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LIOUVILLE THEOREM AND GRADIENT ESTIMATES FOR NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this article we study a nonlinear elliptic equation by using the maximum principle and cutoff functions, We establish related gradient estimates, the Liouville theorem, and the Harnack inequality.

1. Introduction and statement of main results

In 1981, Gidas-Spruck [3] derived the following result.

Theorem 1.1. *Let M^n be a complete manifold with nonnegative Ricci curvature. Assume that $h(x) \in C^2(M^n)$ and $\alpha > 0$ satisfy the following conditions:*

(1) *$h(x) \geq 0$ on M^n;*
(2) *$\Delta h(x) \geq 0$ on M^n;*
(3) *for $r(x)$ large, $|\nabla \log h(x)| \leq C/r(x)$ and if $n \geq 4$, $h(x) \geq C(r(x))^\sigma$ with $\sigma \geq -\frac{2}{n-3}$, where $r(x)$ is the geodesic distance between x and some fixed point p;*
(4) *$1 \leq \alpha \leq \frac{n+2}{n-2}$.*

If $u(x)$ is a nonnegative solution of

$$\Delta u + hu^\alpha = 0,$$

then $u(x) \equiv 0$.

For $\alpha = 1$, Li-Yau [9] demonstrated the same result under the condition that $|\nabla h(x)| = o(r(x))$ as $r(x) \to \infty$. Later, Li [6] proved that as $1 \leq \alpha \leq \frac{n}{n-2}$ $(n \geq 4)$, the condition (3) of Theorem 1.1 is unnecessary. On these conditions were further weakened, see [1, 5, 7]. In 2010, Yang [11] studied the equation

$$\Delta u + cu^{-\alpha} = 0$$

on a noncompact complete Riemannian manifold, where $\alpha > 0$ and c are two real constants. The corresponding gradient estimates and Liouville type theorem are also derived.

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Recently, Wang [10] deduced gradient estimates and Liouville type theorem for positive solutions to the equation

$$\Delta_f u^m + cu = 0$$

on smooth measure space with m-Bakry-Émery curvature bounded by $Ric_{f,m} \geq -(m-1)K$, where $K \geq 0$.

Inspired by the works [3, 8, 10, 11], we investigate the nonlinear elliptic equation

$$\Delta u^m + \lambda(x)u^l = 0, \quad m > 1 \tag{1.1}$$

on a complete Riemannian manifold with Ricci curvature bounded below, where $m > 1$ and l are real numbers, and $\lambda(x) \in C^2(M^n)$. If $M^n = \Omega$ is a bounded smooth domain in $\mathbb{R}^n$ and $\lambda(x) \leq 0$ is a constant, the equation (1.1) is regarded as the thin film equation, which depict a steady state of the thin film. Concrete content can be seen [4]. Our main results reads as follows.

Theorem 1.2. *Let (M^n, g) be a complete Riemannian manifold without boundary. Suppose that B_{2R} is a geodesic ball of radius $2R$ around $p \in M$ and $\mathrm{Ric}(B_{2R}) \geq -K$ with $K \geq 0$. Also suppose that there exist two positive numbers δ and τ such that $|\lambda(x,t)| \leq \delta$ and $|\nabla\lambda|^2 \leq \tau|\lambda|^2$. Let $u(x)$ is a positive solution to the equation (1.1) and $v = \frac{m}{m-1}u^{m-1}$.*

(a) Assume that $l \geq 1$, then

$$\sup_{B_p(R)} \frac{|\nabla v|^2}{v} \leq \frac{C_4(m-1)}{m+1}\Big[\frac{1}{R^2}(1+\sqrt{K}R) + 2K + \tau\Big] \sup_{x\in M^n} v + H_1. \tag{1.2}$$

(b) Assume that $l < 1$, then

$$\sup_{B_p(R)} \frac{|\nabla v|^2}{v} \leq \frac{C_4(m-1)}{m+1}\Big[\frac{1}{R^2}(1+\sqrt{K}R) + 2K + \tau\Big] \sup_{x\in M^n} v + H_2. \tag{1.3}$$

Where C_4 is a constant depending only on n, and

$$H_1 = \frac{(m-1)(n-1)}{m(m+1)}\Big|\frac{2(m+1)}{n-1} + (m-2l+1)\Big|\delta\Big(\frac{m-1}{m}\sup_{x\in M^n} v\Big)^{\frac{l-1}{m-1}},$$
$$H_2 = \frac{(m-1)(n-1)}{m(m+1)}\Big|\frac{2(m+1)}{n-1} + (m-2l+1)\Big|\delta\Big(\frac{m-1}{m}\inf_{x\in M^n} v\Big)^{\frac{l-1}{m-1}}.$$

Moreover, if (M^n, g) has nonnegative Ricci curvature, letting $R \to \infty$, we have following estimate for $l \geq 1$,

$$\frac{|\nabla v|^2}{v} \leq C(m,n,l,K,\delta,\tau,\sup_{x\in M^n} v), \tag{1.4}$$

and for $l < 1$,

$$\frac{|\nabla v|^2}{v} \leq C(m,n,l,K,\delta,\tau,\sup_{x\in M^n} v). \tag{1.5}$$

Where

$$C(m,n,l,K,\delta,\tau,\sup_{x\in M^n} v) = \frac{C_4(m-1)}{m+1}(2K+\tau)\sup_{x\in M^n} v + H_1,$$
$$C'(m,n,l,K,\delta,\tau,\sup_{x\in M^n} v, \inf_{x\in M^n} v) = \frac{C_4(m-1)}{m+1}(2K+\tau)\sup_{x\in M^n} v + H_2.$$

By using (1.4) and (1.5), we derive the related Harnack inequalities.

Corollary 1.3. *Let (M^n,g) be a noncompact complete Riemannian manifold without boundary. Suppose that $\mathrm{Ric}(M^n)\geq 0$. Let $u(x)$ is a positive solution of the equation (1.1), and $v=\frac{m}{m-1}u^{m-1}$.*

If $l\geq 1$, then

$$v(x)\leq v(y)\exp\Big[r(x,y)\sqrt{\frac{C(m,n,l,K,\delta,\tau,\sup_{x\in M^n}v)}{\inf_{x\in M^n}v}}\Big]. \tag{1.6}$$

If $l<1$, then

$$v(x)\leq v(y)\exp\Big[r(x,y)\sqrt{\frac{C'(m,n,l,K,\delta,\tau,\sup_{x\in M^n}v,\inf_{x\in M^n}v)}{\inf_{x\in M^n}v}}\Big]. \tag{1.7}$$

Theorem 1.4. *Let (M^n,g) be a complete Riemannian manifold without boundary. Suppose that B_{2R} is a geodesic ball of radius $2R$ around $p\in M$ and $\mathrm{Ric}(B_{2R})\geq -K$ with $K\geq 0$. Let $u(x)$ is a positive solution of the equation (1.1). Let $v=\frac{m}{m-1}u^{m-1}$ and $|\nabla\lambda|^2\leq\tau|\lambda|^2$ for some positive constant τ. If $\lambda\geq 0$ and $l\leq\frac{(n+1)(m+1)}{2(n-1)}$ or $\lambda\leq 0$ and $l\geq\frac{(n+1)(m+1)}{2(n-1)}$, then we have*

$$\sup_{B_p(R)}\frac{|\nabla v|^2}{v}\leq C_4\big[\frac{1}{R^2}(1+\sqrt{K}R)+2K+\tau\big]\sup_{x\in M^n}v, \tag{1.8}$$

where C_4 is a constant depending only on n.

Letting $R\to\infty$, then we infer on a complete noncompact Riemannian manifold,

$$\frac{|\nabla v|^2}{v}\leq C_4(2K+\tau)\sup_{x\in M^n}v. \tag{1.9}$$

Applying (1.9), we can derive the following Liouville type theorem as $\lambda(x)$ is a constant.

Corollary 1.5. *Let (M^n,g) be a noncompact complete Riemannian manifold without boundary. Suppose that $\mathrm{Ric}(M^n)\geq 0$ and $u(x)$ is a positive solution of the equation (1.1), where $\lambda(x)$ is a constant. If $\lambda\geq 0$ and $l\leq\frac{(n+1)(m+1)}{2(n-1)}$ or $\lambda\leq 0$ and $l\geq\frac{(n+1)(m+1)}{2(n-1)}$, then u is a constant.*

Theorem 1.6. *Let (M^n,g) be a complete noncompact Riemannian manifold with $\mathrm{Ric}(M^n)\geq -K$ with $K\geq 0$. Let $u(x)$ is a positive solution to the equation*

$$\Delta u^m+\lambda u^l=0, \tag{1.10}$$

where $\lambda>0$ is a constant. Let $v=\frac{m}{m-1}u^{m-1}$ and $1\leq l\leq\frac{(n+1)(m+1)}{2(n-1)}$. If

$$\lambda\leq\frac{2(m-1)(n-1)K}{\frac{2(m+1)}{n-1}+(m-2l+1)}\big(\frac{m-1}{m}\big)^{\frac{l-1}{m-1}}\big(\sup_{x\in M^n}v\big)^{\frac{m-l}{m-1}}, \tag{1.11}$$

then for any $x\in M^n$,

$$\begin{aligned}\frac{|\nabla v|^2}{v}&\leq\frac{2(m-1)^2(n-1)^2}{m(m+1)}K\sup_{x\in M^n}v\\&\quad-\frac{(m-1)(n-1)}{m(m+1)}\big[\frac{2(m+1)}{n-1}+(m-2l+1)\big]\lambda\big(\frac{m-1}{m}\sup_{x\in M^n}v\big)^{\frac{l-1}{m-1}}.\end{aligned}$$

If

$$\lambda \geq \frac{2(m-1)(n-1)K}{\frac{2(m+1)}{n-1}+(m-2l+1)}\big(\frac{m-1}{m}\big)^{\frac{l-1}{m-1}}\big(\sup_{x\in M^n} v\big)^{\frac{m-l}{m-1}},$$

then v must be a constant.

Note that by taking $l=1$ in Theorem 1.6, our result partially generalize Wang's result in [10]. By (1.11), we can find the lower bound estimate as $m \geq l$, and the upper bound estimate as $m \leq l$ for positive solutions of (1.10).

2. Preliminaries

To prove our main results, we need the lemma below. Let $v=\frac{m}{m-1}u^{m-1}$, then

$$(m-1)v\Delta v+|\nabla v|^2=-\lambda(m-1)\big(\frac{m-1}{m}\big)^{\frac{l-1}{m-1}}v^{1+\frac{l-1}{m-1}}. \tag{2.1}$$

Lemma 2.1. *Let (M^n,g) be a complete Riemannian manifold without boundary. Suppose that B_{2R} is a geodesic ball of radius $2R$ around $p\in M$ and Ricci$(B_{2R})\geq -K$ with $K\geq 0$. Let $u(x)$ is a positive solution to the equation (1.1) and $v=\frac{m}{m-1}u^{m-1}$. Let $w=\frac{|\nabla v|^2}{v}$ and $G=\varphi w$, where $\varphi(x)$ is a smooth cutoff function (see the proof of Theorem 1.2). Suppose that $G(x)$ reaches the maximum value at x_0 and $\varphi(x_0)>0$. Then at x_0,*

$$\begin{aligned}\varphi\Delta w \geq{}& \big[\frac{n}{2(n-1)}+\frac{2m}{(n-1)(m-1)^2}+\frac{1}{m-1}\big]\frac{G^2}{v\varphi}\\
&+\big[\frac{2m}{m-1}-\frac{n}{n-1}-\frac{2}{(n-1)(m-1)}\big]\frac{\nabla v\nabla\varphi}{v\varphi}G+\frac{n}{2(n-1)}\frac{|\nabla\varphi|^2}{\varphi^2}G\\
&+\big[\frac{2(m+1)}{(n-1)(m-1)}+\frac{m-2l+1}{m-1}\big]\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{v}-2(n-1)KG\\
&-\frac{2}{v}\varphi|\nabla v||\nabla\lambda|\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}-\frac{2}{n-1}\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{\nabla v\nabla\varphi}{v}\\
&+\frac{2}{n-1}\lambda^2\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}}\frac{\varphi}{v}.\end{aligned} \tag{2.2}$$

Proof. After calculations we obtain that

$$\begin{aligned}\Delta w &= \Delta\Big(\frac{|\nabla v|^2}{v}\Big)\\
&=\frac{\Delta|\nabla v|^2}{v}-\frac{2\nabla|\nabla v|^2\Delta v}{v^2}-\frac{|\nabla v|^2\Delta v}{v^2}+\frac{2|\nabla v|^4}{v^3}\\
&=\frac{2}{v}\big[|\operatorname{Hess} v|^2+\nabla v\cdot\nabla\Delta v+\operatorname{Ric}((\nabla v,\nabla v))\big]\\
&\quad -\frac{2\nabla|\nabla v|^2\cdot\nabla v}{v^2}-\frac{|\nabla v|^2\Delta v}{v^2}+\frac{2|\nabla v|^4}{v^3}.\end{aligned} \tag{2.3}$$

Since G reaches at the maximum at x_0, so we have $\nabla G=0$. Then at x_0,

$$\nabla w=-\frac{G\nabla\varphi}{\varphi^2}, \tag{2.4}$$

$$\nabla|\nabla v|^2=-\frac{vG}{\varphi^2}\nabla\varphi+\frac{G}{\varphi}\nabla v. \tag{2.5}$$

Choose an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ around x_0, such that $|\nabla v|e_1 = \nabla v$. Then

$$\frac{|\nabla|\nabla v|^2|^2}{4|\nabla v|^2} = \sum_{j=1}^{n} v_{1j}^2, \tag{2.6}$$

$$\frac{\nabla v \cdot \nabla|\nabla v|^2}{2|\nabla v|^2} = v_{11}. \tag{2.7}$$

On the other hand, we have

$$\begin{aligned}
|\operatorname{Hess} v|^2 &\geq v_{11}^2 + 2\sum_{\alpha=2}^{n} v_{1\alpha}^2 + \sum_{\alpha=2}^{n} v_{\alpha\alpha}^2 \\
&\geq v_{11}^2 + 2\sum_{\alpha=2}^{n} v_{1\alpha}^2 + \frac{1}{n-1}\Big(\sum_{\alpha=2}^{n} v_{\alpha\alpha}\Big)^2 \\
&= v_{11}^2 + 2\sum_{\alpha=2}^{n} v_{1\alpha}^2 + \frac{1}{n-1}(\Delta v - v_{11})^2 \\
&= \frac{n}{n-1}v_{11}^2 + 2\sum_{\alpha=2}^{n} v_{1\alpha}^2 - \frac{2}{n-1}\Delta v v_{11} + \frac{1}{n-1}(\Delta v)^2 \\
&= \frac{n}{n-1}v_{11}^2 + 2\sum_{\alpha=2}^{n} v_{1\alpha}^2 + \frac{2v_{11}}{n-1}\Big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\Big] \\
&\quad + \frac{1}{n-1}\Big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\Big]^2.
\end{aligned} \tag{2.8}$$

Putting (2.6) and (2.7) into (2.8), and using (2.5), we have

$$\begin{aligned}
&|\operatorname{Hess} v|^2 + \operatorname{Ric}(\nabla v, \nabla v) \\
&\geq \frac{n}{n-1}v_{11}^2 + 2\sum_{\alpha=2}^{n} v_{1\alpha}^2 + \frac{2v_{11}}{n-1}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big] \\
&\quad + \frac{1}{n-1}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big]^2 - (n-1)K|\nabla v|^2 \\
&\geq \frac{n}{n-1}\sum_{j=1}^{n} v_{1j}^2 + \frac{2v_{11}}{n-1}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big] \\
&\quad + \frac{1}{n-1}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big]^2 - (n-1)K|\nabla v|^2 \\
&= \frac{n}{n-1}\frac{|\nabla|\nabla v|^2|^2}{4|\nabla v|^2} + \frac{1}{n-1}\cdot\frac{\nabla v \cdot \nabla|\nabla v|^2}{|\nabla v|^2}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big] \\
&\quad + \frac{1}{n-1}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big]^2 - (n-1)K|\nabla v|^2 \\
&= \frac{n}{4(n-1)|\nabla v|^2}\big[\frac{G}{\varphi}\nabla v - \frac{vG}{\varphi^2}\nabla\varphi\big]^2 + \frac{1}{n-1}\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big]^2 \\
&\quad + \frac{1}{(n-1)|\nabla v|^2}\big[\frac{G}{\varphi}|\nabla v|^2 - \frac{vG}{\varphi^2}\nabla v\nabla\varphi\big]\big[\frac{1}{m-1}w + \lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\big] \\
&\quad - (n-1)K|\nabla v|^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{4(n-1)}\big(\frac{G}{\varphi}\big)^2 + \frac{nv^2}{4(n-1)|\nabla v|^2}\frac{G^2}{\varphi^4}|\nabla\varphi|^2 - \frac{n}{2(n-1)|\nabla v|^2}\cdot\frac{vG^2\nabla v\nabla\varphi}{\varphi^3} \\
&\quad + \frac{w^2}{(n-1)(m-1)^2} + \frac{2w}{(n-1)(m-1)}\lambda\big(\frac{m}{m-1}v\big)^{\frac{l-1}{m-1}} \\
&\quad + \frac{1}{n-1}\lambda^2\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}} + \frac{1}{(n-1)(m-1)}\frac{G}{\varphi}w + \frac{1}{n-1}\frac{G}{\varphi}\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}} \\
&- \frac{1}{(n-1)|\nabla v|^2}\frac{vG}{\varphi^2}\nabla v\nabla\varphi\frac{w}{m-1} - \frac{1}{(n-1)|\nabla v|^2}\frac{vG}{\varphi^2}\nabla v\nabla\varphi\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}} \\
&\quad - (n-1)K|\nabla v|^2 \\
&= \frac{n}{4(n-1)}\big(\frac{G}{\varphi}\big)^2 + \frac{nv}{4(n-1)}\frac{|\nabla\varphi|^2}{\varphi^2}\frac{G}{\varphi} - \frac{n}{2(n-1)}\frac{G}{\varphi}\frac{\nabla v\nabla\varphi}{\varphi} \\
&\quad + \frac{1}{(n-1)(m-1)^2}\big(\frac{G}{\varphi}\big)^2 + \frac{2\lambda}{(n-1)(m-1)}\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{\varphi} \\
&\quad + \frac{\lambda^2}{n-1}\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}} + \frac{1}{(n-1)(m-1)}\big(\frac{G}{\varphi}\big)^2 \\
&\quad + \frac{1}{n-1}\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{\varphi} - \frac{1}{(n-1)(m-1)}\frac{G}{\varphi}\frac{\nabla v\nabla\varphi}{\varphi} \\
&\quad - \frac{1}{n-1}\lambda\big(\frac{m}{m-1}v\big)^{\frac{l-1}{m-1}}\frac{\nabla v\nabla\varphi}{\varphi} - (n-1)K|\nabla v|^2 \\
&= \big[\frac{n}{4(n-1)} + \frac{m}{(n-1)(m-1)^2}\big]\frac{G^2}{\varphi^2} - \big[\frac{n}{2(n-1)} + \frac{1}{(n-1)(m-1)}\big]\frac{G}{\varphi^2}\nabla v\nabla\varphi \\
&\quad + \frac{n}{4(n-1)}\frac{|\nabla\varphi|^2}{\varphi^3}vG + \frac{m+1}{(n-1)(m-1)}\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{\varphi} \\
&\quad - \frac{1}{n-1}\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{\nabla v\nabla\varphi}{\varphi} + \frac{\lambda^2}{n-1}\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}} \\
&= -(n-1)K|\nabla v|^2.
\end{aligned}
$$

Noting that $v > 0$ when $m > 1$. Using the above inequality in (2.3) and applying (2.1) and (2.5), then (2.2) can be inferred. □

3. The proof of main results

Proof of Theorem 1.2. Construct a smooth function $\theta(t) : [0, +\infty) \to [0, 1]$

$$
\theta(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & t > 2 \end{cases}
$$

such that

$$
-C_1\sqrt{\theta} \le \theta' \le 0, \quad |\theta''| \le C_2\theta. \tag{3.1}
$$

Define the smooth cutoff function $\varphi : M \to \mathbb{R}$ by $\varphi(x) = \theta(\frac{r(x)}{R})$. We suppose that $G = \varphi w = \varphi\frac{|\nabla v|^2}{v}$ attains its maximal value at $x_0 \in B_{2R}$. We can suppose that $G(x_0) > 0$, because otherwise the proof is trivial. Then at x_0, we have

$$
\begin{aligned}
\Delta G &= \Delta\varphi\cdot w + 2\nabla\varphi\nabla w + \varphi\Delta w \\
&= \Delta\varphi\cdot w - 2G\frac{|\nabla\varphi|^2}{\varphi^2} + \varphi\Delta w
\end{aligned}
$$

$$= \frac{\Delta\varphi}{\varphi}G - 2G\frac{|\nabla\varphi|^2}{\varphi^2} + \varphi\Delta w.$$

Note that

$$\nabla\varphi = \frac{\theta'\nabla r}{R},$$
$$\Delta\varphi = \frac{\theta''}{R^2} + \frac{\theta'\Delta r}{R} \geq \frac{\theta''}{R^2} + \frac{(n-1)(1+\sqrt{K}R)\theta'}{R^2}.$$

Since $\Delta G \leq 0$ is valid at x_0, we have

$$\begin{aligned}
0 \geq{}& \big[\frac{\theta''}{\theta R^2} + \frac{(n-1)(1+\sqrt{K}R)\theta'}{\theta R^2}\big]G - \frac{3n-4}{2(n-1)}\frac{(\theta')^2}{R^2\theta^2}G \\
&+ \frac{(m-1)(mn+n-2)+4m}{2(m-1)^2(n-1)}\frac{G^2}{v\theta} - \frac{(m+1)(n-2)}{(n-1)(m-1)}\frac{G\sqrt{G}|\theta'|}{R\theta\sqrt{v\theta}} \\
&+ \big[\frac{2(m+1)}{(m-1)(n-1)} + \frac{m-2l+1}{m-1}\big]\lambda\big(\frac{m}{m-1}v\big)^{\frac{l-1}{m-1}}\frac{G}{v} \\
&- 2(n-1)KG - \frac{2}{v}\theta|\nabla v||\nabla\lambda|\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}} \\
&- \frac{2|\lambda|}{n-1}\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{\sqrt{G}|\theta'|}{R\sqrt{v\theta}} + \frac{2\lambda^2}{n-1}\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}}\frac{\theta}{v}.
\end{aligned} \tag{3.2}$$

Applying the inequality $ax^2+bx \geq -\frac{b^2}{4a}$ with $a>0$, we have

$$\begin{aligned}
&\frac{\lambda^2}{n-1}\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}}\frac{\theta}{v} - \frac{2|\lambda|}{n-1}\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{\sqrt{G}|\theta'|}{R\sqrt{v\theta}} \\
&\geq -\frac{G(\theta')^2}{(n-1)R^2\theta^2}, \\
&\frac{\lambda^2}{n-1}\big(\frac{m-1}{m}v\big)^{2\frac{l-1}{m-1}}\frac{\theta}{v} - \frac{2}{v}\varphi|\nabla v||\nabla\lambda|\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}} \\
&\geq -(n-1)\frac{|\nabla\lambda|^2}{\lambda^2}G.
\end{aligned} \tag{3.3}$$

By the Cauchy inequality, it follows that

$$-\frac{G\sqrt{G}|\theta'|}{R\theta\sqrt{v\theta}} \geq -\frac{G^2}{2v\theta} - \frac{G(\theta')^2}{2R^2\theta^2}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we obtain

$$\begin{aligned}
0 \geq{}& \Big[\frac{\theta''}{\theta R^2} + \frac{(n-1)(1+\sqrt{K}R)\theta'}{\theta R^2}\Big]G - \frac{3n-2}{2(n-1)}\frac{(\theta')^2}{R^2\theta^2}G \\
&+ \frac{m(m+1)}{(m-1)^2(n-1)}\frac{G^2}{v\theta} - \frac{(m+1)(n-2)}{2(m-1)(n-1)}\frac{G(\theta')^2}{R^2\theta^2} \\
&- 2(n-1)KG - (n-1)\frac{|\nabla\lambda|^2}{\lambda^2}G \\
&+ \big[\frac{2(m+1)}{(m-1)(n-1)} + \frac{m-2l+1}{m-1}\big]\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{v}.
\end{aligned} \tag{3.5}$$

From (3.1) and $|\nabla\lambda|^2 \le \tau\lambda^2$, we have

$$
\begin{aligned}
0 \ge & -\Big[\frac{C_2}{R^2} + \frac{(n-1)(1+\sqrt{K}R)C_1}{\sqrt{\theta}R^2}\Big]G + \frac{m(m+1)}{(m-1)^2(n-1)}\frac{G^2}{v\theta} \\
& - \Big[\frac{2m}{m-1} - \frac{n}{(m-1)(n-1)}\Big]\frac{C_1^2}{R^2\theta}G - 2(n-1)KG - (n-1)\tau G \\
& + \Big[\frac{2(m+1)}{(m-1)(n-1)} + \frac{m-2l+1}{m-1}\Big]\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{v} \\
\ge & -\Big[\frac{C_2}{R^2} + \frac{(n-1)(1+\sqrt{K}R)C_1}{\sqrt{\theta}R^2}\Big]G + \frac{m(m+1)}{(m-1)^2(n-1)}\frac{G^2}{v\theta} \\
& - \frac{2m}{m-1}\frac{C_1^2}{R^2\theta}G - 2(n-1)KG - (n-1)\tau G \\
& + \Big[\frac{2(m+1)}{(m-1)(n-1)} + \frac{m-2l+1}{m-1}\Big]\lambda\big(\frac{m-1}{m}v\big)^{\frac{l-1}{m-1}}\frac{G}{v}.
\end{aligned} \tag{3.6}
$$

Multiply by $v\theta$ to both side of (3.6), and using $0 \le \theta \le 1$ we obtain for $l \ge 1$

$$
\begin{aligned}
0 \ge & \frac{m(m+1)}{(m-1)^2(n-1)}G^2 - \frac{2m}{m-1}\frac{C_1^2}{R^2}\sup_{x\in M^n} vG \\
& - \Big[\frac{C_2}{R^2} + \frac{(n-1)(1+\sqrt{K}R)C_1}{R^2}\Big]G\sup_{x\in M^n} v \\
& - \Big[2(n-1)K + (n-1)\tau\Big]G\sup_{x\in M^n} v \\
& - |\frac{2(m+1)}{(m-1)(n-1)} + \frac{m-2l+1}{m-1}||\lambda|\big(\frac{m-1}{m}\sup_{x\in M^n} v\big)^{\frac{l-1}{m-1}}G.
\end{aligned} \tag{3.7}
$$

Meanwhile, for $l < 1$ we obtain

$$
\begin{aligned}
0 \ge & \frac{m(m+1)}{(m-1)^2(n-1)}G^2 - \frac{2m}{m-1}\frac{C_1^2}{R^2}G\sup_{x\in M^n} v \\
& - \Big[\frac{C_2}{R^2} + \frac{(n-1)(1+\sqrt{K}R)C_1}{R^2}\Big]G\sup_{x\in M^n} v \\
& - \Big[2(n-1)K + (n-1)\tau\Big]G\sup_{x\in M} v \\
& - |\frac{2(m+1)}{(m-1)(n-1)} + \frac{m-2l+1}{m-1}|\,|\lambda|\Big(\frac{m-1}{m}\inf_{x\in M^n} v\Big)^{\frac{l-1}{m-1}}G.
\end{aligned} \tag{3.8}
$$

We observe that

$$
\Big[\frac{C_2}{R^2} + \frac{(n-1)(1+\sqrt{K}R)C_1}{R^2}\Big]\sup_{x\in M^n} v \le \frac{C_3}{R^2}(1+\sqrt{K}R)\sup_{x\in M^n} v, \tag{3.9}
$$

for some constant C_3 depending only on n.

On the other hand, for the equation $Ax^2 - Bx \le 0$ with $A > 0, B > 0$, we have $x \le \frac{B}{A}$. By utilize the equation to (3.7) and (3.8), and noting (3.9) we obtain at the the maximum point x_0 for $l \ge 1$

$$
\sup_{B_p(R)} w(x) \le \varphi w(x_0) = G(x_0)
$$

$$
\begin{aligned}
&\le \frac{2(m-1)(n-1)}{m+1}\frac{C_1^2}{R^2}\sup_{x\in M^n} v + \frac{(m-1)^2(n-1)}{m(m+1)}\frac{C_3}{R^2}(1+\sqrt{K}R)\sup_{x\in M^n} v\\
&\quad + \frac{(m-1)^2(n-1)^2}{m(m+1)}(2K+\tau)\sup_{x\in M^n} v\\
&\quad + \frac{(m-1)(n-1)}{m(m+1)}\Big|\frac{2(m+1)}{n-1}+(m-2l+1)\Big|\,|\lambda|\Big(\frac{m-1}{m}\sup_{x\in M^n} v\Big)^{\frac{l-1}{m-1}},
\end{aligned}
$$

and for $l<1$,

$$
\begin{aligned}
\sup_{B_p(R)} w(x) &\le \varphi w(x_0) = G(x_0)\\
&\le \frac{2(m-1)(n-1)}{m+1}\frac{C_1^2}{R^2}\sup_{x\in M^n} v + \frac{(m-1)^2(n-1)}{m(m+1)}\frac{C_3}{R^2}(1+\sqrt{K}R)\sup_{x\in M^n} v\\
&\quad + \frac{(m-1)^2(n-1)^2}{m(m+1)}(2K+\tau)\sup_{x\in M^n} v\\
&\quad + \frac{(m-1)(n-1)}{m(m+1)}\Big|\frac{2(m+1)}{n-1}+(m-2l+1)\Big|\,|\lambda|\Big(\frac{m-1}{m}\inf_{x\in M^n} v\Big)^{\frac{l-1}{m-1}}.
\end{aligned}
$$

The proof is complete. □

The proof of Theorem 1.4. Simple calculations show that $\frac{2(m+1)}{(m-1)(n-1)}+\frac{m-2l+1}{m-1}\ge 0$ as $\lambda\ge 0$ and $l\le\frac{(n+1)(m+1)}{2(n-1)}$ or $\lambda\le 0$ and $l\ge\frac{(n+1)(m+1)}{2(n-1)}$. Hence, dropping the last term in (3.6) which is nonnegative, we have

$$
\begin{aligned}
0 \ge& -\Big[\frac{C_2}{R^2}+\frac{(n-1)(1+\sqrt{K}R)C_1}{\sqrt{\theta}R^2}\Big]G + \frac{m(m+1)}{(m-1)^2(n-1)}\frac{G^2}{v\theta}\\
&-\frac{2m}{m-1}\frac{C_1^2}{R^2\theta}G-(n-1)(2K+\tau)G.
\end{aligned}
$$

Multiplying by $v\theta$ on both sides, and using $0\le\theta\le 1$, we obtain

$$
\begin{aligned}
0 \ge& \frac{m(m+1)}{(m-1)^2(n-1)}G^2-\frac{2m}{m-1}\frac{C_1^2}{R^2}G\sup_{x\in M^n} v\\
&-\Big[\frac{C_2}{R^2}+\frac{(n-1)(1+\sqrt{K}R)C_1}{R^2}\Big]G\sup_{x\in M^n} v-(n-1)(2K+\tau)G\sup_{x\in M} v.
\end{aligned}
$$

Therefore, at the the maximum point x_0 we obtain

$$
\begin{aligned}
\sup_{B_p(R)} w(x) &\le \varphi w(x_0) = G(x_0)\\
&\le \frac{2(m-1)(n-1)}{m+1)}\frac{C_1^2}{R^2}\sup_{x\in M^n} v + \frac{(m-1)^2(n-1)}{m(m+1)}\frac{C_3}{R^2}(1+\sqrt{K}R)\sup_{x\in M^n} v\\
&\quad + \frac{(m-1)^2(n-1)^2}{m(m+1)}(2K+\tau)\sup_{x\in M^n} v,
\end{aligned}
$$

where we used (3.9). The proof is complete. □

The proof of Theorem 1.6. It is not difficult to find that $\frac{2(m+1)}{(m-1)(n-1)}+\frac{m-2l+1}{m-1}\geq 0$ for $1\leq l\leq \frac{(n+1)(m+1)}{2(n-1)}$. Then we have form (3.6),

$$\begin{aligned} 0 &\geq \frac{m(m+1)}{(m-1)^2(n-1)}G^2-\frac{2m}{m-1}\frac{C_1^2}{R^2}\sup_{x\in M^n} vG \\ &\quad -\Big[\frac{C_2}{R^2}+\frac{(n-1)(1+\sqrt{K}R)C_1}{R^2}\Big]G\sup_{x\in M^n} v \\ &\quad -\big[2(n-1)K+(n-1)\tau\big]G\sup_{x\in M^n} v \\ &\quad +\Big[\frac{2(m+1)}{(m-1)(n-1)}+\frac{m-2l+1}{m-1}\Big]\lambda\Big(\frac{m-1}{m}\sup_{x\in M^n} v\Big)^{\frac{l-1}{m-1}}G. \end{aligned} \tag{3.10}$$

By (3.10), and (3.9) we obtain at the the maximum point x_0,

$$\begin{aligned} &\sup_{B_p(R)} w(x)\leq \varphi w(x_0)=G(x_0)\\ &\leq \frac{2(m-1)(n-1)}{m+1}\frac{C_1^2}{R^2}\sup_{x\in M^n} v+\frac{(m-1)^2(n-1)}{m(m+1)}\frac{C_3}{R^2}(1+\sqrt{K}R)\sup_{x\in M^n} v\\ &\quad +\frac{2(m-1)^2(n-1)^2}{m(m+1)}K\sup_{x\in M^n} v\\ &\quad -\frac{(m-1)(n-1)}{m(m+1)}\Big[\frac{2(m+1)}{n-1}+(m-2l+1)\Big]\lambda\Big(\frac{m-1}{m}\sup_{x\in M^n} v\Big)^{\frac{l-1}{m-1}}. \end{aligned}$$

Letting $R\to\infty$, we infer as

$$\lambda\leq \frac{2(m-1)(n-1)K}{\frac{2(m+1)}{n-1}+(m-2l+1)}\Big(\frac{m-1}{m}\Big)^{\frac{l-1}{m-1}}\big(\sup_{x\in M^n} v\big)^{\frac{m-l}{m-1}},$$

and

$$\begin{aligned} \frac{|\nabla v|^2}{v} &\leq \frac{2(m-1)^2(n-1)^2}{m(m+1)}K\sup_{x\in M^n} v\\ &\quad -\frac{(m-1)(n-1)}{m(m+1)}\Big[\frac{2(m+1)}{n-1}+(m-2l+1)\Big]\lambda\Big(\frac{m-1}{m}\sup_{x\in M^n} v\Big)^{\frac{l-1}{m-1}}. \end{aligned}$$

On the other hand, as

$$\lambda\geq \frac{2(m-1)(n-1)K}{\frac{2(m+1)}{n-1}+(m-2l+1)}\Big(\frac{m-1}{m}\Big)^{\frac{l-1}{m-1}}\big(\sup_{x\in M^n} v\big)^{\frac{m-l}{m-1}},$$

we derive that v must be constant. $\square$

Proof of Corollary 1.3. Let minimal geodesic $\gamma(s):[0,1]\to M^n$, so that $\gamma(0)=y$, $\gamma(1)=x$, then

$$\begin{aligned} \ln\frac{v(x)}{v(y)} &= \int_0^1 \frac{d\ln(v(\gamma(s)))}{ds}=\int_0^1\frac{\nabla v\cdot\gamma'}{v(\gamma(s))}ds\\ &\leq \int_0^1\frac{|\nabla v|\cdot|\gamma'|}{|v(\gamma(s))|}ds=r(x,y)\int_0^1\frac{|\nabla v|}{|v(\gamma(s))|}ds\\ &\leq r(x,y)\int_0^1\sqrt{\frac{C(m,n,l,K,\delta,\tau,\sup_{x\in M^n} v)}{\inf_{x\in M^n} v}}ds \end{aligned}$$

$$= r(x,y)\sqrt{\frac{C(m,n,l,K,\delta,\tau,\sup_{x\in M} v)}{\inf_{x\in M^n} v}}.$$

□

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