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NONEXISTENCE OF SOLUTIONS TO SOME INEQUALITIES AND SYSTEMS WITH SINGULAR COEFFICIENTS ON THE BOUNDARY

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Abstract. We obtain sufficient conditions for the nonexistence of positive solutions to some elliptic inequalities and systems containing the p-Laplace operators and coefficients possessing singularities on the boundary.

1. Introduction

The problem of sufficient conditions for nonexistence of solutions to systems of nonlinear elliptic differential equations and inequalities with singular coefficients has been studied by many authors. For the Laplacian and heat operator with a point singularity inside the domain, pioneering results in this direction were obtained by Brezis and Cabré [1] by means of comparison principles. For higher order operators that do not satisfy the comparison principle, Pohozaev [11] suggested the nonlinear capacity method. Later it was developed in joint works with Mitidieri and other authors (see, in particular, the monograph [10] and references therein). This method allowed one to obtain a number of new sharp sufficient conditions of non-solvability of differential inequalities in various functional classes. The method is based on deriving asymptotically optimal a priori estimates of the solutions by means of algebraic analysis of the integral form of the inequality under consideration with a special choice of test functions. Applications of this method to different types of elliptic inequalities and systems containing degeneracy, point singularities, gradient terms etc. can be found, for example, in [4, 5, 9].

In the present paper, a modification of the nonlinear capacity method is used in order to obtain dimension independent sufficient conditions of non-solvability for some quasilinear elliptic inequalities in a bounded domain with coefficients having singularities near the boundary. This distinguishes the problem setting suggested here from the aforementioned works in this field, where singularities appeared at single points or at infinity. In [9], some results concerning the case of boundary singularities are also obtained, but they are dimension dependent.

For the proof of nonexistence results by the nonlinear capacity method, test functions with different geometrical structure of the support are constructed, which

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takes into account the specific nature of problems under consideration. Our first results in this direction were published in [6, 7].

The rest of the paper consists of two sections. In §2, we establish nonexistence results for scalar quasilinear elliptic inequalities, and in §3, for systems of such inequalities.

From here on, letter c denotes different positive constants, which may depend on the parameters of the problems under consideration.

2. Scalar inequalities

Consider the problem

$$\begin{gathered} -\operatorname{div}(|Du|^{p-2}Du) \geq f(x)u^q|Du|^s, \quad x \in \Omega, \\ u(x) \geq 0, \quad x \in \Omega, \end{gathered} \tag{2.1}$$

where Ω is a bounded domain with a smooth boundary, $f(x) \in C(\Omega)$ is a positive function.

Solutions to (2.1) will be understood in the weak (distributional) sense according to the following definition.

Definition 2.1. A nonnegative function $u \in W^{1,p}_{\rm loc}(\Omega)$ will be called a weak (distributional) solution of (2.1) if $f(x)u^q|Du|^s \in L^1_{\rm loc}(\Omega)$ and for each nonnegative test function $\psi \in C^1_0(\Omega)$ it holds

$$\int_\Omega |Du|^{p-2}(Du, D\psi)\,dx \geq \int_\Omega f(x)u^q|Du|^s\psi\,dx. \tag{2.2}$$

Remark 2.2. Similarly to [10], it can be shown that if such a solution exists and is strictly positive in Ω, then (2.2) still holds for test functions of the form $\psi = u^\gamma\varphi$ with $\gamma \in \mathbb{R}$ and $\varphi \in C^1_0(\Omega)$. If u vanishes somewhere in Ω and $\gamma < 0$, one can use test functions $\psi = (u+\delta)^\gamma\varphi$ and take $\delta \to 0_+$, which yields the same results as in the previous case. Therefore we will assume in the sequel that $u > 0$ whenever it exists.

We use the notation $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, and

$$\Omega_{k\eta} = \{x \in \Omega :\ \rho(x) \geq k\eta\} \quad (\eta > 0,\ k = 1, 2).$$

Theorem 2.3. *Let $f(x) \geq c\rho^{-\alpha}(x)$ $(x \in \Omega)$ with some constant $c > 0$, $p > 1$, $q > p-1$, $s > 0$, and $\alpha > q+1$. Then problem* (2.1) *has no nontrivial (distinct from a constant a.e.) weak solutions.*

For other definitions of a solution, the nonexistence condition can be different. In particular, for the so-called very weak solution in the semilinear case $p = 2$, it becomes $\alpha > 2$ (see, e.g., the survey [3]).

Proof of Theorem 2.3. Assume that there exists a nontrivial weak solution u of inequality (2.1). Introduce a family of functions $\varphi_\eta \in C^1_0(\Omega; [0,1])$ of the form $\varphi_\eta(x) = \xi^\lambda_\eta(x)$ with

$$\xi_\eta(x) = \begin{cases} 1, & x \in \Omega_{2\eta}, \\ 0, & x \notin \Omega_\eta, \end{cases} \tag{2.3}$$

$$|D\xi_\eta(x)| \leq c\eta^{-1} \quad (x \in \Omega) \tag{2.4}$$

and $\lambda > 0$ sufficiently large. Then, using a test function $\psi = u^\gamma \varphi_\eta$ with $1-p < \gamma < 0$ in (2.2), we obtain

$$\begin{aligned}
&\int_\Omega f(x) u^{q+\gamma} |Du|^s \varphi_\eta \,dx \\
&\le \int_\Omega (|Du|^{p-2} Du, D(u^\gamma \varphi_\eta)) \,dx \\
&= \gamma \int_\Omega u^{\gamma-1} |Du|^p \varphi_\eta \,dx + \int_\Omega u^\gamma |Du|^{p-2} (Du, D\varphi_\eta) \,dx \\
&\le \gamma \int_\Omega u^{\gamma-1} |Du|^p \varphi_\eta \,dx + \int_\Omega u^\gamma |Du|^{p-1} |D\varphi_\eta| \,dx,
\end{aligned}$$

whence

$$\int_\Omega f(x) u^{q+\gamma} |Du|^s \varphi_\eta \,dx + |\gamma| \int_\Omega u^{\gamma-1} |Du|^p \varphi_\eta \,dx \le \int_\Omega u^\gamma |Du|^{p-1} |D\varphi_\eta| \,dx.$$

Representing the integrand on the right-hand side of this inequality in the form

$$2^{-y/s} u^{\frac{(q+\gamma)y}{s}} |Du|^y f^{y/s} \varphi_\eta^{y/s} 2^{y/s} u^{\frac{\gamma s-(q+\gamma)y}{s}} |Du|^{p-1-y} |D\varphi_\eta| f^{-y/s} \varphi_\eta^{-y/s},$$

where y will be chosen below, and applying the parametric Young inequality with the exponent s/y, we obtain

$$\begin{aligned}
&\frac{1}{2} \int_\Omega f(x) u^{q+\gamma} |Du|^s \varphi_\eta \,dx + |\gamma| \int_\Omega u^{\gamma-1} |Du|^p \varphi_\eta \,dx \\
&\le c \int_\Omega u^{\frac{\gamma s-(q+\gamma)y}{s-y}} |Du|^{\frac{(p-1-y)s}{s-y}} |D\varphi_\eta|^{\frac{s}{s-y}} f^{-\frac{y}{s-y}} \varphi_\eta^{-\frac{y}{s-y}} \,dx.
\end{aligned}$$

Apply the Young inequality with the exponent z,

$$\begin{aligned}
&c \int_\Omega u^{\frac{\gamma s-(q+\gamma)y}{s-y}} |Du|^{\frac{(p-1-y)s}{s-y}} |D\varphi_\eta|^{\frac{s}{s-y}} f^{-\frac{y}{s-y}} \varphi_\eta^{-\frac{y}{s-y}} \,dx \\
&\le \frac{|\gamma|}{2} \int_\Omega u^{\frac{(\gamma s-(q+\gamma)y)z}{s-y}} |Du|^{\frac{(p-1-y)sz}{s-y}} \varphi_\eta \,dx \\
&\quad + c \int_\Omega |D\varphi_\eta|^{\frac{sz'}{s-y}} f^{-\frac{yz'}{s-y}} \varphi_\eta^{1-\frac{sz'}{s-y}} \,dx,
\end{aligned} \tag{2.5}$$

where $\frac{1}{z} + \frac{1}{z'} = 1$.

We choose y and z so that

$$\begin{gathered}
(p-1-y)sz = p(s-y), \\
\frac{\gamma s-(q+\gamma)y}{s-y} z = \gamma - 1,
\end{gathered}$$

i.e.,

$$\begin{gathered}
y = y_\gamma = \frac{s(p+\gamma-1)}{p(q+\gamma) - s(\gamma-1)}, \\
z = z_\gamma = \frac{p[p(q+\gamma) - s(\gamma-1) - (p+\gamma-1)]}{(p-1)(p(q+\gamma) - s(\gamma-1)) - s(p+\gamma-1)}.
\end{gathered}$$

Note that for $\gamma = 0$, by our assumptions $q > p-1 > 0$ and $s > 0$, we have

$$\frac{s}{y_0} = \frac{pq+s}{p-1} > \frac{pq+s}{q} = p + \frac{s}{q} > p > 1,$$

$$z_0 = \frac{p(q-1)+s+1}{(p-1)q} = 1 + \frac{q-(p-1)+s}{p(q-1)} > 1.$$

Hence by continuity, for $|\gamma|$ sufficiently small, one has $\frac{s}{y_\gamma} > 1$ and $z_\gamma > 1$, as required for applying the Young inequality.

For such y and z, and φ_η with properties (2.3), (2.4) and sufficiently large $\lambda > 0$, (2.5) implies

$$\begin{aligned} &\frac{1}{2}\int_\Omega f(x)u^{q+\gamma}|Du|^s\varphi_\eta\,dx + \frac{|\gamma|}{2}\int_\Omega u^{\gamma-1}|Du|^p\varphi_\eta\,dx \\ &\le c\eta^{\frac{\alpha(p+\gamma-1)-p(q+\gamma)+s\gamma+q-p+1}{q+s-p+1}}. \end{aligned} \tag{2.6}$$

Taking $\eta \to 0_+$, for sufficiently small $\gamma < 0$ we obtain a contradiction to the assumed non-triviality of u, which proves the theorem. □

Similar arguments yield an analogous result for the problem with variable exponents

$$\begin{gathered} -\operatorname{div}(|Du|^{p(x)-2}Du) \ge f(x)u^{q(x)}|Du|^{s(x)}, \quad x \in \Omega, \\ u(x) \ge 0, \quad x \in \Omega, \end{gathered} \tag{2.7}$$

where $p(x), q(x), s(x), f(x) \in C(\Omega)$ are appropriate positive functions. This problem will be considered in detail in future article.

3. Systems of inequalities

In this section we consider the system of inequalities

$$\begin{gathered} -\operatorname{div}(|Du|^{p-2}Du) \ge f(x)v^{q_1}|Dv|^{q_2}, \quad x \in \Omega, \\ -\operatorname{div}(|Dv|^{q-2}Dv) \ge g(x)u^{p_1}|Du|^{p_2}, \quad x \in \Omega, \\ u,\, v \ge 0, \quad x \in \Omega, \end{gathered} \tag{3.1}$$

where Ω is a bounded domain with a smooth boundary.

We assume that $p, q > 1$, and $f, g \in C(\Omega)$ are positive functions such that $f(x) \ge a_0\rho^{-\alpha}(x)$, $g(x) \ge b_0\rho^{-\beta}(x)$ for $x \in \Omega$, where $a_0, b_0 > 0$.

The solutions of (3.1) will be understood in the weak (distributional) sense according to the following definition.

Definition 3.1. A pair of nonnegative functions $(u, v) \in W^{1,p}_{\rm loc}(\Omega) \cap W^{1,q}_{\rm loc}(\Omega)$ are called a weak (distributional) solution of (3.1) if $f(x)v^{q_1}|Dv|^{q_2} \in L^1_{\rm loc}(\Omega)$, $g(x)u^{p_1}|Dv|^{p_2} \in L^1_{\rm loc}(\Omega)$, and for any nonnegative test functions $\psi_1, \psi_2 \in C^1_0(\Omega)$ it holds

$$\begin{gathered} \int_\Omega |Du|^{p-2}(Du, D\psi_1)\,dx \ge \int_\Omega f(x)v^{q_1}|Dv|^{q_2}\psi_1\,dx, \\ \int_\Omega |Dv|^{q-2}(Dv, D\psi_2)\,dx \ge \int_\Omega g(x)u^{p_1}|Du|^{p_2}\psi_2\,dx. \end{gathered} \tag{3.2}$$

Similarly to Remark 2.2, we can assume that $u > 0$ and $v > 0$ whenever they exist, and use test functions of the form $\psi_1 = u^\gamma\varphi$ and $\psi_2 = v^\gamma\varphi$ with $\varphi \in C^1_0(\Omega)$.

Theorem 3.2. *Let $p_1 + p_2 > p - 1$, $q_1 + q_2 > q - 1$ and either*

$$(\beta - 1 - p_1)(q_1 + q_2) + (\alpha - 1 - q_1)(q - 1) > 0 \tag{3.3}$$

or

$$(\alpha - 1 - q_1)(p_1 + p_2) + (\beta - 1 - p_1)(p - 1) > 0. \tag{3.4}$$

Then problem (3.1) has no nontrivial solutions.

Proof. Let (u,v) be a nontrivial solution of system (3.1), and $\varphi_\eta \in C_0^\infty(\Omega;[0,1])$ be functions of the same form as in the proof of Theorem 2.3, which satisfy (2.3) and (2.4).

Using a test function $\psi_1 = u^\gamma\varphi_\eta$ in (3.2), and $\psi_2 = v^\gamma\varphi_\eta$ in (3.2), where γ is a number such that $p_1+p_2-p+1<\gamma<0$, $q_1+q_2-q+1<\gamma<0$, we obtain

$$\int f v^{q_1}|Dv|^{q_2}u^\gamma\varphi_\eta\,dx \le \gamma\int u^{\gamma-1}|Du|^p\varphi_\eta\,dx + \int u^\gamma|Du|^{p-1}|D\varphi_\eta|\,dx, \tag{3.5}$$

$$\int g u^{p_1}|Du|^{p_2}v^\gamma\varphi_\eta\,dx \le \gamma\int v^{\gamma-1}|Dv|^q\varphi_\eta\,dx + \int v^\gamma|Dv|^{q-1}|D\varphi_\eta|\,dx. \tag{3.6}$$

We use the representations

$$u^\gamma|Du|^{p-1} = u^{a_1}|Du|^{b_1}\varphi_\eta^{\frac{1}{c_1}}u^{\gamma-a_1}|Du|^{p-1-b_1}\varphi_\eta^{-\frac{1}{c_1}}, \tag{3.7}$$

$$v^\gamma|Dv|^{q-1} = v^{a_2}|Du|^{b_2}\varphi_\eta^{\frac{1}{c_2}}v^{\gamma-a_2}|Dv|^{q-1-b_2}\varphi_\eta^{-\frac{1}{c_2}}, \tag{3.8}$$

to apply to the right-hand sides of (3.5) and (3.6) the parametric Young inequality with exponents denoted by c_1 and c_2, respectively. We choose the parameters so that

$$\begin{gathered} a_1c_1 = \gamma-1, \quad b_1c_1 = p,\\ \frac{\gamma-a_1}{p-1-b_1} = \frac{p_1}{p_2}, \end{gathered} \tag{3.9}$$

and

$$\begin{gathered} a_2c_2 = \gamma-1, \quad b_2c_2 = q,\\ \frac{\gamma-a_2}{q-1-b_2} = \frac{q_1}{q_2}. \end{gathered} \tag{3.10}$$

The purpose of this choice consists in preparation to the consequent application of the Hölder inequality, in order to obtain, under a suitable choice of the parameters, $\int bu^{p_1}|Du|^{p_2}\varphi_\eta\,dx$ and $\int av^{q_1}|Dv|^{q_2}\varphi_\eta\,dx$.

Solving the systems of equations (3.9) and (3.10), we obtain

$$\begin{gathered} a_1 = \frac{(\gamma-1)((p-1)p_1-\gamma p_2)}{pp_1+p_2(1-\gamma)},\\ b_1 = \frac{p((p-1)p_1-\gamma p_2)}{pp_1+p_2(1-\gamma)},\\ c_1 = \frac{pp_1+p_2(1-\gamma)}{(p-1)p_1-\gamma p_2}, \end{gathered} \tag{3.11}$$

and

$$\begin{gathered} a_2 = \frac{(\gamma-1)((q-1)q_1-\gamma q_2)}{qq_1+q_2(1-\gamma)},\\ b_2 = \frac{q((q-1)q_1-\gamma q_2)}{qq_1+q_2(1-\gamma)},\\ c_2 = \frac{qq_1+q_2(1-\gamma)}{(q-1)q_1-\gamma q_2}. \end{gathered} \tag{3.12}$$

Substituting (3.11) and (3.12) in (3.7) and (3.8), we have the representations

$$\begin{aligned} u^\gamma|Du|^{p-1} &= u^{\frac{(\gamma-1)((p-1)p_1-\gamma p_2)}{pp_1+p_2(1-\gamma)}}|Du|^{\frac{p((p-1)p_1-\gamma p_2)}{pp_1+p_2(1-\gamma)}}\varphi_\eta^{\frac{(p-1)p_1-\gamma p_2}{pp_1+p_2(1-\gamma)}}\\ &\quad\times u^{\frac{p_1(p+\gamma-1)}{pp_1+p_2(1-\gamma)}}|Du|^{\frac{p_2(p+\gamma-1)}{pp_1+p_2(1-\gamma)}}\varphi_\eta^{-\frac{(p-1)p_1-\gamma p_2}{pp_1+p_2(1-\gamma)}}, \end{aligned}$$

$$
\begin{aligned}
v^\gamma |Dv|^{q-1} = {}& v^{\frac{(\gamma-1)((q-1)q_1-\gamma q_2)}{qq_1+q_2(1-\gamma)}} |Dv|^{\frac{q((q-1)q_1-\gamma q_2)}{qq_1+q_2(1-\gamma)}} \varphi_\eta^{\frac{(q-1)q_1-\gamma q_2}{qq_1+q_2(1-\gamma)}} \\
&\times v^{\frac{q_1(q+\gamma-1)}{qq_1+q_2(1-\gamma)}} |Dv|^{\frac{q_2(q+\gamma-1)}{qq_1+q_2(1-\gamma)}} \varphi_\eta^{-\frac{(q-1)q_1-\gamma q_2}{qq_1+q_2(1-\gamma)}}.
\end{aligned}
$$

Note that for $\gamma = 0$ we have

$$
c_1 = \frac{qq_1+q_2}{(q-1)q_1} > \frac{(q-1)q_1+q_2}{(q-1)q_1} = 1 + \frac{q_2}{(q-1)q_1} > 1
$$

and similarly $c_2 > 1$. Hence the same inequalities $c_1 > 1$ and $c_2 > 1$ hold by continuity for $|\gamma|$ sufficiently small. Thus, applying to the right-hand sides of (3.5) and (3.6) the parametric Young inequality with the exponents c_1 and c_2 from (3.11) and (3.12) respectively, we arrive at

$$
\begin{aligned}
&\int f v^{q_1} |Dv|^{q_2} u^\gamma \varphi_\eta \, dx + \frac{|\gamma|}{2} \int u^{\gamma-1} |Du|^p \varphi_\eta \, dx \\
&\le c_\gamma \int u^{\frac{p_1(p+\gamma-1)}{p_1+p_2}} |Du|^{\frac{p_2(p+\gamma-1)}{p_1+p_2}} \frac{|D\varphi_\eta|^{\frac{pp_1+p_2(1-\gamma)}{p_1+p_2}}}{\varphi_\eta^{\frac{pp_1+p_2(1-\gamma)}{p_1+p_2}-1}} \, dx, \\
&\int g u^{p_1} |Du|^{p_2} v^\gamma \varphi_\eta \, dx + \frac{|\gamma|}{2} \int v^{\gamma-1} |Dv|^q \varphi_\eta \, dx \\
&\le d_\gamma \int v^{\frac{q_1(q+\gamma-1)}{q_1+q_2}} |Dv|^{\frac{q_2(q+\gamma-1)}{q_1+q_2}} \frac{|D\varphi_\eta|^{\frac{qq_1+q_2(1-\gamma)}{q_1+q_2}}}{\varphi_\eta^{\frac{qq_1+q_2(1-\gamma)}{q_1+q_2}-1}} \, dx,
\end{aligned}
$$

where the constants c_γ and d_γ depend only on p, q, p_1, q_1, p_2, q_2 and γ. Applying the Hölder inequality with the exponents

$$
\begin{aligned}
d_1 &= \frac{p_1+p_2}{p+\gamma-1}, \quad d_1' = \frac{p_1+p_2}{p_1+p_2-p-\gamma+1}, \\
d_2 &= \frac{q_1+q_2}{q+\gamma-1}, \quad d_2' = \frac{q_1+q_2}{q_1+q_2-q-\gamma+1}
\end{aligned}
$$

respectively (note that under our assumptions for $\gamma = 0$

$$
d_1 = \frac{p_1+p_2}{p-1} > 1, \quad d_2 = \frac{q_1+q_2}{q-1} > 1
$$

and hence by continuity $d_1 > 1$ and $d_2 > 1$ for any $|\gamma|$ sufficiently small), we obtain

$$
\begin{aligned}
&\int f v^{q_1} |Dv|^{q_2} u^\gamma \varphi_\eta \, dx + \frac{|\gamma|}{2} \int u^{\gamma-1} |Du|^p \varphi_\eta \, dx \\
&\le c_\gamma \Big(\int g u^{p_1} |Du|^{p_2} \varphi_\eta \, dx \Big)^{\frac{p+\gamma-1}{p_1+p_2}} \\
&\quad \times \Big(\int g^{-\frac{p+\gamma-1}{p_1+p_2-p-\gamma+1}} \frac{|D\varphi_\eta|^{\frac{pp_1+p_2(1-\gamma)}{p_1+p_2-p-\gamma+1}}}{\varphi_\eta^{\frac{pp_1+p_2(1-\gamma)}{p_1+p_2-p-\gamma+1}-1}} \, dx \Big)^{\frac{p_1+p_2-p-\gamma+1}{p_1+p_2}},
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
&\int g u^{p_1}|Du|^{p_2} v^{\gamma}\varphi_\eta\,dx + \frac{|\gamma|}{2}\int v^{\gamma-1}|Dv|^q\varphi_\eta\,dx \\
&\le d_\gamma\Big(\int f v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx\Big)^{\frac{q+\gamma-1}{q_1+q_2}} \\
&\quad\times\Big(\int f^{-\frac{q+\gamma-1}{q_1+q_2-q-\gamma+1}}\frac{|D\varphi_\eta|^{\frac{qq_1+q_2(1-\gamma)}{q_1+q_2-q-\gamma+1}}}{\varphi_\eta^{\frac{qq_1+q_2(1-\gamma)}{q_1+q_2-q-\gamma+1}-1}}\,dx\Big)^{\frac{q_1+q_2-q-\gamma+1}{q_1+q_2}}.
\end{aligned} \tag{3.14}
$$

Further, using test functions $\psi_1=\psi_2=\varphi_\eta$ in (3.2), we obtain

$$
\int a v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx \le \int |Du|^{p-1}|D\varphi_\eta|\,dx\,, \tag{3.15}
$$

$$
\int b u^{p_1}|Du|^{p_2}\varphi_\eta\,dx \le \int |Dv|^{q-1}|D\varphi_\eta|\,dx\,. \tag{3.16}
$$

We use the representation

$$
|Du|^{p-1} = u^{a_3}|Du|^{b_3}\varphi_\eta^{\frac{1}{c_3}}u^{-a_3}|Du|^{p-1-b_3}(g\varphi_\eta)^{\frac{1}{d_3}}g^{-\frac{1}{d_3}}\varphi_\eta^{-\frac{1}{c_3}-\frac{1}{d_3}}, \tag{3.17}
$$

$$
|Dv|^{q-1} = v^{a_4}|Dv|^{b_4}\varphi_\eta^{\frac{1}{c_4}}v^{-a_4}|Dv|^{q-1-b_4}(a\varphi_\eta)^{\frac{1}{d_4}}f^{-\frac{1}{d_4}}\varphi_\eta^{-\frac{1}{c_4}-\frac{1}{d_4}}, \tag{3.18}
$$

for applying to the right-hand sides of (3.15) and (3.16) the triple Young inequality, with the exponents c_3, d_3, e_3 and c_4, d_4, e_4 respectively. Here we choose the parameters so that

$$
\begin{gathered}
a_3c_3=\gamma-1,\quad b_3c_3=p,\quad a_3d_3=-p_1,\\
(p-1-b_3)d_3=p_2,\quad \frac{1}{c_3}+\frac{1}{d_3}+\frac{1}{e_3}=1,
\end{gathered} \tag{3.19}
$$

and

$$
\begin{gathered}
a_4c_4=\gamma-1,\quad b_4c_4=q,\quad a_4d_4=-q_1,\\
(q-1-b_4)d_4=q_2,\quad \frac{1}{c_4}+\frac{1}{d_4}+\frac{1}{e_4}=1.
\end{gathered} \tag{3.20}
$$

Solving the systems of equations (3.19) and (3.20), we obtain

$$
\begin{gathered}
a_3=\frac{(\gamma-1)p_1(p-1)}{pp_1+p_2(1-\gamma)},\\
b_3=\frac{pp_1(p-1)}{pp_1+p_2(1-\gamma)},\\
c_3=\frac{pp_1+p_2(1-\gamma)}{p_1(p-1)},\\
d_3=\frac{pp_1+p_2(1-\gamma)}{(p-1)(1-\gamma)},\\
e_3=\frac{pp_1+p_2(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)},
\end{gathered} \tag{3.21}
$$

and

$$
\begin{aligned}
a_4 &= \frac{(\gamma-1)q_1(q-1)}{qq_1+q_2(1-\gamma)},\\
b_4 &= \frac{qq_1(q-1)}{qq_1+q_2(1-\gamma)},\\
c_4 &= \frac{qq_1+q_2(1-\gamma)}{q_1(q-1)},\\
d_4 &= \frac{qq_1+q_2(1-\gamma)}{(q-1)(1-\gamma)},\\
e_4 &= \frac{qq_1+q_2(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}.
\end{aligned} \tag{3.22}
$$

Note that for $\gamma = 0$ one has

$$
\begin{gathered}
c_3 = \frac{pp_1+p_2}{p_1(p-1)} = \frac{p_1(p-1)+p_1+p_2}{p_1(p-1)} = 1 + \frac{p_1+p_2}{p_1(p-1)} > 1,\\
d_3 = \frac{pp_1+p_2}{p-1} = \frac{p_1(p-1)+p_1+p_2}{p-1} = p_1 + \frac{p_1+p_2}{p-1} > p_1 > 1,\\
e_3 = \frac{pp_1+p_2}{p_1+p_2-p+1} > \frac{p_1+p_2}{p_1+p_2-p+1} > 1,
\end{gathered}
$$

and similar estimates for c_4, d_4, e_4. Then it follows by continuity that for $|\gamma|$ sufficiently small all these exponents also exceed 1, similarly to the previous arguments.

Substituting (3.21) and (3.22) in (3.17) and (3.18), we have the representations

$$
\begin{aligned}
|Du|^{p-1} &= u^{\frac{(\gamma-1)p_1(p-1)}{pp_1+p_2(1-\gamma)}} |Du|^{\frac{pp_1(p-1)}{pp_1+p_2(1-\gamma)}} \varphi_\eta^{\frac{p_1(p-1)}{pp_1+p_2(1-\gamma)}}\\
&\quad \times u^{\frac{p_1(p-1)(1-\gamma)}{pp_1+p_2(1-\gamma)}} |Du|^{\frac{p_2(p-1)(1-\gamma)}{pp_1+p_2(1-\gamma)}} (b\varphi_\eta)^{\frac{(p-1)(1-\gamma)}{pp_1+p_2(1-\gamma)}}\\
&\quad \times g^{-\frac{(p-1)(1-\gamma)}{pp_1+p_2(1-\gamma)}} \varphi_\eta^{\frac{(\gamma-p_1-1)(p-1)}{pp_1+p_2(1-\gamma)}},\\
|Dv|^{q-1} &= v^{\frac{(\gamma-1)q_1(q-1)}{qq_1+q_2(1-\gamma)}} |Dv|^{\frac{qq_1(q-1)}{qq_1+q_2(1-\gamma)}} \varphi_\eta^{\frac{q_1(q-1)}{qq_1+q_2(1-\gamma)}}\\
&\quad \times v^{\frac{q_1(q-1)(1-\gamma)}{qq_1+q_2(1-\gamma)}} |Dv|^{\frac{q_2(q-1)(1-\gamma)}{qq_1+q_2(1-\gamma)}} (b\varphi_\eta)^{\frac{(q-1)(1-\gamma)}{qq_1+q_2(1-\gamma)}}\\
&\quad \times g^{-\frac{(q-1)(1-\gamma)}{qq_1+q_2(1-\gamma)}} \varphi_\eta^{\frac{(\gamma-q_1-1)(q-1)}{qq_1+q_2(1-\gamma)}}.
\end{aligned}
$$

Applying to the right-hand sides of (3.15) and (3.16) the triple Young inequality with the exponents c_3, d_3, e_3, c_4, d_4, e_4 from (3.21), (3.22) respectively, we arrive at

$$
\begin{aligned}
&\int f v^{q_1} |Dv|^{q_2} \varphi_\eta \, dx\\
&\le \Big(\int u^{\gamma-1} |Du|^p \varphi_\eta \, dx\Big)^{\frac{p_1(p-1)}{pp_1+p_2(1-\gamma)}}\\
&\quad \times \Big(\int g u^{p_1} |Du|^{p_2} \varphi_\eta \, dx\Big)^{\frac{(p-1)(1-\gamma)}{pp_1+p_2(1-\gamma)}}\\
&\quad \Big(\int g^{-\frac{(p-1)(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)}} \frac{|D\varphi_\eta|^{\frac{pp_1+p_2(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)}}}{\varphi_\eta^{\frac{pp_1+p_2(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)}-1}} \, dx\Big)^{\frac{p_1+(p_2-p+1)(1-\gamma)}{pp_1+p_2(1-\gamma)}},
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&\int g u^{p_1}|Du|^{p_2}\varphi_\eta\,dx \\
&\le \Big(\int v^{\gamma-1}|Dv|^{q}\varphi_\eta\,dx\Big)^{\frac{q_1(q-1)}{qq_1+q_2(1-\gamma)}} \\
&\quad\times\Big(\int f v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx\Big)^{\frac{(q-1)(1-\gamma)}{qq_1+q_2(1-\gamma)}} \\
&\quad\times\Big(\int g^{-\frac{(q-1)(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}}\frac{|D\varphi_\eta|^{\frac{qq_1+q_2(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}}}{\varphi_\eta^{\frac{qq_1+q_2(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}-1}}\,dx\Big)^{\frac{q_1+(q_2-q+1)(1-\gamma)}{qq_1+q_2(1-\gamma)}}.
\end{aligned}
\tag{3.24}
$$

Using (3.13) and (3.14), from the previous estimates we derive

$$
\begin{aligned}
&\int f v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx \\
&\le D_\gamma\Big(\int g u^{p_1}|Du|^{p_2}\varphi_\eta\,dx\Big)^{\frac{p_1(p-1)(p+\gamma-1)+(p_1+p_2)(p-1)(1-\gamma)}{(pp_1+p_2(1-\gamma))(p_1+p_2)}} \\
&\quad\times\Big(\int \frac{g^{-\frac{p+\gamma-1}{p_1+p_2-p-\gamma+1}}|D\varphi_\eta|^{\frac{pp_1+p_2(1-\gamma)}{p_1+p_2-p-\gamma+1}}}{\varphi_\eta^{\frac{pp_1+p_2(1-\gamma)}{p_1+p_2-p-\gamma+1}-1}}\,dx\Big)^{\frac{p_1(p-1)(p_1+p_2-p-\gamma+1)}{(pp_1+p_2(1-\gamma))(p_1+p_2)}} \\
&\quad\times\Big(\int g^{-\frac{(p-1)(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)}}\frac{|D\varphi_\eta|^{\frac{pp_1+p_2(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)}}}{\varphi_\eta^{\frac{pp_1+p_2(1-\gamma)}{p_1+(p_2-p+1)(1-\gamma)}-1}}\,dx\Big)^{\frac{p_1+(p_2-p+1)(1-\gamma)}{pp_1+p_2(1-\gamma)}},
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&\int g u^{p_1}|Du|^{p_2}\varphi_\eta\,dx \\
&\le E_\gamma\Big(\int f v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx\Big)^{\frac{q_1(q-1)(q+\gamma-1)+(q_1+q_2)(q-1)(1-\gamma)}{(qq_1+q_2(1-\gamma))(q_1+q_2)}} \\
&\quad\times\Big(\int \frac{g^{-\frac{q+\gamma-1}{q_1+q_2-q-\gamma+1}}|D\varphi_\eta|^{\frac{qq_1+q_2(1-\gamma)}{q_1+q_2-q-\gamma+1}}}{\varphi_\eta^{\frac{qq_1+q_2(1-\gamma)}{q_1+q_2-q-\gamma+1}-1}}\,dx\Big)^{\frac{q_1(q-1)(q_1+q_2-q-\gamma+1)}{(qq_1+q_2(1-\gamma))(q_1+q_2)}} \\
&\quad\times\Big(\int g^{-\frac{(q-1)(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}}\frac{|D\varphi_\eta|^{\frac{qq_1+q_2(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}}}{\varphi_\eta^{\frac{qq_1+q_2(1-\gamma)}{q_1+(q_2-q+1)(1-\gamma)}-1}}\,dx\Big)^{\frac{q_1+(q_2-q+1)(1-\gamma)}{qq_1+q_2(1-\gamma)}},
\end{aligned}
\tag{3.26}
$$

where D_γ and $E_\gamma > 0$ depend only on p, q, p_1, q_1, p_2, q_2 and γ.

Then by (2.3) and (2.4) we have

$$
\int f v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx \le c\Big(\int g u^{p_1}|Du|^{p_2}\varphi_\eta\,dx\Big)^{\mu_1}\eta^{\nu_1}, \tag{3.27}
$$

$$
\int g u^{p_1}|Du|^{p_2}\varphi_\eta\,dx \le c\Big(\int f v^{q_1}|Dv|^{q_2}\varphi_\eta\,dx\Big)^{\mu_2}\eta^{\nu_2}, \tag{3.28}
$$

where after simplifying the obtained expressions one gets

$$
\begin{gathered}
\mu_1 = \frac{p-1}{p_1+p_2}, \quad \nu_1 = \frac{(\beta-1-p_1)(p-1)}{p_1+p_2}, \\
\mu_2 = \frac{q-1}{q_1+q_2}, \quad \nu_2 = \frac{(\alpha-1-q_1)(q-1)}{q_1+q_2}.
\end{gathered}
$$

Substituting (3.27) in (3.28) and vice versa, we obtain

$$\int f v^{q_1} |Dv|^{q_2} \varphi_\eta \, dx \le c\eta^{\frac{[(\beta-1-p_1)(q_1+q_2)+(\alpha-1-q_1)(q-1)](p-1)}{(p_1+p_2)(q_1+q_2)-(p-1)(q-1)}},$$

$$\int g u^{p_1} |Du|^{p_2} \varphi_\eta \, dx \le c\eta^{\frac{[(\alpha-1-q_1)(p_1+p_2)+(\beta-1-p_1)(p-1)](q-1)}{(p_1+p_2)(q_1+q_2)-(p-1)(q-1)}}.$$

Passing to the limit as $\eta \to 0_+$, due to (3.3) and (3.4) we obtain a contradiction, which completes the proof. □

Similar necessary conditions for existence of solutions can be formulated for higher order equations and systems [2], [8], as well as for systems of quasilinear elliptic inequalities with variable exponents. We leave the latter subject for a future article.

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