,\mu}(u_n) \to c, \quad \bar{I}'_{\lambda,\mu}(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.11}$$

Lemma 3.5. *The sequence $\{u_n\}$ in (3.11) is bounded in $X_0^s(\Omega)$.*

Proof. Since $\{\bar{I}_{\lambda,\mu}(u_n)\}$ is bounded, for n big enough, we have

$$\frac{1}{2}\|u_n\|_\lambda^2 - \int_\Omega G(u_n, \underline{u}_\mu)dx \le c + 1$$

Take $\varepsilon > 0$ arbitrarily, from $\bar{I}_{\lambda,\mu}(u_n) \to 0$ as $n \to \infty$, for sufficiently large n, we have

$$\Big|\int_{\mathbb{R}^N} (-\Delta)^{s/2} u_n (-\Delta)^{s/2}\psi dx + \lambda \int_\Omega u_n\psi dx - \int_\Omega g(u_n, \underline{u}_\mu)\psi dx\Big| \le \varepsilon\|\psi\|_{X_0^s(\Omega)}$$

for any $\psi \in X_0^s(\Omega)$. Putting $\psi = u_n$, we obtain

$$\|u_n\|_\lambda^2 \ge \int_\Omega g(u_n, \underline{u}_\mu)u_n dx - \varepsilon\|u_n\|_{X_0^s(\Omega)}.$$

Then we obtain

$$\begin{aligned}
&(2 + c_p)(c + 1) + \varepsilon\|u_n\|^2_{X_0^s(\Omega)} \\
&\ge (2 + c_p)\bar{I}_{\lambda,\mu}(u_n) - \bar{I}'_{\lambda,\mu}(u_n)u_n \\
&= \frac{c_p}{2}\|u_n\|_\lambda^2 + \int_\Omega \big[(c_p + 2)g(u_n, \underline{u}_\mu)u_n - G(u_n, \underline{u}_\mu)\big]\, dx
\end{aligned}$$

where $c_p = \min\{1, p - 1\}$. From Lemma 3.1 (iv) and Lemma 2.6, it follows that

$$\begin{aligned}
(2 + c_p)(c + 1) &\ge \frac{c_p}{2}\Big(\|u_n\|_\lambda^2 - \int_\Omega p(\underline{u}_\mu)^{p-1}u_n^2 dx\Big) - \varepsilon\|u_n\|^2_{X_0^s(\Omega)} \\
&\ge \frac{c_p}{2}(1 - \frac{1}{\kappa_1(\mu)})\|u_n\|_\lambda^2 - \varepsilon\|u_n\|^2_{X_0^s(\Omega)},
\end{aligned}$$

with $\kappa_1(\mu) > 1$. Since the norm $\|\cdot\|_\lambda$ is equivalent with $\|\cdot\|_{X_0^s(\Omega)}$, we see that $\{u_n\}$ is bounded in $X_0^s(\Omega)$. □

Recall that $S(N, s)$ denotes the best Sobolev constant of the embedding $X_0^s(\Omega) \hookrightarrow L^{p+1}(\Omega)$, see Section 1, (1.12)-(1.14). Let us now introduce a cut-off function $\phi_0(t) \in C^\infty(\mathbb{R}_+)$, which is non-increasing and satisfies

$$\phi_0(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2}, \\ 0 & \text{if } t \ge 1. \end{cases}$$

Assume without loss of generality that $0 \in \Omega$. For some fixed $r > 0$ small enough such that $\overline{B}_r \subset \Omega$, set $\phi(x) = \phi_r(x) = \phi_0(\frac{|x|}{r})$ and consider the family of nonnegative truncated functions

$$\eta_\varepsilon(x) = \frac{\phi u_\varepsilon(x)}{\|\phi u_\varepsilon(x)\|_{L^{p+1}}} \tag{3.12}$$

The following lemma, proved in [28], is important in proving Lemma 3.7.

Lemma 3.6. *For $\varepsilon > 0$ small enough, we obtain*

$$\|\eta_\varepsilon\|^2_{X_0^s(\Omega)} = S(N,s) + O(\varepsilon^{N-2s}),$$

$$\|\eta_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} O(\varepsilon^{2s}), & \text{if } N > 4s,\\ O(\varepsilon^{2s}|\ln\varepsilon|), & \text{if } N = 4s,\\ O(\varepsilon^{N-2s}), & \text{if } N < 4s,\end{cases}$$

$$\int_\Omega |\eta_\varepsilon|^{2_s^*-1}dx = O(\varepsilon^{(N-2s)/2}), \quad \int_\Omega |\eta_\varepsilon| dx = O(\varepsilon^{(N-2s)/2}).$$

The next lemma plays an important role in the proof of Theorem 1.5 below.

Lemma 3.7. *Assume that either $\lambda \in (-\lambda_1, 0]$ and $N > 2s$; or $\lambda > 0$ and $2s < N < 6s$ in Theorem 1.5 holds. Let $\mu \in (0, \bar\mu)$, then for $\varepsilon > 0$ small*

$$0 < \sup_{t>0} \bar I_{\lambda,\mu}(t\eta_\varepsilon) < \frac{s}{N} S(N,s)^{N/2s} \tag{3.13}$$

where η_ε id defined in (3.12)*, $\bar I_{\lambda,\mu}$ is defined in* (3.2)*.*

Proof. We consider now

$$\bar I_{\lambda,\mu}(t\eta_\varepsilon) = \frac{t^2}{2}\Big(\|\eta_\varepsilon\|^2_{X_0^s(\Omega)} + \lambda\|\eta_\varepsilon\|^2_{L^2(\Omega)}\Big) - \int_\Omega G(t\eta_\varepsilon, \underline{u}_\mu)dx.$$

Clearly, $\lim_{n\to+\infty} \bar I_{\lambda,\mu}(t\xi_\varepsilon) = -\infty$, Then $\sup_{t>0} \bar I_{\lambda,\mu}(t\eta_\varepsilon)$ is attained at some value $t_\varepsilon > 0$. This implies

$$\begin{aligned} \frac{d}{dt}\bar I_{\lambda,\mu}(t\eta_\varepsilon)\Big|_{t=t_\varepsilon} &= \bar I'_{\lambda,\mu}(t_\varepsilon\eta_\varepsilon)[\eta_\varepsilon]\\ &= t_\varepsilon\big[\|\eta_\varepsilon\|^2_{X_0^s(\Omega)} + \lambda\|\eta_\varepsilon\|^2_{L^2(\Omega)}\big] - \int_\Omega g(t_\varepsilon\eta_\varepsilon, \underline{u}_\mu)\eta_\varepsilon dx = 0. \end{aligned} \tag{3.14}$$

Combining this equality and Lemmas 3.1 and 3.6, we can easily conclude that for $\varepsilon > 0$ small enough, it holds that

$$A_1 < t_\varepsilon < A_2 \tag{3.15}$$

where A_1, A_2 ere positive constants independent of ε. By Lemma 3.1 (ii), we have

$$\begin{aligned} &\sup_{t>0} \bar I_{\lambda,\mu}(t\eta_\varepsilon)\\ &= \bar I_{\lambda,\mu}(t_\varepsilon\eta_\varepsilon)\\ &= \frac{t_\varepsilon^2}{2}\Big[\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\eta_\varepsilon|^2 dx + \lambda\int_\Omega \eta_\varepsilon^2 dx\Big] - \int_\Omega G(t_\varepsilon v_\varepsilon, \underline{u}_\mu)dx\\ &= \frac{t_\varepsilon^2}{2}\Big[\int_{\mathbb{R}^N} |(-\Delta)^{s/2}\eta_\varepsilon|^2 dx + \lambda\int_\Omega \eta_\varepsilon^2 dx\Big] - \frac{1}{p+1}t_\varepsilon^{p+1} - \int_\Omega H(t_\varepsilon v_\varepsilon, \underline{u}_\mu)dx. \end{aligned}$$

Since $\underline{u}_\mu > 0$ in Ω, there exists $s_0 > 0$ such that $\underline{u}_\mu > s_0$ for $|x| < r$. From Lemma 3.2 (ii), we obtain, for $\varepsilon \le \frac{r}{2}$

$$\begin{aligned} \int_\Omega H(t_\varepsilon\eta_\varepsilon, \underline{u}_\mu)dx &= \int_{B_r(0)} H(t_\varepsilon\eta_\varepsilon, \underline{u}_\mu)dx \ge C\int_{|x|\le\varepsilon} (t_\varepsilon\eta_\varepsilon)^p dx\\ &\ge C\int_{|x|\le\varepsilon}\Big(\frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2 + |x|^2)^{(N-2s)/2}}\Big)^p dx \end{aligned}$$

$$= C\varepsilon^{(N-2s)/2}\int_{|y|\le 1}\frac{dy}{(1+|y|^2)^{(N+2s)/2}}.$$

Using the estimates in Lemma 3.6, we obtain

$$\begin{aligned}\sup_{t>0}\bar{I}_{\lambda,\mu}(t\eta_\varepsilon) &\le \frac{s}{N}S(N,s)^{N/2s}+O(\varepsilon^{N-2s})-O(\varepsilon^{(N-2s)/2})\\ &\quad + C\lambda\begin{cases}O(\varepsilon^{2s}), & \text{if } N>4s,\\ O(\varepsilon^{2s}|\ln\varepsilon|), & \text{if } N=4s,\\ O(\varepsilon^{N-2s}), & \text{if } N<4s.\end{cases}\end{aligned}$$

For $\mu\in(0,\bar{\mu})$ and either (i) or (ii) in Theorem 1.5 holds, then we obtain

$$0<\sup_{t>0}\bar{I}_{\lambda,\mu}(t\eta_\varepsilon)<\frac{s}{N}S(N,s)^{N/2s}$$

for $\varepsilon>0$ sufficiently small. □

Proof of (i) of Theorem 1.5. Lemma 3.5 implies that $\{u_n\}$ is bounded in $X_0^s(\Omega)$. Thus, there exist a subsequence, still denote by $\{u_n\}$, and some $u\in X_0^s(\Omega)$ such that

$$\begin{gathered} u_n\rightharpoonup u\quad \text{weakly in } X_0^s(\Omega);\\ u_n\to u\text{strongly in } L^r(\Omega),\ \forall 1\le r<2_s^*;\\ u_n\to u\text{a.e. in } \Omega.\end{gathered}$$

Since $\bar{I}'_{\lambda,\mu}(w_n)\to 0$ as $n\to\infty$, for any $\psi\in X_0^s(\Omega)$, we have

$$\int_{\mathbb{R}^N}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\psi dx+\lambda\int_\Omega u_n\psi dx-\int_\Omega g(u_n,\underline{u}_\mu)\psi dx=o(1)\|\psi\|_{X_0^s(\Omega)}.$$

Letting $n\to\infty$, from Lemma 3.3, we have

$$\int_{\mathbb{R}^N}(-\Delta)^{s/2}u(-\Delta)^{s/2}\psi dx+\lambda\int_\Omega u\psi dx-\int_\Omega g(u,\underline{u}_\mu)\psi dx=0.$$

Putting $\psi=u$, we have

$$\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2dx+\lambda\int_\Omega u^2dx-\int_\Omega g(u,\underline{u}_\mu)udx=0.$$

From Lemma 3.1 (ii), we find that

$$\bar{I}_{\lambda,\mu}(u)=\frac{1}{2}\int_\Omega g(u,\underline{u}_\mu)wdx-\int_\Omega G(u,\underline{u}_\mu)dx\ge 0. \tag{3.16}$$

Now we show that $u_n\to u$ strongly in $X_0^s(\Omega)$. Set $v_n=u_n-u$, then

$$\begin{gathered} v_n\rightharpoonup 0\quad \text{weakly in } X_0^s(\Omega);\\ v_n\to 0\quad \text{strongly in } L^r(\Omega),\ \forall 1\le r<2_s^*;\\ v_n\to 0\quad \text{a.e. in } \Omega.\end{gathered}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u_n|^2dx+\lambda\int_\Omega |u_n|^2dx\\ &=\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2dx+\lambda\int_\Omega|u|^2dx+\int_{\mathbb{R}^N}|(-\Delta)^{s/2}v_n|^2dx+o(1).\end{aligned} \tag{3.17}$$

By the Brézis-Lieb Lemma [3], we have

$$\int_\Omega (u_n^+)^{p+1}dx = \int_\Omega (u^+)^{p+1}dx + \int_\Omega (v_n^+)^{p+1}dx + o(1). \tag{3.18}$$

Then from (3.6), (3.8) and (3.18),we obtain

$$\begin{aligned}\int_\Omega g(u_n,\underline{u}_\lambda)u_n dx &= \int_\Omega h(u_n,\underline{u}_\lambda)u_n dx + \int_\Omega (u_n^+)^{p+1}dx \\ &= \int_\Omega h(u,\underline{u}_\lambda)u dx + \int_\Omega (u^+)^{p+1}dx + \int_\Omega (v_n^+)^{p+1}dx + o(1) \\ &= \int_\Omega g(u,\underline{u}_\lambda)u dx + \int_\Omega (v_n^+)^{p+1}dx + o(1).\end{aligned} \tag{3.19}$$

Similarly, it follows from (3.6), (3.7) and (3.18) that

$$\begin{aligned}\int_\Omega G(u_n,\underline{u}_\lambda)dx &= \int_\Omega H(u_n,\underline{u}_\lambda)dx + \frac{1}{p+1}\int_\Omega (u_n^+)^{p+1}dx \\ &= \int_\Omega G(u,\underline{u}_\lambda)dx + \frac{1}{p+1}\int_\Omega (v_n^+)^{p+1}dx + o(1).\end{aligned} \tag{3.20}$$

Then, combining (3.17) and (3.20), we obtain

$$\begin{aligned}\bar{I}_{\lambda,\mu}(u_n) &= \bar{I}_{\lambda,\mu}(u) + \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}v_n|^2 dx - \frac{1}{p+1}\int_\Omega (v_n^+)^{p+1}dx \\ &= c + o(1)\end{aligned} \tag{3.21}$$

as $n\to\infty$. Since $\bar{I}'_{\lambda,\mu}(u_n)\to 0$ and $\{u_n\}$ is bounded in $X_0^s(\Omega)$, we obtain

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_n|^2 dx + \lambda\int_\Omega |u_n|^2 dx - \int_\Omega g(u_n,\underline{u}_\lambda)u_n dx = o(1).$$

From (3.17) and (3.19), we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}v_n|^2 dx - \int_\Omega (v_n^+)^{p+1}dx = o(1).$$

Since $\{v_n\}$ is bounded in $X_0^s(\Omega)$, we may assume that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}v_n|^2 dx \to \ell \quad\text{and}\quad \int_\Omega (v_n^+)^{p+1}dx \to \ell$$

for some $\ell\ge 0$. By the definition of $S(N,s)$, we obtain $S(N,s)\ell^{(N-2s)/N}\le \ell$. Assume that $\ell>0$, then, $\ell\ge S(N,s)^{N/2s}$. Letting $n\to\infty$ in (3.21) and combine (3.16), we have

$$c = \bar{I}_{\lambda,\mu}(u) + \frac{s}{N}S(N,s)^{N/2s} \ge \frac{s}{N}S(N,s)^{N/2s},$$

which contradicts with the definition of c in (3.10). Thus, $\ell=0$, and

$$\bar{I}_{\lambda,\mu}(u) = c > 0 \quad\text{and}\quad \bar{I}'_{\lambda,\mu}(u) = 0,$$

which gives that u is a nontrivial solution of (1.15), and $u\ge 0$. By Proposition 2.1, we have $u>0$ in Ω. The proof is complete. □

Proof of Theorem 1.5 (ii). Let $\lambda>0$ and $N\ge 6s$. As the proof of (i) of Theorem 1.5, we only need to prove that there exist $\mu^*\in(0,\bar{\mu})$ and $v_\mu\in X_0^s(\Omega)$ such that

$$\sup_{t>0}\bar{I}_{\lambda,\mu}(tv_\mu) < \frac{s}{N}S(N,s)^{N/2s} \quad\text{for } \mu^*\le\mu\le\bar{\mu}. \tag{3.22}$$

Indeed, let $\underline{u}_{\bar\mu}$ be the unique positive solution of (1.1) with $\mu = \bar\mu$, and denote $\phi_\mu \in X_0^s(\Omega)$ is the eigenfunction corresponding to the first eigenvalue $\kappa_1(\mu)$ to the problem (2.15). We also assume that $\phi_\mu > 0$ in Ω and $\|\phi_\mu\|_{L^{p+1}} = 1$. Take $\varepsilon > 0$ small, such that

$$\varepsilon(2p\|\underline{u}_{\bar\mu}\|_{L^{p+1}}^{p-1})^{N/2s} < S(N,s)^{N/2s}. \tag{3.23}$$

Note that $1 < p \le 2$ if $N \ge 6s$. By Lemma 3.2 (iii) we have, for $v \in X_0^s(\Omega)$,

$$\begin{aligned}
\bar I_{\lambda,\mu}(v) &= \frac{1}{2}\Big(\|v\|_\lambda^2 - p\int_\Omega \underline{u}_\mu^{p-1}v^2dx\Big) - \frac{1}{p+1}\int_\Omega v^{p+1}dx \\
&\quad + \frac{p}{2}\int_\Omega \underline{u}_\mu^{p-1}v^2dx - \int_\Omega H(v,\underline{u}_\mu) \\
&\le \frac{1}{2}\Big(\|v\|_\lambda^2 - (1-\varepsilon)p\int_\Omega \underline{u}_\mu^{p-1}v^2dx\Big) - \frac{\varepsilon}{p+1}\int_\Omega v^{p+1}dx.
\end{aligned}$$

Putting

$$P_\varepsilon(v) := \|v\|_\lambda^2 - (1-\varepsilon)p\int_\Omega \underline{u}_\mu^{p-1}v^2dx,$$

it follows that

$$\sup_{t>0}\bar I_{\lambda,\mu}(t\phi_\mu) \le \sup_{t>0}\Big(\frac{t^2}{2}P_\varepsilon(\phi_\mu) - \frac{\varepsilon t^{p+1}}{p+1}\Big) = \frac{s[P_\varepsilon(\phi_\mu)]^{N/2s}}{N\varepsilon^{(N-2s)/2s}}. \tag{3.24}$$

Recall that ϕ_μ attains the infimum $\kappa_1(\mu)$ to the minimization problem (2.20); that is,

$$\|\phi_\mu\|_\lambda^2 = \kappa_1(\mu)\int_\Omega \underline{u}_\mu^{p-1}\phi_\mu^2dx.$$

Thus,

$$P_\varepsilon(\phi_\mu) = (\kappa_1(\mu) - 1 + \varepsilon)p\int_\Omega \underline{u}_\mu^{p-1}\phi_\mu^2dx.$$

By Hölder's inequality, we have

$$\int_\Omega \underline{u}_\mu^{p-1}\phi_\mu^2dx \le \|\underline{u}_\mu\|_{L^{p+1}}^{p-1}\|\phi_\mu\|_{L^{p+1}}^2 = \|\underline{u}_\mu\|_{L^{p+1}}^{p-1}.$$

Since $\underline{u}_\mu$ is increasing in $\mu \in (0,\bar\mu)$ for each $x \in \Omega$, we have

$$P_\varepsilon(\phi_\mu) \le (\kappa_1(\mu) - 1 + \varepsilon)p\|\underline{u}_{\bar\mu}\|_{L^{p+1}}^{p-1}.$$

By Lemma 2.10, there exists $\mu^* \in (0,\bar\mu)$ such that $\kappa_1(\mu) - 1 < \varepsilon$ for $\mu \in [\mu^*,\bar\mu)$. Then, we have $P_\varepsilon(\phi_\mu) \le 2\varepsilon p\|\underline{u}_{\bar\mu}\|_{L^{p+1}}^{p-1}$ for $\mu \in [\mu^*,\bar\mu)$. It follows from (3.23) and (3.24) that

$$\sup_{t>0}\bar I_{\lambda,\mu}(t\phi_\mu) \le \frac{s(2p\|\underline{u}_{\bar\mu}\|_{L^{p+1}}^{p-1})^{N/2s}\varepsilon}{N} < \frac{s}{N}S(N,s)^{N/2s}$$

for $\mu \in [\mu^*,\bar\mu)$. Thus, we obtain (3.22) and complete the proof. $\square$

4. Proof of theorem 1.6

In the following, let $\Omega = \{x \in \mathbb{R}^N : \ |x| < R\}$, and assume that $f = f(|x|)$ is radially symmetric about the origen, f satisfies (A1) and $f(r)$ is decreasing in $r \in [0, R]$. Let $\underline{u}_\mu$ be the minimal solution obtained in Theorem 1.3. Then $\underline{u}_\mu \in C^{0,\gamma}(\Omega)$ for any $0 < \gamma < 2s$ if $2s \le 1$, or $\underline{u}_\mu \in C^{1,\gamma}(\Omega)$ for any $0 < \gamma < 2s - 1$ if $2s > 1$, and the results in [16] show that $\underline{u}_\mu = \underline{u}_\mu(r)$ must be radially symmetric about the origin and strictly decreasing in $r \in [0, R]$ by the method of moving planes.

Let us consider the problem

$$(-\Delta)^s v + \lambda v = g(v, \underline{u}_\mu) \quad \text{in } \Omega,\ v \in X_0^s(\Omega), \tag{4.1}$$

where $g(v, \underline{u}_\mu) = G'_v(v, \underline{u}_\mu)$ which is defined by (3.3).

Thanks to the Pohozaev identity for the fractional Laplacian obtained by Ros-Oton et al. [27], from the similar calculations we can get the following lemma which shows a Pohozaev identity for (4.1).

Lemma 4.1. *If v is a solution of (4.1), then*

$$\begin{aligned} &\int_\Omega \Big[\frac{2N}{N-2s} G(v, \underline{u}_\mu) - g(v, \underline{u}_\mu) v\Big] dx + \frac{2}{N-2s} \int_\Omega G_\tau(v, \underline{u}_\mu) \nabla(\underline{u}_\mu) \cdot x dx \\ &- \frac{2\lambda s}{N-2s} \int_\Omega v^2 dx \\ &= C_s \int_{\partial\Omega} (\frac{v}{\delta^s})^2 (x \cdot \gamma) ds, \end{aligned} \tag{4.2}$$

where $G_\tau(\sigma, \tau) = (\partial/\partial\tau) G(\sigma, \tau)$, $\delta(x) = dist(x, \partial\Omega)$, C_s is a constant related to s and γ is the unit outward normal to $\partial\Omega$ at x.

Proof. By [1, Proposition 2], we have $v \in L^\infty(\Omega)$. Then we can easily deduce the result by [27, Proposition 1.12]. □

Lemma 4.2. *If $\underline{u}_\mu$ is the minimal solution of problem (1.1), and $f \in C^\alpha(\Omega)$, then*

$$\|\underline{u}_\mu\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \mu \to 0. \tag{4.3}$$

Proof. If we can deduce that there exists a super-solution u^*_μ of problem (1.1) and $\|u^*_\mu\|_{L^\infty(\Omega)} \to 0$ as $\mu \to 0$, then by the definition of $\underline{u}_\mu$, the proof is done. To achieve this, we denote $e \in X_0^s(\Omega)$ is the solution of

$$\begin{gathered} (-\Delta)^s e + \lambda e = 1, \quad \text{in } \Omega, \\ e > 0 \quad \text{in } \Omega, \\ e = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega. \end{gathered}$$

Since $f \in C^\alpha(\Omega)$ and $p > 1$, we can find $\mu_0 > 0$ such that for all $0 < \mu \le \mu_0$, there exists $M = M(\mu) > 0$ satisfying

$$\begin{gathered} M \ge \mu \|f\|_{L^\infty(\Omega)} + M^p \|e\|^p_{L^\infty(\Omega)}, \\ M(\mu) \to 0 \text{ as } \mu \to 0. \end{gathered}$$

As a consequence, the function Me satisfies

$$M = (-\Delta)^s (Me) + \lambda(Me) \ge \mu \|f\|_{L^\infty(\Omega)} + M^p \|e\|^p_{L^\infty(\Omega)}.$$

and hence it is a super-solution of (1.1). Moreover, by Lemma 2.3, we can obtain $\underline{u}_\mu \le M(\mu) e$. This immediately implies (4.3) since $e \in L^\infty(\Omega)$. □

From two lemmas above, we can achieve the following proposition which plays an important role in proving Theorem 1.6.

Proposition 4.3. *Let $N \geq 6s$ and $\lambda > 0$. Then there exists a $\mu^* > 0$ such that problem* (4.1) *has no positive solution for $0 < \mu < \mu^*$,*

Proof. Assume that there exists a positive solution v of (4.1) for some $\mu > 0$. By the definition of G, we have $G_\tau(\sigma, \tau) \geq 0$ for $\sigma, \tau \geq 0$. Since $\underline{u}_\mu(r)$ is decreasing in $r \in [0, R]$, then $(\underline{u}_\mu)_r(r) \leq 0$ a.e. in $[0, R]$, we obtain

$$\int_\Omega G_\tau(v, \underline{u}_\mu)\nabla(\underline{u}_\mu) \cdot x dx = \int_0^R r^N G_\tau(v(r), \underline{u}_\mu(r))(\underline{u}_\mu)_r(r) \leq 0. \tag{4.4}$$

Substituting (4.4) into (4.2), we deduce that

$$\frac{2\lambda s}{N - 2s} \int_\Omega v^2 dx \leq \int_\Omega \Big[(p + 1)G(v, \underline{u}_\mu) - g(v, \underline{u}_\mu)v\Big] dx. \tag{4.5}$$

Note that $p = (N + 2s)/(N - 2s) \leq 2$ when $N \geq 6s$. Then, it follows from Lemma 3.1 (v) that

$$(p + 1)G(v, \underline{u}_\mu) - g(v, \underline{u}_\mu)v \leq \frac{p(p - 1)}{2} \|\underline{u}_\mu\|^{p-1}_{L^\infty(\Omega)} v^2 \quad \text{for } 0 \leq r \leq R. \tag{4.6}$$

From 4.2, there exists $\mu_* > 0$ such that

$$\|\underline{u}_\mu\|^{p-1}_{L^\infty(\Omega)} < \frac{4\lambda s}{(N - 2s)p(p - 1)} \tag{4.7}$$

for $\mu \in (0, \mu_*]$. Then, it follows from (4.5) and (4.6) that

$$(p + 1)G(v, \underline{u}_\mu) - g(v, \underline{u}_\mu)v < \frac{2\lambda s}{N - 2s} v^2 \quad \text{for } 0 \leq r \leq R \tag{4.8}$$

if $\mu \in (0, \mu_*]$. This contradicts with (4.5). Therefore, (4.1) has no positive solution for $\mu \in (0, \mu_*]$. □

Proof of Theorem 1.6. Assume to the contrary that (1.1) has an another positive solution u with $u \not\equiv \underline{u}_\mu$ for some $\mu \in (0, \mu_*]$. Put $v = u - \underline{u}_\mu$. Since $\underline{u}_\mu$ is the minimal solution, v is non-negative and satisfies (4.1) with $g(v, \underline{u}_\mu) \geq 0$. By Proposition 2.1, $v > 0$ in Ω. Therefore, v is a positive solution of (4.1). This contradicts to Proposition 4.3. Thus, (1.1) has a unique positive solution $\underline{u}_\mu$ for $\mu \in (0, \mu_*]$. □

Acknowledgments. This research is supported by NSFC grants No. 11561024, 11701178 and 11371160, by the Science and Technology project of Jiangxi Provinvial Education Department (GJJ150537). The authors are grateful to Professor Yinbin Deng and Professor Shuangjie Peng for their fruitful discussions on the subject.

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