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ANALYSIS OF STAGNATION POINT FLOW OF AN UPPER-CONVECTED MAXWELL FLUID

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Abstract. Several recent papers have investigated the two-dimensional stagnation point flow of an upper-convected Maxwell fluid by employing a similarity change of variable to reduce the governing PDEs to a nonlinear third order ODE boundary value problem (BVP). In these previous works, the BVP was studied numerically and several conjectures regarding the existence and behavior of the solutions were made. The purpose of this article is to mathematically verify these conjectures. We prove the existence of a solution to the BVP for all relevant values of the elasticity parameter. We also prove that this solution has monotonically increasing first derivative, thus verifying the conjecture that no "overshoot" of the boundary condition occurs. Uniqueness results are presented for a large range of parameter space and bounds on the skin friction coefficient are calculated.

1. Introduction

This article analyzes the boundary value problem (BVP) governing two-dimensional stagnation-point flow of a fluid obeying the upper-convected Maxwell model [7], [8], [11]. Models such as this have been developed to describe the viscoelastic properties that certain fluids exhibit, and which not captured using standard Newtonian theory. Several previous studies concerning stagnation-point flow of viscoelastic fluids investigated a second-grade fluid model [3, 10]. However, Sadeghy et al. [11] note that the second-grade model is valid only for slow flows involving small levels of elasticity. Also, overshoot of the free stream velocity inside the boundary layer has been observed for the second-grade model [1, 3]. A later study [6] suggested that the second-grade model of [3] needed to be augmented and in this augmented model no overshoot was observed [2, 12].

Given these limitations and uncertainties surrounding the behavior of stagnation-point flow of a second-grade fluid, Sadeghy et al. [11] conducted an investigation of the same physical configuration using an upper-convected Maxwell (UCM) model. They note that the upper-convected Maxwell model is valid for much larger values of elasticity than is the second-grade model. They endeavored to determine whether the behavior observed in the second-grade model regarding the velocity profiles and the skin friction coefficient are present in the UCM model.

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Specifically, the problem derived in [11] (see also [7, 8]) is

$$[1+kf(\eta)^2]f'''(\eta)+[1-2kf'(\eta)]f(\eta)f''(\eta)+1-f'(\eta)^2=0, \quad 0<\eta<\infty, \tag{1.1}$$

subject to

$$f(0)=0, \quad f'(0)=0, \quad f'(\infty)=1. \tag{1.2}$$

where $k>0$ is the elasticity parameter

Sadeghy et al. [11] investigate this BVP numerically and report a unique solution for each value of k. They also detect no overshoot in the velocity profiles for any value of k. (Overshoot would involve a solution to the BVP with velocities that exceed the free-stream velocity far from the plate and would be exhibited by $f'(\eta)>1$ on some interval(s) of η.) Sadeghy et al. also find that the skin friction coefficient, which is proportional to $f''(0)$, decreases as the elasticity parameter k increases.

The objective of this paper is to determine whether these numerical observations can be verified mathematically from direct analysis of the BVP. We note that since the BVP (1.1)-(1.2) is nonlinear, there is no guarantee that a solution even exists, or that it is unique. Thus our first goal (Section 2) is to prove that for any $k>0$, a solution to the BVP does indeed exist. This solution will be shown to satisfy $0<f'(\eta)<1$ and $f''(\eta)>0$ for all $\eta>0$. Thus a solution without overshoot does exist for this model. In fact, since it can be seen from the ODE (1.1) that f' cannot have a maximum above $f'=1$, solutions with overshoot are not possible in this model. Next, for $k\geq k_0\approx 3.5584$ (defined in Section 3) we prove that there cannot be two solutions which both satisfy $0<f'<1$ and $f''>0$ for all $\eta>0$. We show that any second (or further) solutions must have very specific behavior, including a minimum of f' below -1. Thus, any further solutions would have to exhibit the physically dubious behavior of flow reversal ($f'<0$). While such solutions are not unknown in some flow configurations [13], we conjecture, based on our numerical investigations, that they do not occur in the current model. In Section 4, we discuss the behavior of the skin friction coefficient as a function of the elasticity parameter as well as other qualitative properties of the solution. Finally, in Section 5, we discuss open problems for the current model and how the analysis presented here might be applied to other models involving the upper-convected Maxwell fluid model. We note that techniques employed to other stagnation point problems, such as those used in [4], might fruitfully be applied to the current problem.

To investigate the BVP (1.1)-(1.2) we study a family of initial value problems (IVPs) given by the ODE (1.1) along with the conditions at $\eta=0$, (1.2)a,b. To this we add a third condition at $\eta=0$, namely,

$$f''(0)=\alpha. \tag{1.3}$$

Equations (1.1),(1.2)a,b and (1.3) constitute a well-posed IVP. By standard existence and uniqueness theory, for each value of α, this IVP will have a unique local solution on some open interval containing $\eta=0$. Denote this solution by $f(\eta;\alpha)$. Occasionally the dependence of f on η or α or both will be dropped for convenience and ease of reading. In the next section we show that a value α^* exists such that $f(\eta;\alpha^*)$ exists for all $\eta>0$ and also satisfies the boundary condition at infinity, (1.2)c, giving a solution to the BVP.

Nomenclature:

- f : dimensionless stream function
- η : similarity variable

- k : elasticity parameter
- α : topological shooting parameter
- v : derivative of f with respect to α
- $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \overline{\eta}, \overline{\overline{\eta}}$: various values of η used in the proofs
- $k_0, k_1, k_2, \hat{k}$: various values of k used in the proofs
- $\alpha_1, \alpha_2, \overline{\alpha}, \alpha^*$: various values of α used in the proofs

2. Existence and qualitative properties

Theorem 2.1. *For every $k > 0$ there exists a solution to the BVP* (1.1)-(1.2). *Further, this solution satisfies $0 < f'(\eta) < 1$ and $f''(\eta) > 0$ for all $\eta > 0$.*

Proof. The proof of existence will involve the following subsets of $(0, \infty)$:

$$\mathcal{A} = \big\{\alpha > 0 : f''(\eta; \alpha) = 0 \text{ before } f'(\eta; \alpha) = 1\big\},$$
$$\mathcal{B} = \big\{\alpha > 0 : f'(\eta; \alpha) = 1 \text{ before } f''(\eta; \alpha) = 0\big\}.$$

We will show that each of these sets is non-empty and open. For $\mathcal{A}$ this is just a matter of continuity of the solutions of the IVP (1.1), (1.2)a,b and (1.3) in its initial conditions. We claim that for all $\alpha > 0$ sufficiently small, $\alpha \in \mathcal{A}$. To see this, first note that

$$f'''(0; \alpha) = -1 < 0. \tag{2.1}$$

If we take $\alpha = 0$, then for small $\eta > 0$ we have $f'(\eta; 0) < 1$ and $f''(\eta; 0) < 0$; say on $(0, \varepsilon]$ for some $\varepsilon > 0$. By continuity of the solutions of the initial value problem in its initial conditions, for $\alpha > 0$ sufficiently small, $f'(\eta; \alpha)$ will stay close to $f'(\eta; 0)$, i.e. we can arrange that $f'(\eta; \alpha) < 1$ on $(0, \varepsilon]$ with $f''(\varepsilon; \alpha) < 0$. But as $f''(0; \alpha) > 0$, there must exist a first $\eta_0 \in (0, \varepsilon)$ such that $f''(\eta_0; \alpha) = 0$ with $f'(\eta; \alpha) < 1$ on $[0, \eta_0]$. Thus for $\alpha > 0$ sufficiently small we have $\alpha \in \mathcal{A}$. To show that $\mathcal{A}$ is open, consider $\overline{\alpha} \in \mathcal{A}$. We will show that all α sufficiently close to $\overline{\alpha}$ are also in $\mathcal{A}$. At $\eta_0(\overline{\alpha})$ we have $0 < f'(\eta_0; \overline{\alpha}) < 1$ and $f''(\eta_0; \overline{\alpha}) = 0$. Evaluating (1.1) at $\eta_0(\overline{\alpha})$ implies that

$$f'''(\eta_0; \overline{\alpha}) = \frac{f'(\eta_0; \overline{\alpha})^2 - 1}{1 + kf(\eta_0; \overline{\alpha})^2} \neq 0. \tag{2.2}$$

Thus, by continuity of the solutions of the IVP in its initial conditions, for α sufficiently close to $\overline{\alpha}$, $f''(\eta; \alpha)$ will also have a root, $\eta_0(\alpha)$, near $\eta_0(\overline{\alpha})$ with $f'(\eta_0; \alpha) < 1$. Thus $\alpha \in \mathcal{A}$ and $\mathcal{A}$ is open.

The study of the set $\mathcal{B}$ will require bounds on f and f'. First note that integrating (1.1), (by parts where necessary), from 0 to η gives

$$f''(\eta)[1 + kf(\eta)^2] = \alpha - \eta + (2kf' - 1)ff' + 2\int_0^\eta f'(t)^2[1 - kf'(t)]\, dt \tag{2.3}$$

We claim that for large positive α, $f' = 1$ in the interval $(0, 1]$ strictly before $f'' = 0$. (In fact in this case, if $f' = 1$ before $f'' = 0$, then $f'' = 0$ cannot occur at all.) Suppose that the assertion is false. Then one of the following must occur: (i) $f'' = 0$ at some first point in $(0, 1]$ with $f' < 1$, (ii) $f'' > 0$ and $f' < 1$ for all $\eta \in (0, 1]$, or (iii) $f'' = 0$ and $f' = 1$ simultaneously. We eliminate each of these in turn. To begin with (i), suppose that there exists a first $\eta_1 \in (0, 1]$ with

$$f''(\eta_1) = 0 \tag{2.4}$$

with $0 \le f' < 1$ for $\eta \in [0, \eta_1]$. Integrating $0 \le f' < 1$ from 0 to η gives $0 \le f < \eta \le 1$ on $[0, \eta_1] \subset [0, 1]$. Using these bounds on f and f' and ignoring the positive terms other than α on the right hand side of (2.3) we obtain:

$$f''(\eta) \ge \frac{\alpha - 2(k+1)}{k+1} \quad \eta \in [0, \eta_1]. \tag{2.5}$$

Thus if we choose $\alpha > 2(k+1)$ then $f''(\eta_1) > 0$ contradicting (2.4). A similar argument shows that if $\alpha > 3(k+1)$ then we cannot have (ii) $f'' > 0$ and $f' < 1$ on all of $(0, 1]$. (i.e. $f'(1)$ will be greater than 1.) This leaves only case (iii) $f' = 1$ and $f'' = 0$ simultaneously; however, substituting this information into (1.1) gives $f''' = 0$ implying that $f'(\eta) \equiv 1$, contradicting the basic existence and uniqueness theorem for initial value problems, as $f'(0) = 0 \ne 1$. Thus if $\alpha > 3(k+1)$ then we must have $f' = 1$ strictly before $f'' = 0$ and therefore $\alpha \in \mathcal{B}$. An argument similar to that for the set $\mathcal{A}$ shows that $\mathcal{B}$ is also open.

Thus, the sets $\mathcal{A}$ and $\mathcal{B}$ are non-empty and open. They are also obviously disjoint. But the interval $(0, \infty)$ is connected and thus $\mathcal{A} \cup \mathcal{B} \ne (0, \infty)$. Therefore, there exists some α^* such that $\alpha^* \notin \mathcal{A}$ and $\alpha^* \notin \mathcal{B}$. For such a value of α^* the only possibility is for the solution $f(\eta; \alpha^*)$ to exist for all $\eta > 0$ with $0 < f'(\eta; \alpha^*) < 1$ and $f''(\eta; \alpha^*) > 0$.

It remains to show that $f'(\infty; \alpha^*) = 1$, satisfying the boundary condition at infinity (1.2)c. Since $f'(\eta; \alpha^*)$ is positive, increasing, and bounded above by 1 we conclude that $f'(\infty; \alpha^*) = L \le 1$ exists. Suppose for contradiction that $0 < L < 1$.

To begin, we claim that $f''' \le 0$ for all $\eta > 0$. Since $f'''(0) = -1$, f''' starts off negative. Suppose that f''' were to assume a positive value at some point, say $f'''(\eta_2) > 0$ for some $\eta_2 > 0$. We could not have $f''' > 0$ for all $\eta > \eta_2$ since two integrations from η_2 to $\eta > \eta_2$ of the inequality $f''' > 0$ would imply that $f' \to +\infty$ as $\eta \to \infty$ contradicting the boundedness of f'. Thus f''' would at some point have to decrease back through zero. Recalling the other properties of f from above, this would require a point at which $f > 0$, $0 < f' < 1$, $f'' > 0$, $f''' = 0$ and $f^{(4)} \le 0$. Differentiating (1.1) and evaluating at such a point implies that

$$(1 + kf^2)f^{(4)} - f'f'' - 2k(f')^2 f'' - 2kf(f'')^2 = 0. \tag{2.6}$$

However, each term on the left side of (2.6) is non-positive with the last three necessarily negative and we have a contradiction. Thus f''' cannot become positive and we have $f''' \le 0$ for all $\eta > 0$. This along with $f'' > 0$ for all $\eta > 0$, implies that $f''(\infty)$ exists and since $f'(\infty)$ also exists we must have $f''(\infty) = 0$.

Next note that from the properties above on f through f''' we can conclude from the equation for the fourth derivative,

$$(1 + kf^2)f^{(4)} + ff''' - (1 + 2kf')f'f'' - 2kf(f'')^2 = 0, \tag{2.7}$$

that $f^{(4)} > 0$ for all $\eta > 0$. This along with $f''' \le 0$ implies that $f'''(\infty)$ exists, and must be zero since $f''(\infty)$ also exists.

Next, rewrite the ODE (1.1) as

$$(1 - 2kf')ff'' + kf^2 f''' = (f')^2 - 1 - f'''. \tag{2.8}$$

On the right side of (2.8) we have that $(f')^2 - 1 - f''' \to L^2 - 1 = -m < 0$ as $\eta \to \infty$. Thus there exists an $\eta_3 > 0$ such that

$$(1 - 2kf')ff'' + kf^2 f''' < -\frac{m}{2} \quad \forall \eta > \eta_3. \tag{2.9}$$

Dividing by $f > 0$ we have

$$f'' - 2kf'f'' + kff''' < -\frac{m}{2}\frac{1}{f} \quad \forall \eta > \eta_3. \tag{2.10}$$

Since $0 < f' < L$ for all $\eta > 0$ (not just $\eta > \eta_3$) we have on integration from 0 to $\eta > 0$ that

$$f < L\eta, \quad \forall \eta > 0, \tag{2.11}$$

or, after some rearrangement,

$$-\frac{1}{f} < -\frac{1}{L\eta}, \quad \forall \eta > 0. \tag{2.12}$$

Using (2.12) in (2.10) we obtain

$$f'' - 2kf'f'' + kff''' < -\frac{m}{2L\eta}, \quad \forall \eta > \eta_3. \tag{2.13}$$

Integrating (by parts where necessary) from η_3 to $\eta > \eta_3$, collecting some terms and aggregating all constants on the left into a quantity called C we obtain

$$f' - \frac{3k}{2}(f')^2 + kff'' + C < -\frac{m}{2L}\left(\ln \eta - \ln \eta_3\right), \quad \forall \eta > \eta_3. \tag{2.14}$$

Since f' is bounded, letting $\eta \to \infty$ we conclude that $ff'' \to -\infty$, giving a contradiction since $f > 0$ and $f'' > 0$. Thus the assumption that $f' \to L < 1$ leads to a contradiction and we must have $f'(\infty; \alpha^*) = 1$ giving a solution to the BVP. □

3. Uniqueness results

Theorem 3.1. *If $k \geq k_0 \approx 3.5584$, then there cannot be two solutions $f(\eta)$ of the BVP (1.1)-(1.2) both of which satisfy $0 < f'(\eta) < 1$ and $f''(\eta) > 0$ for all $\eta > 0$. The value k_0 will be defined precisely in the proof.*

Proof. By contradiction, assume the the existence of two solutions, $f(\eta; \alpha_1)$ and $f(\eta; \alpha_2)$, of the BVP (1.1)-(1.2) both of which satisfy $0 < f'(\eta; \alpha_i) < 1$ and $f''(\eta; \alpha_i) > 0$ for all $\eta > 0$, $i = 1, 2$. Without loss of generality assume that $\alpha_2 > \alpha_1$.

To arrive at a contradiction we will use the quantity $v = \partial f / \partial \alpha$. Differentiating the ODE (1.1) along with the initial conditions for $f(\eta; \alpha)$ with respect to α we see that $v(\eta; \alpha)$ satisfies the following IVP:

$$(1 + kf^2)v''' + (1 - 2kf')fv'' - 2(kff'' + f')v' + [(1 - 2kf')f'' + 2kff''']v = 0, \tag{3.1}$$

subject to

$$v(0) = 0, \quad v'(0) = 0, \quad v''(0) = 1. \tag{3.2}$$

Evaluating this at $\eta = 0$ gives $v'''(0) = 0$. Differentiating (3.1) with respect to η gives

$$\begin{aligned} &(1 + kf^2)v^{(4)} + fv''' - [(1 + 2kf')f' + 4kff'']v'' - (4kf' + 1)f''v' \\ &+ [2kff^{iv} + f''' - 2kf''^2]v = 0. \end{aligned} \tag{3.3}$$

Evaluating at $\eta = 0$ we have $v^{(4)}(0) = 0$. Finally, differentiating (3.3) with respect to η and evaluating at $\eta = 0$ gives $v^{(5)}(0) = 2\alpha > 0$. Our main interest is in the behavior of $v(\eta; \alpha)$ and its derivatives for $\alpha_1 \leq \alpha \leq \alpha_2$ and $\eta > 0$. Note that for small $\eta > 0$ we have $v > 0$, $v' > 0$, $v'' > 0$, $v''' > 0$ and $v^{(4)} > 0$. Ultimately, we wish to show that v' cannot have a maximum and thus $v' > 0$ will be bounded away from zero as $\eta \to \infty$. Also note, that before v' can have a maximum, v'' must first have a maximum.

So for the purpose of contradiction suppose that v'' has a first maximum at $\overline{\eta}$ and v' has a first maximum at $\overline{\overline{\eta}} > \overline{\eta}$. At $\overline{\eta}$ we have $v(\overline{\eta};\alpha) > 0$, $v'(\overline{\eta};\alpha) > 0$, $v''(\overline{\eta};\alpha) > 0$, $v'''(\overline{\eta};\alpha) = 0$, and $v^{(4)}(\overline{\eta};\alpha) \le 0$. Note also that, up until this point, $v(\eta;\alpha)$ and all its derivative through $v'''(\eta;\alpha)$ are positive. Thus $f(\eta;\alpha)$ and all its derivatives through $f'''(\eta;\alpha)$ are increasing functions of α. Thus we conclude that

$$0 < f(\eta;\alpha_1) \le f(\eta;\alpha) \le f(\eta;\alpha_2), \tag{3.4}$$

$$0 < f'(\eta;\alpha_1) \le f'(\eta;\alpha) \le f'(\eta;\alpha_2) < 1, \tag{3.5}$$

$$0 < f''(\eta;\alpha_1) \le f''(\eta;\alpha) \le f''(\eta;\alpha_2), \tag{3.6}$$

$$f'''(\eta;\alpha_1) \le f'''(\eta;\alpha) \le f'''(\eta;\alpha_2) < 0, \tag{3.7}$$

for all $0 < \eta \le \overline{\eta}$. With these inequalities in place, we see from (2.7) that

$$f^{(4)}(\eta;\alpha) > 0 \tag{3.8}$$

for all $0 < \eta \le \overline{\eta}$ and $\alpha_1 \le \alpha \le \alpha_2$.

Evaluating (3.1) at $\overline{\eta}$ gives

$$(1-2kf')fv'' - 2(kff''+f')v' + [(1-2kf')f'' + 2kff''']v = 0 \quad \text{at } \overline{\eta}. \tag{3.9}$$

Given the conditions listed in the last paragraph, a necessary condition for (3.9) to hold is that $1-2kf'(\overline{\eta}) > 0$. Thus $f'(\overline{\eta}) < 1/2k$. But since f' is strictly increasing we have

$$f'(\eta) < \frac{1}{2k} \text{on } [0,\overline{\eta}]. \tag{3.10}$$

Next, evaluating the ODE for $v^{(4)}$ (3.3) at $\overline{\eta}$ we have

$$\begin{aligned}&(1+kf^2)v^{(4)} - [(1+2kf')f' + 4kff'']v'' - (4kf'+1)f''v' \\ &+ [2kff^{(4)} + f''' - 2kf''^2]v = 0 \quad \text{at } \overline{\eta}.\end{aligned} \tag{3.11}$$

The terns involving v' and v'' are strictly negative and the term involving $v^{(4)}$ is less than or equal to zero. Since $v > 0$, the only way for (3.11) to hold is if

$$2kff^{(4)} + f''' - 2kf''^2 > 0 \quad \text{at } \overline{\eta}. \tag{3.12}$$

Using (2.7), the inequality (2.14) can be rewritten as

$$\begin{aligned}&\frac{kf''^2(4kf^2-1)}{1+kf^2} + \frac{f'''(1-kf^2)}{1+kf^2} + \frac{kf''(2ff'+4kff'^2-f'')}{1+kf^2} \\ &- \frac{2k^2f^2f''^2}{1+kf^2} > 0 \quad \text{at } \overline{\eta}.\end{aligned} \tag{3.13}$$

The last term of (3.13) is negative. Next, several technical lemmas will derive bounds on α, $\overline{\eta}$ and $\overline{\overline{\eta}}$ as well as show that the first two terms of (3.13) are nonpositive. We begin with several bounds on f and its derivatives.

Using (3.4) through (3.8) along with (3.10) we conclude

$$-1 < f''' < 0 \quad \text{on } (0,\overline{\eta}], \tag{3.14}$$

$$\alpha - \eta < f'' < \alpha \quad \text{on } (0,\overline{\eta}], \tag{3.15}$$

$$\alpha\eta - \frac{\eta^2}{2} < f' < \min\big\{\alpha\eta, \frac{1}{2k}\big\} \quad \text{on } (0,\overline{\eta}], \tag{3.16}$$

$$\frac{\alpha\eta^2}{2} - \frac{\eta^3}{6} < f < \min\Big\{\frac{\alpha\eta^2}{2}, \frac{\eta}{2k}\Big\} \quad \text{on } (0,\overline{\eta}]. \tag{3.17}$$

□

Lemma 3.2. *The quantity*

$$\frac{kf''^2(4kf^2-1)}{1+kf^2} \quad \text{at } \overline{\eta} \tag{3.18}$$

is nonpositive.

Proof. For contradiction suppose that

$$\frac{kf''^2(4kf^2-1)}{1+kf^2} > 0 \quad \text{at } \overline{\eta} \tag{3.19}$$

For (3.19) to hold we must have $4kf(\overline{\eta})^2 - 1 > 0$, or

$$f(\overline{\eta}) > \frac{1}{2\sqrt{k}}. \tag{3.20}$$

From (3.17) we have

$$f(\eta) < \frac{\eta}{2k} \quad \text{on } [0,\overline{\eta}]. \tag{3.21}$$

Combining (3.20) and (3.21) we have

$$\frac{1}{2\sqrt{k}} < f(\overline{\eta}) < \frac{\overline{\eta}}{2k}, \tag{3.22}$$

which implies that

$$\overline{\eta} > \sqrt{k}. \tag{3.23}$$

We next show that $\alpha \geq \sqrt{k}$. For contradiction suppose that $\alpha < \sqrt{k} < \overline{\eta}$. The last part of this inequality using (3.23). Then from (3.16) we have

$$\frac{\alpha^2}{2} < f'(\alpha) < f'(\overline{\eta}) < \frac{1}{2k}, \tag{3.24}$$

from which we conclude that

$$\alpha < \frac{1}{\sqrt{k}}. \tag{3.25}$$

Recall that

$$f''(\eta)[1+kf(\eta)^2] = \alpha - \eta + (2kf'-1)ff' + 2\int_0^\eta f'(t)^2[1-kf'(t)]\,dt. \tag{3.26}$$

Using (3.10) and (3.25) we obtain

$$f''(\eta)[1+kf(\eta)^2] < \frac{1}{\sqrt{k}} + \Big(\frac{1-2k^2}{2k^2}\Big)\eta \quad \text{on } [0,\overline{\eta}], \tag{3.27}$$

which is less than or equal to zero if $\eta \geq 2k^2/(\sqrt{k}(2k^2-1))$. Since $f'' > 0$ we must therefore have

$$\overline{\eta} < \frac{2k^2}{\sqrt{k}(2k^2-1)}. \tag{3.28}$$

Combining (3.28) with (3.23) we have

$$\sqrt{k} < \overline{\eta} < \frac{2k^2}{\sqrt{k}(2k^2-1)}, \tag{3.29}$$

which implies

$$h(k) = 2k^3 - 2k^2 - k < 0. \tag{3.30}$$

The positive root of $h(k)$ is $\hat{k} = (1+\sqrt{3})/2 \approx 1.366$ with $h(k) < 0$ if $k < \hat{k}$. Since $k \ge k_0 > \hat{k}$ we arrive at a contradiction.

Thus if $k > k_0$, we must have

$$\alpha \ge \sqrt{k} \tag{3.31}$$

Using (3.31) in (3.16) we obtain

$$f' > \alpha\eta - \frac{\eta^2}{2} > \sqrt{k}\eta - \frac{\eta^2}{2}. \tag{3.32}$$

Since f' is increasing and $\overline{\eta} > \sqrt{k}$ we have

$$f'(\eta) > f'(\sqrt{k}) > \frac{k}{2} \text{ on } [\sqrt{k}, \overline{\eta}]. \tag{3.33}$$

Combining (3.33) with (3.10) we have

$$\frac{k}{2} < f'(\overline{\eta}) < \frac{1}{2k}, \tag{3.34}$$

which implies that $k < 1$ giving a contradiction since $k \ge k_0 > 1$, thus the proof is complete. □

Thus if $k \ge k_0$, we have

$$\frac{kf''^2(4kf^2-1)}{1+kf^2} \le 0 \quad \text{at } \overline{\eta}. \tag{3.35}$$

Note that this implies that $f(\overline{\eta}) \le 1/2\sqrt{k}$, and since f is increasing we have

$$f(\eta) \le \frac{1}{2\sqrt{k}} \quad \text{on } [0, \overline{\eta}]. \tag{3.36}$$

Lemma 3.3. *The quantity*

$$\frac{f'''(1-kf^2)}{1+kf^2} \quad \textit{at } \overline{\eta} \tag{3.37}$$

is nonpositive.

Proof. Note that $f'''(\overline{\eta}) < 0$. Also, from (3.36) we have $f(\overline{\eta}) < 1/2\sqrt{k}$. Thus $1 - kf(\overline{\eta})^2 > 1 - k(1/2\sqrt{k})^2 = 3/4 > 0$. Thus

$$\frac{f'''(1-kf^2)}{1+kf^2} < 0 \qquad \text{at } \overline{\eta} \tag{3.38}$$

which completes the proof. □

Lemma 3.4.

$$f''(\overline{\overline{\eta}}) < \frac{1}{k^{3/2}} \quad \textit{and} \quad \overline{\overline{\eta}} > \alpha - \frac{1}{k^{3/2}} \tag{3.39}$$

Proof. By Lemmas 3.2 and 3.3, in order for (3.13) to hold we must have

$$\frac{kf''(2ff'(1+2kf')-f'')}{1+kf^2} > 0 \quad \text{at } \overline{\eta}. \tag{3.40}$$

Since $f''(\overline{\eta}) > 0$, (3.40) implies that

$$2ff'(1+2kf') - f'' > 0 \quad \text{at } \overline{\eta}. \tag{3.41}$$

Using (3.10) and (3.36) in the quantity on the left-hand side of (3.41) we have

$$\begin{aligned}&2f(\overline{\eta})f'(\overline{\eta})(1+2kf'(\overline{\eta}))-f''(\overline{\eta})\\&<2\left(\frac{1}{2\sqrt{k}}\right)\left(\frac{1}{2k}\right)\left(1+2k\left(\frac{1}{2k}\right)\right)-f''(\overline{\eta})=\frac{1}{k^{3/2}}-f''(\overline{\eta}).\end{aligned}\tag{3.42}$$

We obtain a contradiction of (3.40) if the right most term of (3.42) is less than or equal to zero. Thus we must have $f''(\overline{\eta})<1/k^{3/2}$. Since f'' is decreasing and $\overline{\overline{\eta}}>\overline{\eta}$ we obtain

$$f''(\overline{\overline{\eta}})<\frac{1}{k^{3/2}}.\tag{3.43}$$

For the lower bound on $\overline{\overline{\eta}}$, we first consider $\overline{\eta}$. Suppose that $\overline{\eta}\le\alpha$. Then, since $\alpha-\eta<f''(\eta)$ on $[0,\overline{\eta}]$ we have

$$2f(\overline{\eta})f'(\overline{\eta})(1+2kf'(\overline{\eta}))-f''(\overline{\eta})<\frac{1}{k^{3/2}}-f''(\overline{\eta})<\frac{1}{k^{3/2}}+\overline{\eta}-\alpha.\tag{3.44}$$

Again, we have a contradiction if the right most term of (3.44) is less than or equal to zero. Thus

$$\overline{\overline{\eta}}>\overline{\eta}>\alpha-\frac{1}{k^{3/2}}.\tag{3.45}$$

□

Lemma 3.5. *For $\alpha_1\le\alpha\le\alpha_2$,*

$$\alpha>\frac{\sqrt{54k^2-2}}{9k^{3/2}}.\tag{3.46}$$

Proof. Multiplying (1.1) by f'' and integrating (by parts where necessary) from 0 to η we have

$$\frac{1}{2}(1+kf^2)f''^2-\frac{1}{2}\alpha^2+\int_0^\eta(1-3kf')ff''^2\,dt+f'-\frac{1}{3}f'^3=0.\tag{3.47}$$

Let η_4 be the point where $f(\eta;\alpha_1)$ increases through $1/3k$. (Note that $k\ge k_0>1/3$ so that $1/3k<1$). Then

$$\frac{1}{2}\alpha_1^2=\frac{27k^2-1}{81k^3}+\frac{1}{2}(1+kf(\eta_4)^2)f''(\eta_4)^2+\int_0^{\eta_4}(1-3kf')ff''^2\,dt.\tag{3.48}$$

Since all three terms on the right side of (3.48) are positive we have

$$\frac{1}{2}\alpha_1^2>\frac{27k^2-1}{81k^3},\tag{3.49}$$

or, for $\alpha_1\le\alpha\le\alpha_2$,

$$\alpha\ge\alpha_1>\frac{\sqrt{54k^2-2}}{9k^{3/2}}.\tag{3.50}$$

Recall that at a first maximum of v' at $\overline{\overline{\eta}}$, we have $v>0$, $v'>0$, $v''=0$ and $v'''\le 0$. Evaluating (3.1) at $\overline{\overline{\eta}}$ we have

$$(1+kf^2)v'''-2(kff''+f')v'+[(1-2kf')f''+2kff''']v=0\quad\text{at }\overline{\overline{\eta}}.\tag{3.51}$$

The first term of (3.51) is nonpositive and the second is strictly negative. Since $v>0$, we must therefore have

$$(1-2kf')f''+2kff'''>0\quad\text{at }\overline{\overline{\eta}}\tag{3.52}$$

in order for (3.51) to hold. Recall that $f > 0$, $f'' > 0$ and $f''' < 0$. Thus a necessary condition for (3.51) to hold is that $f'(\bar{\bar{\eta}}) < 1/2k$. Since f' is increasing, we thus have

$$f'(\eta) < \frac{1}{2k} \quad \text{on } [0, \bar{\bar{\eta}}]. \tag{3.53}$$

Next we use (1.1) to rewrite (3.52) as

$$(1-2kf')f'' + 2kf\frac{f'^2 - 1 + (2kf'-1)ff''}{1+kf^2} > 0 \quad \text{at } \bar{\bar{\eta}}, \tag{3.54}$$

or

$$\frac{1}{1+kf^2}\left[(1-2kf')(1-kf^2)f'' + 2kf(f'^2-1)\right] > 0 \quad \text{at } \bar{\bar{\eta}}. \tag{3.55}$$

We will show that (3.55) cannot hold, giving us our final contradiction to the assumption that v' can have a positive maximum. To this end, first note that since $f(\bar{\bar{\eta}}) > 0$, $f'(\bar{\bar{\eta}}) < 1/2k < 1$ and $f''(\bar{\bar{\eta}}) > 0$, a necessary condition for (3.55) to hold is that

$$f(\bar{\bar{\eta}}) < 1/\sqrt{k}. \tag{3.56}$$

The argument will take two paths depending on whether $\bar{\bar{\eta}} > \alpha$ or $\bar{\bar{\eta}} \le \alpha$.

If $\bar{\bar{\eta}} > \alpha$, then since f is increasing we have from (3.17)

$$\frac{\alpha^3}{3} < f(\alpha) < f(\bar{\bar{\eta}}). \tag{3.57}$$

Using the facts that $\alpha^3/3 < f(\bar{\bar{\eta}}) < 1/\sqrt{k}$, $f'(\bar{\bar{\eta}}) < 1/2k < 1$ and $f''(\bar{\bar{\eta}}) < 1/k^{3/2}$ in the quantity in brackets in (3.55) we have

$$\begin{aligned} &(1-2kf'(\bar{\bar{\eta}}))(1-kf(\bar{\bar{\eta}})^2)f''(\bar{\bar{\eta}}) + 2kf(\bar{\bar{\eta}})(f'(\bar{\bar{\eta}})^2-1) \\ &< \frac{1}{k^{3/2}} + 2k\frac{\alpha^3}{3}\Big(\frac{1}{4k^2} - 1\Big). \end{aligned} \tag{3.58}$$

Next, using (3.46) we have

$$\begin{aligned} &(1-2kf'(\bar{\bar{\eta}}))(1-kf(\bar{\bar{\eta}})^2)f''(\bar{\bar{\eta}}) + 2kf(\bar{\bar{\eta}})(f'(\bar{\bar{\eta}})^2-1) \\ &< \frac{1}{k^{3/2}} + \frac{(54k^2-2)^{3/2}(1-4k^2)}{6\cdot 9^3\cdot k^{11/2}}. \end{aligned} \tag{3.59}$$

We obtain a contradiction of (3.55) if the right side of (3.59) is less than or equal to zero. This occurs if $k \ge k_1 >\approx 2.8618$ where k_1 is the root of

$$\frac{1}{k^{3/2}} + \frac{(54k^2-2)^{3/2}(1-4k^2)}{6\cdot 9^3\cdot k^{11/2}} = 0. \tag{3.60}$$

Finally, if $\bar{\bar{\eta}} \le \alpha$, then from Lemma 3.4 we have $\bar{\bar{\eta}} > \alpha - 1/k^{3/2}$. Using this in (3.17) and the fact that f is increasing we have

$$\frac{\alpha}{3}\Big(\alpha - \frac{1}{k^{3/2}}\Big)^2 < f(\bar{\bar{\eta}}). \tag{3.61}$$

Using the facts that $\alpha(\alpha - 1/k^{3/2})^2/3 < f(\bar{\bar{\eta}}) < 1/\sqrt{k}$, $f'(\bar{\bar{\eta}}) < 1/2k < 1$ and $f''(\bar{\bar{\eta}}) < 1/k^{3/2}$ in the quantity in brackets in (3.55) we have

$$\begin{aligned} &(1-2kf'(\bar{\bar{\eta}}))(1-kf(\bar{\bar{\eta}})^2)f''(\bar{\bar{\eta}}) + 2kf(\bar{\bar{\eta}})(f'(\bar{\bar{\eta}})^2-1) \\ &< \frac{1}{k^{3/2}} + 2k\frac{\alpha}{3}\Big(\alpha - \frac{1}{k^{3/2}}\Big)^2\Big(\frac{1}{4k^2} - 1\Big). \end{aligned} \tag{3.62}$$

Using the result of Lemma 3.5 in (3.62), at $\overline{\overline{\eta}}$ we have

$$\begin{aligned}&(1-2kf')(1-kf^2)f''+2kf(f'^2-1)\\&<\frac{1}{k^{3/2}}+\frac{2k}{3}\frac{\sqrt{54k^2-2}}{9k^{3/2}}\Big(\frac{\sqrt{54k^2-2}}{9k^{3/2}}-\frac{1}{k^{3/2}}\Big)^2\Big(\frac{1}{4k^2}-1\Big).\end{aligned}\tag{3.63}$$

We obtain a contradiction of (3.55) if the right side of (3.63) is less than or equal to zero. This occurs if $k \geq k_2 > \approx 4.9377$ where k_2 is the root of

$$\frac{1}{k^{3/2}}+\frac{2k}{3}\frac{\sqrt{54k^2-2}}{9k^{3/2}}\Big(\frac{\sqrt{54k^2-2}}{9k^{3/2}}-\frac{1}{k^{3/2}}\Big)^2\Big(\frac{1}{4k^2}-1\Big)=0. \tag{3.64}$$

The analysis of the last paragraph used the rather crude bounds $1-kf(\overline{\overline{\eta}})^2<1$ and $1-2kf'(\overline{\overline{\eta}})<1$. An improved value for k can be obtained by employing the lower bounds in (3.17) and (3.16). We thus obtain a contradiction of (3.55) if k is larger than the root of

$$\begin{aligned}&\Big(1-\frac{\sqrt{54k^2-2}(\sqrt{54k^2-2}-9)}{81k^2}\Big)\Big(1-\frac{\sqrt{54k^2-2}(\sqrt{54k^2-2}-9)^2}{3\cdot 9^3\cdot k^{7/2}}\Big)\big(\frac{1}{k^{3/2}}\big)\\&+\frac{2k}{3}\frac{\sqrt{54k^2-2}}{9k^{3/2}}\Big(\frac{\sqrt{54k^2-2}}{9k^{3/2}}-\frac{1}{k^{3/2}}\Big)^2\Big(\frac{1}{4k^2}-1\Big)=0.\end{aligned}$$

The root of this equation defines the value $k_0 \approx 3.5584$ mentioned in the statement of the theorem.

This final contradiction proves that v' cannot have a maximum. Thus $v''>0$ for all $\eta>0$, and since $v'(0)=0$ we conclude that $v'>0$ is bounded away from zero for η large.

Next, if two solutions with the properties given in the statement of Theorem 3.1 were to exist, then by the Mean Value Theorem we would have

$$f'(\eta;\alpha_2)-f'(\eta;\alpha_1)=\Big(\frac{\partial f'(\eta;\alpha)}{\partial\alpha}\Big)_{\alpha=\hat{\alpha}}(\alpha_2-\alpha_1)=v'(\eta;\hat{\alpha})(\alpha_2-\alpha_1), \tag{3.65}$$

for $\eta>0$ where $\hat{\alpha}\in(\alpha_1,\alpha_2)$. Since $v'(\eta;\hat{\alpha})$ is bounded away from zero for η large, there exists a constant $M>0$ such that

$$0=f'(\infty;\alpha_2)-f'(\infty;\alpha_1)=\lim_{\eta\to\infty}v'(\eta;\hat{\alpha})(\alpha_2-\alpha_1)>M(\alpha_2-\alpha_1)>0. \tag{3.66}$$

This contradiction proves Theorem 3.1. □

Our numerical investigations indicate solutions that violate the inequalities $0<f'<1$ and $f''>0$ do not exist, however we have not been able to prove this. We end this section by discussing the properties that any second (or further) solutions must have. First note that from the ODE (1.1), f' can only have a maximum in the range $-1<f'<1$ and can only have a minimum if either $f'>1$ or $f'<-1$. Thus, as mentioned in the introduction, there can be no solutions exhibiting overshoot (values of f' above 1). Further, any second solution would have to have at least one minimum below -1, and thus would exhibit flow reversal ($f'<0$), which is unlikely in this physical configuration.

4. Bounds on the skin friction coefficient

A quantity of much physical interest is the skin friction coefficient which is proportional to $f''(0)=\alpha$. The following result gives bounds on α. (In this section, α corresponds to the value $\alpha(k)$ that gives the solution to the BVP (1.1)-(1.2) with the properties $0<f'(\eta,\alpha)<1$ and $f''(\eta,\alpha)>0$ for all $\eta>0$.)

Theorem 4.1. *For $0 < k \le 1/3$ we have*

$$f''(0) = \alpha > \frac{2}{\sqrt{3}}. \tag{4.1}$$

For $k > 1/3$ we have

$$f''(0) = \alpha > \frac{\sqrt{54k^2 - 2}}{9k^{3/2}} \tag{4.2}$$

For $k \ge \hat{k} \approx 2.2825$ we have

$$f''(0) = \alpha < \frac{2}{\sqrt{3}}. \tag{4.3}$$

The exact value of $\hat{k}$ will be defined in the proof.

Proof. Using the fact that f' approaches one from below in the ODE (1.1), we conclude that ff'' tends to zero. Letting $\eta \to \infty$ in (3.47) we conclude that

$$f''(0) = \alpha^* > \frac{2}{\sqrt{3}} \quad \text{for } 0 < k \le \frac{1}{3}, \tag{4.4}$$

whereas the result of Lemma 3.5 gives

$$f''(0) = \alpha^* > \frac{\sqrt{54k^2 - 2}}{9k^{3/2}} \quad \text{for } k > \frac{1}{3}. \tag{4.5}$$

To obtain the upper bound on α for $k > \hat{k}$ we again let $\eta \to \infty$ in (3.47) to obtain

$$\int_0^\infty (1 - 3kf')ff''^2\,dt = \frac{1}{2}\Big(\alpha^2 - \frac{4}{3}\Big). \tag{4.6}$$

If the integral on the left hand side of (4.6) is negative the result follows. Suppose for contradiction that the integral is non-negative. Then $\alpha \ge 2/\sqrt{3}$. From §2 we have that for a solution that satisfies $0 < f' < 1$ and $f'' > 0$ for all $\eta > 0$, we also have $f^{(4)} > 0$ for all $\eta > 0$. This again leads to the bounds

$$-1 < f'''(\eta) < 0, \tag{4.7}$$

$$\alpha - \eta < f''(\eta) < \alpha, \tag{4.8}$$

$$\alpha\eta - \frac{\eta^2}{2} < f'(\eta) < \alpha\eta, \tag{4.9}$$

$$\frac{\alpha\eta^2}{2} - \frac{\eta^3}{6} < f(\eta) < \frac{\alpha\eta^2}{2}. \tag{4.10}$$

Using $\alpha \ge 2/\sqrt{3}$ and (4.9) we have $f'(2/\sqrt{3}) > 2/3$. Since $f''' < 0$ we have

$$\frac{\eta}{\sqrt{3}} = l(\eta) < f'(\eta) \quad \text{on } [0, 2/\sqrt{3}], \tag{4.11}$$

where $l(\eta) = \eta/\sqrt{3}$ is the line through $(0,0)$ and $(2/\sqrt{3}, 2/3)$. Let η_5 be the point where f' increases through $1/3k$. Using (4.11) we have that $\eta_5 < 1/\sqrt{3}k$. Using (4.8) through (4.11) we have

$$\begin{aligned} \int_0^{\eta_5} (1 - 3kf')ff''^2\,dt &< \int_0^{1/\sqrt{3}k} \Big(1 - 3k\big(\frac{t}{\sqrt{3}}\big)\Big)\Big(\frac{\alpha t^2}{2}\Big)(\alpha)^2\,dt \\ &= \frac{\alpha^3}{72\sqrt{3}k^3}. \end{aligned} \tag{4.12}$$

Next, again employing the bounds (4.8) through (4.11) we have

$$\begin{aligned}\int_{\eta_5}^{1/\sqrt{3}} (1-3kf')ff''^2\,dt &< \int_{1/\sqrt{3}k}^{1/\sqrt{3}} (1-3k(\alpha t))\Big(\frac{6k-1}{18\sqrt{3}k^3}\Big)\Big(\frac{1}{\sqrt{3}}\Big)^2\,dt \\ &= \frac{(6k-1)(1+k-k^2)}{162k^4}.\end{aligned} \tag{4.13}$$

Combining (4.12) and (4.13) we have

$$\int_0^{1/\sqrt{3}} (1-3kf')ff''^2\,dt < \frac{\alpha^3}{72\sqrt{3}k^3} + \frac{(6k-1)(1+k-k^2)}{162k^4}. \tag{4.14}$$

We arrive at a contradiction (of the assumption that $\int_0^\infty (1-3kf')ff''^2\,dt \geq 0$) if the right hand side of (4.14) is negative and thus we must have

$$\alpha > \Big(\frac{4\sqrt{3}(6k-1)(k^2-k-1)}{9k}\Big)^{1/3}. \tag{4.15}$$

Using (4.15) in (4.9) we have

$$f'(1/\sqrt{3}) > \Big(\frac{4\sqrt{3}(6k-1)(k^2-k-1)}{9k}\Big)^{1/3}\frac{1}{\sqrt{3}} - \frac{1}{6}. \tag{4.16}$$

We obtain a contradiction (of $0 < f' < 1$ for all $\eta > 0$) if the right side of (4.16) is greater or equal to 1. This occurs if $k \geq \hat{k} \approx 2.2825$ where $\hat{k}$ is the positive root of

$$\Big(\frac{4\sqrt{3}(6k-1)(k^2-k-1)}{9k}\Big)^{1/3}\frac{1}{\sqrt{3}} - \frac{7}{6} = 0. \tag{4.17}$$

Thus for $k \geq \hat{k}$ we have $\alpha < 2/\sqrt{3}$. □

5. Discussion and open problems

Through direct analysis of the BVP (1.1)-(1.2) we have proven the existence of a solution for stagnation point flow of an upper-convected Maxwell fluid. This solution is shown to satisfy $0 < f' < 1$ and $f'' > 0$ for all $\eta > 0$. For $k \geq k_0$ we have also shown that a solution *with these properties* is unique. Any further solutions must exhibit the physically unrealistic property of flow reversal in the boundary layer.

The analysis presented here should prove useful in the study of generalizations of the stagnation point problem posed in Sadeghy et al. [11]. Kumari and Nath [7] extended this model by considering the effects of heat transfer and an induced magnetic field on the flow. (See equations (8) and (11) in [7].) More recently, Lok et al. [8] considered a generalization that incorporated the effects of a shrinking sheet with suction. (See equations (7) and (9) in [8].) Straightforward extensions of the arguments given here should yield existence results for both of these problems. The uniqueness results will no doubt prove more problematic. One reason is that the already technical nature of Theorem 3.1 will no doubt become more involved in these more complicated problems. Another reason is that the numerical results of Lok et al. [8] indicate that no solution to their boundary value problem exists for values of the shrinking parameter λ less than a critical value. There are really no "standard" methods for proving nonexistence of solutions to BVPs. Each problem generally has to be approached on an ad-hoc basis. However, the techniques of [9] may prove applicable to the problem posed in [8]. Finally, we mention that the techniques employed here involved a third order ODE with two boundary conditions at $\eta = 0$ and one boundary condition at infinity. The method does generalize to higher dimensional ODEs where the dimension of the topological shooting space may also increase depending on the number of boundary conditions at each boundary. Thus, the methods could be applied to the higher dimensional problems considered in [4], [5], [7] and [8].

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