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LEAST ENERGY SIGN-CHANGING SOLUTIONS FOR THE NONLINEAR SCHRÖDINGER-POISSON SYSTEM

CHAO JI, FEI FANG, BINLIN ZHANG

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Abstract. This article concerns the existence of the least energy sign-changing solutions for the Schrödinger-Poisson system

$$
\begin{gathered}
-\Delta u + V(x)u + \lambda\phi(x)u = f(u), \quad \text{in } \mathbb{R}^3, \\
-\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3.
\end{gathered}
$$

Because the so-called nonlocal term $\lambda\phi(x)u$ is involved in the system, the variational functional of the above system has totally different properties from the case of $\lambda = 0$. By constraint variational method and quantitative deformation lemma, we prove that the above problem has one least energy sign-changing solution. Moreover, for any $\lambda > 0$, we show that the energy of a sign-changing solution is strictly larger than twice of the ground state energy. Finally, we consider λ as a parameter and study the convergence property of the least energy sign-changing solutions as $\lambda \searrow 0$.

1. Introduction

In this article, we are interested in the existence, energy property of sign-changing solution u_λ and a convergence property of u_λ as $\lambda \searrow 0$ for the nonlinear Schrödinger-Poisson system

$$
\begin{gathered}
-\Delta u + V(x)u + \lambda\phi(x)u = f(u), \quad \text{in } \mathbb{R}^3, \\
-\Delta\phi = u^2, \quad \text{in } \mathbb{R}^3,
\end{gathered}
\tag{1.1}
$$

where $\lambda > 0$ is a parameter. We assume that $f \in C^1(\mathbb{R},\mathbb{R})$ and satisfies the following hypotheses:

- (H1) $f(t) = o(|t|)$ as $t \to 0$.
- (H2) $|f(t)| \le C(1 + |t|^p)$ for all $t \in \mathbb{R}$ and $3 < p < 5$.
- (H3) $\lim_{t\to\infty} F(t)/t^4 = +\infty$, where $F(t) = \int_0^t f(s)ds$.
- (H4) $f(t)/|t|^3$ is an increasing function of t on $\mathbb{R} \setminus \{0\}$.

We assume the potential $V(x)$, satisfies

- (H5) $V(x) \in C(\mathbb{R}^3,\mathbb{R})$, $\inf_{x\in\mathbb{R}^3} V(x) > 0$ and $\lim_{|x|\to\infty} V(x) = +\infty$.

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We define the Sobolev space

$$H = \big\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2dx < \infty\big\}$$

with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx\Big)^{1/2}, \quad \forall u \in H.$$

By (H5), it follows that for $2 \le q < 6$, the embedding $H \hookrightarrow L^q(\mathbb{R}^3)$ is compact, see [7]. In fact, the condition (H5) may be weaken, for example, we refer to [6, 7] for more details.

In recent years, there has been a great deal work dealing with problem (1.1), specially on the existence of positive solutions, ground states and semiclassical states, see for examples, [2, 3, 4, 11, 12, 13, 18, 19, 21, 22, 25, 28], etc. To the best of our knowledge, there are very few results about the existence of sign-changing solutions for problem (1.1). Recently, in [14], the authors study the infinitely many sign-changing solutions for the nonlinear Schrödinger–Poisson system. And in [26], the authors studied the existence of sign-changing solutions for a Schrödinger–Poisson system with pure power nonlinearity $|u|^{p-1}u$, moreover, only when $\lambda > 0$ is small enough, the authors showed that the energy of any sign-changing solution is strictly larger than the least energy. However, their method strongly depends on the fact that the nonlinearity is homogeneous, so it is difficult to apply their method to our problem.

For $u \in H$. Let ϕ_u be unique solution of $-\Delta\phi = u^2$ in $D^{1,2}(\mathbb{R}^3)$, then

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|}dy.$$

The weak solutions to problem (1.1) are the critical points of the functional defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)|u|^2)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(u)dx.$$

Then $I_\lambda \in C^1(H, \mathbb{R})$ and for any $\psi \in H$,

$$\langle I'_\lambda(u), \psi\rangle = \int_{\mathbb{R}^3} (\nabla u \nabla\psi + V(x)u\psi)dx + \lambda\int_{\mathbb{R}^3} \phi_u u\psi dx - \int_{\mathbb{R}^3} f(u)\psi dx.$$

We call u a least energy sign-changing solution to problem (1.1) if u is a solution of problem (1.1) with $u^\pm \neq 0$ and

$$I_\lambda(u) = \inf\{I_\lambda(v) : v^\pm \neq 0, I'_\lambda(v) = 0\},$$

where $u^+ = \max\{u(x), 0\}$ and $u^- = \min\{u(x), 0\}$.

When $\lambda = 0$, problem (1.1) does not depend on the nonlocal term $\phi_u(x)$ any more, that is, it becomes to the following semilinear local equation

$$-\Delta u + V(x)u = f(u), \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

There are several ways in the literature to obtain sign-changing solutions for equation (1.2), see for instance [5, 8, 10, 16, 17, 20, 29, 30, 31]. However, all these methods heavily relay on the following decompositions:

$$I_0(u) = I_0(u^+) + I_0(u^-), \tag{1.3}$$

$$\langle I'_0(u), u^+\rangle = \langle I'_0(u^+), u^+\rangle \quad \text{and} \quad \langle I'_0(u), u^-\rangle = \langle I'_0(u^-), u^-\rangle, \tag{1.4}$$

where

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^3} F(u)dx.$$

Furthermore, (1.3) and (1.4) imply that the energy of any sign-changing solution to (1.2) is larger than two times the least energy in H. However, for the case $\lambda > 0$, due to the effect of the nonlocal term, the functional I_λ no longer possesses the same decompositions as (1.3), (1.4). Indeed, we have

$$I_\lambda(u) = I_\lambda(u^+) + I_\lambda(u^-) + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u^-}(u^+)^2dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2dx, \tag{1.5}$$

$$\langle I_\lambda'(u), u^+\rangle = \langle I_\lambda'(u^+), u^+\rangle + \lambda\int_{\mathbb{R}^3}\phi_{u^-}(u^+)^2dx, \tag{1.6}$$

$$\langle I_\lambda'(u), u^-\rangle = \langle I_\lambda'(u^-), u^-\rangle + \lambda\int_{\mathbb{R}^3}\phi_{u^+}(u^-)^2dx. \tag{1.7}$$

So the methods to obtain sign-changing solutions of the local problem (1.2) and to estimate the energy of the sign-changing solutions seem not suitable for our nonlocal one (1.1).

To obtain a sign-changing solution for problem (1.1), borrowing the idea in [23], we first try to seek a minimizer of the energy functional I_λ over the following constraint:

$$\mathcal{M}_\lambda = \{u \in H : u^\pm \neq 0, \langle I_\lambda'(u), u^+\rangle = \langle I_\lambda'(u), u^-\rangle = 0\}$$

and then we show that the minimizer is a sign-changing solution of (1.1). Note that the paper [8] is concerned with equation (1.2), but in our problem (1.1) the nonlocal term is involved such that the properties (1.3), (1.4) fail, and it is rather difficult to show that $\mathcal{M}_\lambda \neq \emptyset$. To prove it, in [24], the authors used the parametric method and implicit function theorem, this makes the problem very complicated, here we use Miranda's Theorem in [15], which was first used in [1] for the least energy sign-changing solution to Schrödinger-Poisson system on bounded domain and can greatly simplify the proof in [24]. To show that the minimizer of the constrained problem is a sign-changing solution, we will use the quantitative deformation lemma and degree theory.

The following are the main results of this article.

Theorem 1.1. *Let* (H1)–(H5) *hold. Then for any $\lambda > 0$, problem* (1.1) *has a least energy sign-changing solution u_λ, which has precisely two nodal domains.*

In [26] the authors proved that the energy of any sign-changing solution is strictly larger than the least energy only when $\lambda > 0$ is small enough, here we improve it to the case for any $\lambda > 0$. In order to describe our result, some notations are needed. Let

$$\mathcal{N}_\lambda := \{u \in H \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\}, \tag{1.8}$$

$$c_\lambda := \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u) \tag{1.9}$$

Let $u_\lambda \in H$ be a sign-changing solution of problem (1.1), it is clear from (1.6) and (1.7) that $u_\lambda^\pm \notin \mathcal{N}_\lambda$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, $c_\lambda > 0$ is achieved and $I_\lambda(u_\lambda) > 2c_\lambda$, where u_λ is the least energy sign-changing solution obtained in Theorem 1.1. In particular, $c_\lambda > 0$ is achieved either by a positive or a negative function.*

It is evident that the energy of the sign-changing solution u_λ obtained in Theorem 1.1 depends on λ. As a by-product of this paper, we give a convergence property of u_λ as $\lambda \searrow 0$, which reflects some relationship between $\lambda > 0$ and $\lambda = 0$ in problem (1.1).

Theorem 1.3. *If the assumptions of Theorem 1.1 hold, then for any sequence λ_n with $\lambda_n \searrow 0$ as $n \to \infty$, there exists a subsequence, still denoted by λ_n, such that $u_{\lambda_n} \to u_0$ strongly in H as $n \to \infty$, where u_0 is a least energy sign-changing solution of the problem*

$$\begin{gathered} -\Delta u + V(x)u = f(u), \quad \text{in } \mathbb{R}^3, \\ u \in H, \end{gathered} \tag{1.10}$$

which has precisely two nodal domains.

This paper is organized as follows. In Section2, we present some preliminary lemmas which are essential for this paper. In Section 3, we give the proofs of Theorems 1.1–1.3 respectively.

2. Some technical lemmas

In the sequel, we will use constraint minimization on $\mathcal{M}_\lambda$ to look for a critical point of I_λ. For this, we start with this section by claiming that the set $\mathcal{M}_\lambda$ is nonempty in H.

Lemma 2.1. *Assume that* (H1)–(H5) *hold, if $u \in H$ with $u^\pm \neq 0$, then there exists a unique pair $(s_u, t_u) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.*

Proof. Fixed $u \in H$ with $u^\pm \neq 0$. We first establish the existence of s_u, t_u. Let

$$\begin{aligned} g(s,t) &= \langle I'_\lambda(su^+ + tu^-), su^+\rangle \\ &= s^2\|u^+\|^2 + s^4\lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 dx + s^2t^2\lambda \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx \\ &\quad - \int_{\mathbb{R}^3} f(su^+)su^+ dx, \end{aligned} \tag{2.1}$$

$$\begin{aligned} h(s,t) &= \langle I'_\lambda(su^+ + tu^-), tu^-\rangle \\ &= t^2\|u^-\|^2 + t^4\lambda \int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 dx + s^2t^2\lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx \\ &\quad - \int_{\mathbb{R}^3} f(tu^-)tu^- dx. \end{aligned} \tag{2.2}$$

From (f_1) and (H3), it is easy to obtain that $g(s,s) = 0$, $h(s,s) > 0$ for $s > 0$ small and $g(t,t) < 0$, $h(t,t) > 0$ for $t > 0$ large. Hence there exist $0 < r < R$ such that

$$g(r,r) > 0, \quad h(r,r) > 0, \quad g(R,R) < 0, \quad h(R,R) < 0. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have

$$\begin{gathered} g(r,\beta) > 0, \quad g(\beta,R) < 0, \quad \forall \beta \in [r,R], \\ h(\alpha,r) > 0, \quad h(R,\alpha) < 0, \quad \forall \alpha \in [r,R]. \end{gathered}$$

By Miranda's Theorem [15], there exists some point (s_u, t_u) with $\alpha < s_u, t_u < \beta$ such that $g(s_u, t_u) = h(s_u, t_u) = 0$. So $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.

Now we show that the pair (s_u, t_u) is unique and consider it in two cases. If $u \in \mathcal{M}_\lambda$, then $u^+ + u^- = u \in \mathcal{M}_\lambda$. It means that

$$\langle I'_\lambda(u), u^+\rangle = \langle I'_\lambda(u), u^-\rangle = 0;$$

that is,

$$\|u^+\|^2 + \lambda\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2dx + \lambda\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2dx = \int_{\mathbb{R}^3} f(u^+)u^+dx, \tag{2.4}$$

and

$$\|u^-\|^2 + \lambda\int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2dx + \lambda\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2dx = \int_{\mathbb{R}^3} f(u^-)u^-dx. \tag{2.5}$$

We show that $(s_u, t_u) = (1, 1)$ is the unique pair of numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.

Assume that $(\tilde{s}_u, \tilde{t}_u)$ is another pair of numbers such that $\tilde{s}_u u^+ + \tilde{t}_u u^- \in \mathcal{M}_\lambda$. By the definition of $\mathcal{M}_\lambda$, we have

$$\begin{aligned} &\tilde{s}_u^2\|u^+\|^2 + \tilde{s}_u^4\lambda\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2dx + \tilde{s}_u^2\tilde{t}_u^2\lambda\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2dx \\ &= \int_{\mathbb{R}^3} f(\tilde{s}_u u^+)\tilde{s}_u u^+dx, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\tilde{t}_u^2\|u^-\|^2 + \tilde{t}_u^4\lambda\int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2dx + \tilde{s}_u^2\tilde{t}_u^2\lambda\int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2dx \\ &= \int_{\mathbb{R}^3} f(\tilde{t}_u u^-)\tilde{t}_u u^-dx. \end{aligned} \tag{2.7}$$

Without loss of generality, we may assume that $0 < \tilde{s}_u \le \tilde{t}_u$. Then, from (2.6), we have

$$\tilde{s}_u^2\|u^+\|^2 + \tilde{s}_u^4\lambda\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2dx + \tilde{s}_u^4\lambda\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2dx \le \int_{\mathbb{R}^3} f(\tilde{s}_u u^+)\tilde{s}_u u^+dx,$$

Moreover, we have

$$\tilde{s}_u^{-2}\|u^+\|^2 + \lambda\int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2dx + \lambda\int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2dx \le \int_{\mathbb{R}^3} \frac{f(\tilde{s}_u u^+)\tilde{s}_u}{\tilde{s}_u^3}u^+dx, \tag{2.8}$$

By (2.8) and (2.4), one has

$$(\tilde{s}_u^{-2} - 1)\|u^+\|^2 \le \int_{\mathbb{R}^3} \Big(\frac{f(x, \tilde{s}_u u^+)}{(\tilde{s}_u u^+)^3} - \frac{f(x, u^+)}{(u^+)^3}\Big)(u^+)^4dx. \tag{2.9}$$

It follows from (H4) and (2.9) that $1 \le \tilde{\alpha}_u \le \tilde{\beta}_u$. By the same method, we may get $\tilde{\beta}_u \le 1$ by (H4), (2.5) and (2.7), which shows that $\tilde{\alpha}_u = \tilde{\beta}_u = 1$.

If $u \notin \mathcal{M}_\lambda$, then there exists a pair of positive numbers (α_u, β_u) such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$. Suppose that there exists another pair of positive numbers (α'_u, β'_u) such that $\alpha'_u u^+ + \beta'_u u^- \in \mathcal{M}_\lambda$. Set $v := \alpha_u u^+ + \beta_u u^-$ and $v' := \alpha'_u u^+ + \beta'_u u^-$, we have

$$\frac{\alpha'_u}{\alpha_u}v^+ + \frac{\beta'_u}{\beta_u}v^- = \alpha'_u u^+ + \beta'_u u^- = v' \in \mathcal{M}_\lambda.$$

Since $v \in \mathcal{M}_\lambda$, we obtain that $\alpha_u = \alpha'_u$ and $\beta_u = \beta'_u$, which implies that (α_u, β_u) is the unique pair of numbers such that $\alpha_u u^+ + \beta_u u^- \in \mathcal{M}_\lambda$. The proof is complete. $\square$

Lemma 2.2. *Assume that* (H1)–(H5) *hold. For a fixed* $u \in H$ *with* $u^{\pm} \neq 0$. *If* $g_1(1,1) \leq 0$ *and* $h_1(1,1) \leq 0$, *then there exists a unique pair* $(s_u, t_u) \in (0,1] \times (0,1]$ *such that* $g_1(s_u, t_u) = h_1(s_u, t_u) = 0$.

Proof. Suppose that $s_u \geq t_u > 0$. By Lemma 2.1, we know that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$, then

$$
\begin{aligned}
& s_u^2 \|u^+\|^2 + s_u^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 dx + s_u^4 \lambda \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx \\
& \geq s_u^2 \|u^+\|^2 + s_u^4 \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 dx + s_u^2 t_u^2 \lambda \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx \\
& = \int_{\mathbb{R}^3} f(s_u u^+) s_u u^+ dx.
\end{aligned} \tag{2.10}
$$

Moreover, $g_1(1,1) \leq 0$ implies that

$$
\|u^+\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx \leq \int_{\mathbb{R}^3} f(u^+) u^+ dx. \tag{2.11}
$$

Combining (2.4) with (2.5), we have

$$
\Big(\frac{1}{s_u^2} - 1\Big) \|u^+\|^2 \geq \int_{\mathbb{R}^3} \Big(\frac{f(s_u u^+)}{(s_u u^+)^3} - \frac{f(u^+)}{(u^+)^3}\Big) |u^+|^4 dx.
$$

If $s_u > 1$, the left-hand side of this inequality is negative. But from (H4), the right-hand side of this inequality is positive, so we have $s_u \leq 1$. The proof is thus complete. □

Lemma 2.3. *For a fixed* $u \in H$ *with* $u^{\pm} \neq 0$, *then* (s_u, t_u) *obtained in Lemma 2.1 is the unique maximum point of the function* $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *defined as* $\phi(s,t) = I_\lambda(su^+ + tu^-)$.

Proof. From the proof of Lemma 2.1, we know that (s_u, t_u) is the unique critical point of ϕ in $\mathbb{R}_+ \times \mathbb{R}_+$. By (H3), we conclude that $\phi(s,t) \to -\infty$ uniformly as $|(s,t)| \to \infty$, so it is sufficient to show that a maximum point cannot be achieved on the boundary of $(\mathbb{R}_+, \mathbb{R}_+)$. If we assume that $(0, \bar{t})$ is a maximum point of ϕ. Then since

$$
\begin{aligned}
\phi(s, \bar{t}) &= I_\lambda(su^+ + \bar{t}u^-) \\
&= \frac{s^2}{2} \|u^+\|^2 + \frac{\lambda}{4} s^4 \int_{\mathbb{R}^3} \phi_{u^+}(u^+)^2 dx - \int_{\mathbb{R}^3} F(su^+) dx \\
&\quad + \frac{\lambda}{4} \Big(s^2 \bar{t}^2 \int_{\mathbb{R}^3} \phi_{u^-}(u^+)^2 dx + s^2 \bar{t}^2 \int_{\mathbb{R}^3} \phi_{u^+}(u^-)^2 dx \Big) \\
&\quad + \frac{\bar{t}^2}{2} \|u^-\|^2 + \frac{\lambda}{4} \bar{t}^4 \int_{\mathbb{R}^3} \phi_{u^-}(u^-)^2 dx - \int_{\mathbb{R}^3} F(\bar{t}u^-) dx
\end{aligned}
$$

is an increasing function with respect to s if s is small enough, the pair $(0, \bar{t})$ is not a maximum point of ϕ in $\mathbb{R}_+ \times \mathbb{R}_+$. The proof is now finished. □

By Lemma 2.1, we define the minimization problem

$$
m_\lambda := \inf \Big\{ I_\lambda(u) : u \in \mathcal{M}_\lambda \Big\}.
$$

Lemma 2.4. *Assume that* (H1)–(H5) *hold, then* $m_\lambda > 0$ *can be achieved for any* $\lambda > 0$.

Proof. For every $u \in \mathcal{M}_\lambda$, we have $\langle I'_\lambda(u), u\rangle = 0$. From (f_1), (f_2), for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$f(s)s \le \epsilon s^2 + C_\epsilon |s|^{p+1} \quad \text{for all } s \in \mathbb{R}. \tag{2.12}$$

By Sobolev embedding theorem, we obtain

$$\begin{aligned} \|u\|^2 &\le \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)|u|^2)dx + \lambda \int_{\mathbb{R}^3} \phi_u u^2 dx = \int_{\mathbb{R}^3} f(u)u dx \\ &\le \epsilon \int_{\mathbb{R}^3} |u|^2 dx + C_\epsilon \int_{\mathbb{R}^3} |u|^{p+1} dx \\ &\le C_2 \epsilon \|u\|^2 + C'_\epsilon \|u\|^{p+1}. \end{aligned} \tag{2.13}$$

Pick $\epsilon = 1/(2C_2)$. So there exists a constant $\alpha > 0$ such that $\|u\|^2 > \alpha$. By (2.3), we have

$$f(s)s - 4F(s) \ge 0.$$

Then

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \ge \frac{\|u\|^2}{4} \ge \frac{\alpha}{4}.$$

This implies that $I_\lambda(u)$ is coercive in $\mathcal{M}_\lambda$ and $m_\lambda \ge \frac{\alpha}{4} > 0$.

Let $\{u_n\}_n \subset \mathcal{M}_\lambda$ be such that $I_\lambda(u_n) \to m_\lambda$. Then $\{u_n\}_n$ is bounded in H and there exists $u_\lambda \in H$ such that $u_n^\pm \rightharpoonup u_\lambda^\pm$ weakly in H. Since $u_n \in \mathcal{M}_\lambda$, we have $\langle I'_\lambda(u_n), u_n^\pm\rangle = 0$, that is

$$\|u_n^\pm\|^2 + \lambda \int_{\mathbb{R}^3} \phi_{u_n^\pm}(u_n^\pm)^2 dx + \lambda \int_{\mathbb{R}^3} \phi_{u_n^\mp}(u_n^\pm)^2 dx - \int_{\mathbb{R}^3} f(u_n^\pm)u_n^\pm dx = 0.$$

Similar as (2.7) we also have $\|u_n^\pm\|^2 \ge \beta$ for all $n \in N$, where β is a constant.

Since $u_n \in \mathcal{M}_\lambda$, by (2.6) again, we have

$$\beta \le \|u_n^\pm\|^2 < \int_{\mathbb{R}^3} f(u_n^\pm)u_n^\pm dx \le \epsilon \int_{\mathbb{R}^3} |u_n^\pm|^2 dx + C_\epsilon \int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx.$$

In view of the boundedness of $\{u_n\}_n$, there is $C_2 > 0$ such that

$$\beta \le \epsilon C_2 + C_\epsilon \int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx.$$

Choosing $\epsilon = \beta/(2C_2)$, we obtain

$$\int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx \ge \frac{\beta}{2\bar{C}}. \tag{2.14}$$

where $\bar{C}$ is a positive constant, in fact, $\bar{C} = C_{\frac{\beta}{2C_2}}$.

By (2.8) and the compact embedding $H \hookrightarrow L^q(\mathbb{R}^3)$ for $2 \le q < 6$, we obtain

$$\int_{\mathbb{R}^3} |u_\lambda^\pm|^{p+1} dx \ge \frac{\beta}{2\bar{C}}.$$

Thus, $u_\lambda^\pm \ne 0$. By (f_1), (f_2), the compact embedding and [27, Theorem A.4],

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^3} f(u_n^\pm)u_n^\pm dx &= \int_{\mathbb{R}^3} f(u_\lambda^\pm)u_\lambda^\pm dx, \\ \lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n^\pm) dx &= \int_{\mathbb{R}^3} F(u_\lambda^\pm) dx. \end{aligned} \tag{2.15}$$

By the weak semicontinuity of norm and Fatou's Lemma, we have

$$\begin{aligned}&\|u_\lambda^\pm\|^2+\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^\pm}(u_\lambda^\pm)^2dx+\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^\mp}(u_\lambda^\pm)^2dx\\&\le\liminf_{n\to\infty}\Big\{\|u_n^\pm\|^2+\lambda\int_{\mathbb{R}^3}\phi_{u_n^\pm}(u_n^\pm)^2dx+\lambda\int_{\mathbb{R}^3}\phi_{u_n^\mp}(u_n^\pm)^2dx\Big\}.\end{aligned}$$

From (2.9), we have

$$\|u_\lambda^\pm\|^2+\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^\pm}(u_\lambda^\pm)^2dx+\lambda\int_{\mathbb{R}^3}\phi_{u_\lambda^\mp}(u_\lambda^\pm)^2dx\le\int_{\mathbb{R}^3}f(u_\lambda^\pm)u_\lambda^\pm dx \tag{2.16}$$

From (2.10) and Lemma 2.2, there exists $(s_{u_\lambda},t_{u_\lambda})\in(0,1]\times(0,1]$ such that

$$\bar{u}_\lambda:=s_{u_\lambda}u_\lambda^++t_{u_\lambda}u_\lambda^-\in\mathcal{M}_\lambda.$$

Condition (H4) implies that $H(s):=sf(s)-4F(s)$ is a non-negative function, increasing in $|s|$, so we have

$$\begin{aligned}m_\lambda&\le I_\lambda(\bar{u}_\lambda)=I_\lambda(\bar{u}_\lambda)-\frac14\langle I_\lambda'(\bar{u}_\lambda),\bar{u}_\lambda\rangle\\&=\frac14\|\bar{u}_\lambda\|^2+\frac14\int_{\mathbb{R}^3}\Big(f(\bar{u}_\lambda)\bar{u}_\lambda-4F(\bar{u}_\lambda)\Big)dx\\&=\frac14\|s_{u_\lambda}u_\lambda^+\|^2+\frac14\int_{\mathbb{R}^3}\Big(f(s_{u_\lambda}u_\lambda^+)s_{u_\lambda}u_\lambda^+-4F(s_{u_\lambda}u_\lambda^+)\Big)dx\\&\quad+\frac14\|t_{u_\lambda}u_\lambda^-\|^2+\frac14\int_{\mathbb{R}^3}\Big(f(t_{u_\lambda}u_\lambda^-)t_{u_\lambda}u_\lambda^--4F(t_{u_\lambda}u_\lambda^-)\Big)dx\\&\le\frac14\|u_\lambda\|^2+\frac14\int_{\mathbb{R}^3}\Big(f(u_\lambda)u_\lambda-4F(u_\lambda)\Big)dx\\&\le\liminf_{n\to\infty}\Big[I_\lambda(u_n)-\frac14\langle I_\lambda'(u_n),u_n\rangle\Big]=m_\lambda.\end{aligned}$$

Then we conclude that $s_{u_\lambda}=t_{u_\lambda}=1$. Thus, $\bar{u}_\lambda=u_\lambda$ and $I_\lambda(u_\lambda)=m_\lambda$. □

3. Proof of main results

Proof of Theorem 1.1. We first prove that the minimizer u_λ for the minimization problem is indeed a sign-changing solution of problem (1.1); that is, $I_\lambda'(u_\lambda)=0$. For it, we will use the quantitative deformation lemma.

It is obvious that $I_\lambda'(u_\lambda)u_\lambda^+=0=I_\lambda'(u_\lambda)u_\lambda^-$. From Lemma 2.3, for any $(s,t)\in\mathbb{R}_+\times\mathbb{R}_+$ and $(s,t)\ne(1,1)$,

$$I_\lambda(su_\lambda^++tu_\lambda^-)<I_\lambda(u_\lambda^++u_\lambda^-)=m_\lambda.$$

If $I_\lambda'(u_\lambda)\ne0$, then there exist $\delta>0$ and $\kappa>0$ such that

$$\|I_\lambda'(v)\|\ge\kappa\quad\text{for all }\|v-u_\lambda\|\le3\delta.$$

Let $D:=(\frac12,\frac32)\times(\frac12,\frac32)$ and $g(s,t):=su_\lambda^++tu_\lambda^-$. From Lemma 2.3, we also have

$$\bar{m}_\lambda:=\max_{\partial D}I_\lambda\circ g<m_\lambda.$$

For $\epsilon:=\min\{(m_\lambda-\bar{m}_\lambda)/2,\kappa\delta/8\}$ and $S:=B(u_\lambda,\delta)$, there is a deformation η such that

(a) $\eta(1,u)=u$ if $u\notin I_\lambda^{-1}([m_\lambda-2\epsilon,m_\lambda+2\epsilon])\cap S_{2\delta}$;
(b) $\eta(1,I_\lambda^{m_\lambda+\epsilon}\cap S)\subset I_\lambda^{m_\lambda-\epsilon}$;
(c) $I_\lambda(\eta(1,u)))\le I_\lambda(u)$ for all $u\in H$.

See [27] for more details. It is clear that
$$\max_{(s,t)\in\bar{D}} I_\lambda(\eta(1,g(s,t)))) < m_\lambda.$$
Now we prove that $\eta(1,g(D))\cap\mathcal{M}_\lambda\neq\emptyset$ which contradicts to the definition of m_λ. Let us define $h(s,t)=\eta(1,g(s,t)))$ and
$$\Psi_0(s,t) := \Big(I'_\lambda(g(s,t))u_\lambda^+, I'_\lambda(g(s,t))u_\lambda^-\Big) = \Big(I'_\lambda(su_\lambda^+ + tu_\lambda^-)u_\lambda^+, I'_\lambda(su_\lambda^+ + tu_\lambda^-)u_\lambda^-\Big),$$
$$\Psi_1(s,t) := \Big(\frac{1}{s}I'_\lambda(h(s,t))h^+(s,t), \frac{1}{t}I'_\lambda(h(s,t))h^-(s,t)\Big).$$
Lemma 2.1 and the degree theory imply that $\deg(\Psi_0,D,0)=1$. It follows from that $g=h$ on ∂D. Consequently, we obtain
$$\deg(\Psi_1,D,0)=\deg(\Psi_0,D,0)=1.$$
Thus, $\Psi_1(s_0,t_0)=0$ for some $(s_0,t_0)\in D$, so that
$$\eta(1,g(s_0,t_0)))=h(s_0,t_0)\in\mathcal{M}_\lambda,$$
which is a contradiction. From this, u_λ is a critical point of I_λ, moreover, it is a sign-changing solution for problem (1.1).

Now we prove that u_λ has exactly two nodal domains. By contradiction, we assume that u_λ has at least three nodal domains Ω_1, Ω_2, Ω_3. Without loss generality, we may assume that $u_\lambda>0$ a.e. in Ω_1 and $u_\lambda<0$ a.e. in Ω_2. Set
$$u_{\lambda_i} := \chi_{\Omega_i}u_\lambda, \qquad i=1,2,3,$$
where
$$\chi_{\Omega_i} = \begin{cases} 1 & x\in\Omega_i,\\ 0 & x\in\mathbb{R}^N\setminus\Omega_i. \end{cases}$$
Then $u_{\lambda_i}\neq 0$ and $\langle I'(u_\lambda),u_{\lambda_i}\rangle=0$ for $i=1,2,3$, so we have
$$\langle I'(u_{\lambda_1}+u_{\lambda_2}),(u_{\lambda_1}+u_{\lambda_2})^\pm\rangle<0.$$
By Lemma 2.2, there exists $(s_v,t_v)\in(0,1]\times(0,1]$ such that $s_vu_{\lambda_1}+t_vu_{\lambda_2}\in\mathcal{M}_\lambda$. Since
$$\begin{aligned}
0 &= \frac{1}{4}\langle I'_\lambda(u_\lambda),u_{\lambda_3}\rangle\\
&= \frac{1}{4}\|u_{\lambda_3}\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_\lambda}{u_{\lambda_3}}^2dx - \frac{1}{4}\int_{\mathbb{R}^3}f(u_{\lambda_3})u_{\lambda_3}dx\\
&\leq \frac{1}{4}\|u_{\lambda_3}\|^2 + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_\lambda}{u_{\lambda_3}}^2dx - \frac{1}{4}\int_{\mathbb{R}^3}F(u_{\lambda_3})dx\\
&< I_\lambda(u_{\lambda_3}) + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_1}}{u_{\lambda_3}}^2dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_2}}{u_{\lambda_3}}^2dx.
\end{aligned}$$
From (H4), we have
$$\begin{aligned}
m_\lambda &\leq I_\lambda(s_vu_{\lambda_1}+t_vu_{\lambda_2})\\
&= I_\lambda(s_vu_{\lambda_1}+t_vu_{\lambda_2}) - \frac{1}{4}\langle I'_\lambda(s_vu_{\lambda_1}+t_vu_{\lambda_2}), s_vu_{\lambda_1}+t_vu_{\lambda_2}\rangle\\
&= \frac{s_v^2\|u_{\lambda_1}\|^2+t_v^2\|u_{\lambda_2}\|^2}{4} + \int_{\mathbb{R}^3}\Big(\frac{1}{4}f(s_vu_{\lambda_1})s_vu_{\lambda_1}-F(s_vu_{\lambda_1})\Big)dx\\
&\quad + \int_{\mathbb{R}^3}\Big(\frac{1}{4}f(t_vu_{\lambda_2})t_vu_{\lambda_2}-F(t_vu_{\lambda_2})\Big)dx
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\|u_{\lambda_1}\|^2+\|u_{\lambda_2}\|^2}{4}+\int_{\mathbb{R}^3}\Big(\frac{1}{4}f(u_{\lambda_1})u_{\lambda_1}-F(u_{\lambda_1})\Big)dx \\
&\quad+\int_{\mathbb{R}^3}\Big(\frac{1}{4}f(u_{\lambda_2})u_{\lambda_2}-F(u_{\lambda_2})\Big)dx \\
&= I_\lambda(u_{\lambda_1})+I_\lambda(u_{\lambda_2})+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_2}}{u_{\lambda_1}}^2dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_3}}{u_{\lambda_1}}^2dx \\
&\quad+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_1}}{u_{\lambda_2}}^2dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_3}}{u_{\lambda_2}}^2dx \\
&< I_\lambda(u_{\lambda_1})+I_\lambda(u_{\lambda_2})+I_\lambda(u_{\lambda_3})+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_2}}{u_{\lambda_1}}^2dx \\
&\quad+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_3}}{u_{\lambda_1}}^2dx\quad+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_1}}{u_{\lambda_2}}^2dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_3}}{u_{\lambda_2}}^2dx \\
&\quad+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_1}}{u_{\lambda_3}}^2dx+\frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_{u_{\lambda_2}}{u_{\lambda_3}}^2dx \\
&= I_\lambda(u_\lambda)=m_\lambda,
\end{aligned}
$$

which is impossible, so u_λ has exactly two nodal domains. □

Proof of Theorem 1.2. As in the proof of Lemma 2.4, for each $\lambda>0$, we can get a $v_\lambda\in\mathcal{N}_\lambda$ such that $I_\lambda(v_\lambda)=c_\lambda>0$, where $\mathcal{N}_\lambda$ and c_λ are defined by (1.8) and (1.9), respectively. Moreover, the critical points of I_λ on $\mathcal{N}_\lambda$ are the critical points of I_λ in H. Thus, v_λ is a ground state solution of problem (1.1).

From Theorem 1.1, we know that problem (1.1) has a least energy sign-changing solution u_λ which changes sign only once. Suppose that $u_\lambda=u_\lambda^++u_\lambda^-$. As the proof of Step 1 in Lemma 2.1, there exist unique $s_{u_\lambda^+}>0$ and $t_{u_\lambda^-}>0$ such that

$$
s_{u_\lambda^+}u_\lambda^+\in\mathcal{N}_\lambda,\quad t_{u_\lambda^-}u_\lambda^-\in\mathcal{N}_\lambda.
$$

From (1.6) and (1.7), we have

$$
\langle I_\lambda'(u_\lambda^+),u_\lambda^+\rangle<0,\quad \langle I_\lambda'(u_\lambda^-),u_\lambda^-\rangle<0.
$$

So, by (H1)–(H4), one has $s_{u_\lambda^+}\in(0,1)$ and $t_{u_\lambda^-}\in(0,1)$. Then, by Lemma 2.3, we obtain

$$
2c_\lambda\le I_\lambda(s_{u_\lambda^+}u_\lambda^+)+I_\lambda(t_{u_\lambda^-}u_\lambda^-)\le I_\lambda(s_{u_\lambda^+}u_\lambda^++t_{u_\lambda^-}u_\lambda^-)<I_\lambda(u_\lambda^++u_\lambda^-)=m_\lambda,
$$

that is $I_\lambda(u_\lambda)>2c_\lambda$, which implies that $c_\lambda>0$ can not be achieved by a sign-changing function. This completes the proof. □

Now we prove Theorem 1.3. In the following, we regard $\lambda>0$ as a parameter in problem (1.1). We shall study the convergence property of u_λ as $\lambda\searrow 0$.

Proof of Theorem 1.3. For any $\lambda>0$, let $u_\lambda\in H$ be the least energy sign-changing solution of problem (1.1) obtained in Theorem 1.1, which has exactly two nodal domains.

Step 1. We show that, for any sequence $\{\lambda_n\}_n$ with $\lambda_n\searrow 0$ as $n\to\infty$, $\{u_{\lambda_n}\}_n$ is bounded in H. Choose a nonzero function $\varphi\in C_0^\infty(\mathbb{R}^3)$ with $\varphi^\pm\ne 0$. From $f(s)s-4F(s)\ge 0$, for $s\ne 0$, we have $f(s)s>4F(s)$. Then, (H3) implies that, for any $\lambda \in [0,1]$, there exists a pair $(\theta_1,\theta_2) \in (\mathbb{R}_+ \times \mathbb{R}_+)$, which does not depend on λ, such that

$$\begin{aligned}
&\theta_1^2\|\varphi^+\|^2 + \theta_1^4\lambda\int_{\mathbb{R}^3}\phi_{\varphi^+}(\varphi^+)^2dx + \theta_1^2\theta_2^2\lambda\int_{\mathbb{R}^3}\phi_{\varphi^-}(\varphi^+)^2dx\\
&-\int_{\mathbb{R}^3} f(\theta_1\varphi^+)\theta_1\varphi^+dx < 0,\\
&\theta_2^2\|\varphi^-\|^2 + \theta_2^4\lambda\int_{\mathbb{R}^3}\phi_{\varphi^-}(\varphi^-)^2dx + \theta_2^2\theta_1^2\lambda\int_{\mathbb{R}^3}\phi_{\varphi^+}(\varphi^-)^2dx\\
&-\int_{\mathbb{R}^3} f(\theta_2\varphi^-)\theta_2\varphi^-dx < 0.
\end{aligned}$$

In view of Lemmas 2.1 and 2.2, for any $\lambda \in [0,1]$, there is a unique pair $(s_\varphi(\lambda), t_\varphi(\lambda)) \in (0,1]\times(0,1]$ such that $\bar\varphi := s_\varphi(\lambda)\theta_1\varphi^+ + t_\varphi(\lambda)\theta_2\varphi^- \in \mathcal{M}_\lambda$. Thus, for all $\lambda \in [0,1]$, we have

$$\begin{aligned}
I_\lambda(u_\lambda) &\le I_\lambda(\bar\varphi) = I_\lambda(\bar\varphi) - \frac14\langle I'_\lambda(\bar\varphi),\bar\varphi\rangle\\
&= \frac{\|\bar\varphi\|^2}{4} + \int_{\mathbb{R}^3}\Big(\frac14 f(\bar\varphi)\bar\varphi - F(\bar\varphi)\Big)dx\\
&\le \frac{\|\bar\varphi\|^2}{4} + \int_{\mathbb{R}^3}\Big(C_3\bar\varphi^2 + C_4|\bar\varphi|^{p+1}\Big)dx\\
&\le \frac{\|\theta_1\varphi^+\|^2}{4} + \frac{\|\theta_2\varphi^-\|^2}{4} + \int_{\mathbb{R}^3}\Big(C_3(\theta_1\varphi^+)^2 + C_4|\theta_1\varphi^+|^{p+1}\\
&\quad + C_3(\theta_2\varphi^-)^2 + C_4|\theta_2\varphi^-|^{p+1}\Big)dx\\
&= C_0.
\end{aligned}$$

Moreover, for n large enough, we obtain

$$C_0 + 1 \ge I_{\lambda_n}(u_{\lambda_n}) = I_{\lambda_n}(u_{\lambda_n}) - \frac14\langle I'_{\lambda_n}(u_{\lambda_n}), u_{\lambda_n}\rangle \ge \frac14\|u_{\lambda_n}\|^2.$$

So $\{u_{\lambda n}\}_n$ is bounded in H.

Step 2. There exists a subsequence of $\{\lambda_n\}_n$, still denoted by $\{\lambda_n\}_n$, such that $u_{\lambda_n} \rightharpoonup u_0$ weakly in H. Then, u_0 is a weak solution of (1.10). Since u_{λ_n} is the least energy sign-changing solution of (1.1) with $\lambda = \lambda_n$, then by the compactness of the embedding $H \hookrightarrow L^q(\mathbb{R}^3)$ for $2 \le q < 2^*$, we obtain that $u_{\lambda_n} \to u_0$ strongly in H as $n \to \infty$. In fact,

$$\begin{aligned}
\|u_{\lambda_n} - u_0\|^2 &= \langle I'_{\lambda_n}(u_{\lambda_n}) - I'_0(u_0), u_{\lambda_n} - u_0\rangle - \lambda_n\int_{\mathbb{R}^3}\phi_{u_{\lambda_n}}u_{\lambda_n}(u_{\lambda_n} - u_0)dx\\
&\quad + \int_{\mathbb{R}^3} f(u_{\lambda_n})(u_{\lambda_n} - u_0)dx - \int_{\mathbb{R}^3} f(u_0)(u_{\lambda_n} - u_0)dx.
\end{aligned}$$

Then $u_0 \ne 0$ and u_0 has exactly two nodal domains.

Step 3. Suppose that v_0 is a least energy sign-changing solution of (1.10), we may refer to [9] for the existence of v_0. By Lemma 2.1, for each $\lambda_n > 0$, there is a unique pair $(s_{\lambda_n}, t_{\lambda_n}) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that $s_{\lambda_n} v_0^+ + t_{\lambda_n} v_0^- \in \mathcal{M}_{\lambda_n}$. So we have

$$
\begin{aligned}
&s_{\lambda_n}^2 \|v_0^+\|^2 + s_{\lambda_n}^4 \lambda_n \int_{\mathbb{R}^3} \phi_{v_0^+} (v_0^+)^2 dx + s_{\lambda_n}^2 t_{\lambda_n}^2 \lambda_n \int_{\mathbb{R}^3} \phi_{v_0^-} (v_0^+)^2 dx \\
&= \int_{\mathbb{R}^3} f(s_{\lambda_n} v_0^+) s_{\lambda_n} v_0^+ dx, \\
&t_{\lambda_n}^2 \|v_0^-\|^2 + t_{\lambda_n}^4 \lambda_n \int_{\mathbb{R}^3} \phi_{v_0^-} (v_0^-)^2 dx + s_{\lambda_n}^2 t_{\lambda_n}^2 \lambda_n \int_{\mathbb{R}^3} \phi_{u^+} (u^-)^2 dx \\
&= \int_{\mathbb{R}^3} f(t_{\lambda_n} v_0^-) t_{\lambda_n} v_0^- dx.
\end{aligned}
$$

We know that $v_0^{\pm}$ satisfies $\|v_0^{\pm}\|^2 = \int_{\mathbb{R}^3} f(v_0^{\pm}) v_0^{\pm} dx$. It is easy to check that

$$
(s_{\lambda_n}, t_{\lambda_n}) \to (1,1), \quad \text{as } n \to \infty. \tag{3.1}
$$

From this limit and Lemma 2.3, we have

$$
\begin{aligned}
I_0(v_0) \le I_0(u_0) &= \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) = \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}^+ + u_{\lambda_n}^-) \\
&\le \lim_{n\to\infty} I_{\lambda_n}(s_{\lambda_n} u_{\lambda_n}^+ + t_{\lambda_n} u_{\lambda_n}^-) \\
&= I_0(v_0).
\end{aligned}
$$

This means that u_0 is a least energy sign-changing solution of (1.10) which has precisely two nodal domains. The proof is complete. □

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Chao Ji
Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China
E-mail address: jichao@ecust.edu.cn

Fei Fang
Department of Mathematics, Beijing Technology and Business University, Beijing 100048, China
E-mail address: fangfei68@163.com

Binlin Zhang (corresponding author)
Department of Mathematics, Heilongjiang Institute of Technology, Harbin 150050, China
E-mail address: zhangbinlin2012@163.com