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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS TO CHEMOREPULSION SYSTEMS WITH NONLINEAR SENSITIVITY

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Abstract. This article concerns the chemorepulsion system with nonlinear sensitivity and nonlinear secretion

$$
\begin{gathered}
u_t = \Delta u + \nabla \cdot (\chi u^m \nabla v), \quad x \in \Omega,\ t > 0, \\
0 = \Delta v - v + u^\alpha, \quad x \in \Omega,\ t > 0,
\end{gathered}
$$

under homogeneous Neumann boundary conditions, where $\chi > 0$, $m > 0$, $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. The existence and uniform boundedness of a classical global solutions are obtained. Furthermore, it is shown that for any given u_0, if $\alpha > m$ or $\alpha \geq 1$, the corresponding solution (u, v) converges to $(\bar{u}_0, \bar{u}_0^\alpha)$ as time goes to infinity, where $\bar{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0 dx$.

1. Introduction

Chemotaxis plays essential roles in various biological processes, which directs the movement of cells or organisms in response to the chemical stimuli. The first mathematical study of chemotaxis was the celebrated work by Keller and Segel in the '70s [11, 12] where they proposed the model

$$
\begin{aligned}
u_t &= \Delta u - \nabla \cdot (\chi u \nabla v) \\
\tau v_t &= \Delta v - v + u
\end{aligned}
\tag{1.1}
$$

to describe the aggregation of slime mold Dictyostelium discoideum and traveling pulses of bacteria Escherichia coli, where u denotes the bacteria density, v represents the chemical concentration, respectively, and χ is the chemotactic coefficient. The case that $\chi > 0$ means that bacteria are attracted by the chemical stimuli and the corresponding model is so called the chemoattractive model. The other case that $\chi < 0$ means that bacteria are repulsed by the chemical stimuli, and the corresponding model is so called the chemorepulsive model. The main feature of the chemoattractive models is the blow-up of solutions in finite time in space dimension greater or equal to two; see for instance [3, 4, 5, 9, 10, 17, 25]. Since the blow-up is unrealistic in the real biological processes, various mechanisms are introduced into the chemoattractive models to prevent the blow-up of solutions,

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see [13, 18, 23, 19, 24], for instance. In particular, in [14], the authors used a nonlinear form denoted by a function $f(u)$ to describe the production of the chemical cue, i.e, the second equation in (1.1) was replaced by $v_t = \Delta v - v + f(u)$, where $0 < f(u) < Ku^\alpha$ with some positive constant K and $0 < \alpha < \frac{n}{2}$ (where n denotes the space dimension), and obtained the global existence of classical solutions under some regularity assumptions on the initial data. For the chemorepulsive models, since bacteria are repulsed by the chemical stimuli which may prevent the aggregation of bacteria, the blow-up of solutions is not expected to take place for these models. Indeed, for the chemorepulsive model under homogeneous Neumann boundary conditions for u and v in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, when $\tau = 0$, it was shown in [15, 16] that there exist global in time solutions which are uniformly bounded and converge to the steady state exponentially. When $\tau = 1$, for the space dimension $n = 2$, based on a Lyapunov functional approach, it was proved in [6] that there exists a unique global smooth classical solution, and global weak solutions were also obtained in space dimension $n = 3, 4$. Considering the cross-diffusion term may be dependent on u nonlinearly, Tao in [20] studied the chemorepulsive system

$$\begin{gathered} u_t = \Delta u + \nabla \cdot (f(u)\nabla v), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, \quad x \in \Omega,\ t > 0 \end{gathered} \tag{1.2}$$

under homogeneous Neumann boundary conditions in a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, where $f(u) \leq K(u+1)^m$ with $0 < m < \frac{4}{n+2}$. Under some assumptions on the initial data, the uniformly bounded global solutions are obtained and the large time behavior of solutions is also given. However, the global existence of this repulsive model with $m \geq \frac{4}{n+2}$ is still open.

The purpose of this article is to study a repulsive system with nonlinear sensitivity which also involves nonlinear secretion:

$$\begin{gathered} u_t = \Delta u + \chi \nabla \cdot (u^m \nabla v), \quad x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u^\alpha, \quad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n\,(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. We assume that the chemotactic parameter χ is positive, which shows that the chemical signal with concentration $v = v(x,t)$ is repulsive. We remark that, in this model, the equation of v is an elliptic equation rather than a parabolic equation. Therefore, the global existence can be expected to obtain for more general m and α.

The main result of this article is as follows.

Theorem 1.1. *Let $\chi > 0$, $m > 0$, $\alpha > 0$, $\Omega \subset \mathbb{R}^n\,(n \geq 2)$ be a bounded domain with smooth boundary. Then for any nonnegative $u_0 \in C^0(\bar\Omega)$ ($u_0 \not\equiv 0$), problem* (1.3) *possesses a global in time classical solution, which is nonnegative and bounded in $\Omega \times (0,\infty)$. Furthermore, if $\alpha > m$ or $\alpha \geq 1$, then we have*

$$u(\cdot,t) \to \bar u_0 \quad \textit{and} \quad v(\cdot,t) \to \bar u_0^\alpha \quad \textit{in } L^\infty(\Omega) \quad \textit{as } t \to \infty, \tag{1.4}$$

where

$$\bar{u}_0 := \frac{1}{|\Omega|}\int_\Omega u_0 dx. \tag{1.5}$$

Remark 1.2. If $m = 1$ and $\alpha = 1$, the results obtained in Theorem 1.1 are in agreement with those in [15, 16].

As we know the proof in [20] heavily relies on $0 < m < \frac{4}{n+2}$, however, we only require $m > 0$ in this paper. Moreover, the convexity of domain is not required, which is indispensable in [20].

The condition $\chi > 0$ is crucial, otherwise, the system will become the chemoattractive system, and then, the solutions may blow up in finite time.

This article is organized as follows. In Section 2, we state the local and global existence, and then in Section 3, we deal with the large time behavior of solutions to (1.3) and give the proof of Theorem 1.1.

2. Existence of local and global solutions

In this section, we first state the existence of classical local solutions to system (1.3), then establish some a priori estimates which are the core of the argument concerning the existence and boundedness of global solutions.

Lemma 2.1. *Let $\chi > 0$, $m > 0$, $\alpha > 0$, $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary. Assume that the initial datum $u_0 \in C^0(\bar{\Omega})$ ($u_0 \not\equiv 0$) is nonnegative. Then there exist $T^* \in (0,\infty]$ and a pair of nonnegative functions $(u,v) \in C^0(\bar{\Omega}\times[0,T^*)) \cap C^{2,1}(\bar{\Omega}\times(0,T^*))$ solving (1.3) classically in $\Omega\times(0,T^*)$. Moreover, if $T^* < \infty$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad \textit{as } t \nearrow T^*. \tag{2.1}$$

Proof. The existence of a local classical solutions is based on a fixed point theorem. One can refer to [21, Lemma 2.1] for more details. Moreover, the nonnegativity of u and of v follow from the maximum principle. $\square$

The following L^1 estimates can be easily checked.

Lemma 2.2. *The solution (u,v) of (1.3) satisfies the mass conservation property*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \textit{for all } t \in [0,T^*). \tag{2.2}$$

Proof. Integrating the first equation of (1.3) with respect to space, we get

$$\frac{d}{dt}\int_\Omega u dx \equiv 0, \quad \text{for all } t \in (0,T^*),$$

which implies (2.2) directly. $\square$

The following Lemma is the core of the argument concerning existence and boundedness of global solutions.

Lemma 2.3. *Let $\chi > 0$, $m > 0$, $\alpha > 0$. $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain with smooth boundary. Then for any nonnegative $u_0 \in C^0(\bar{\Omega})$ ($u_0 \not\equiv 0$), any $k > 1$, the solution of (1.3) satisfies*

$$\|u(\cdot,t)\|_{L^k(\Omega)} \leq \|u_0\|_{L^k(\Omega)} \quad \textit{for all } t \in (0,T^*), \tag{2.3}$$

$$\int_0^t\int_\Omega |\nabla u^{\frac{k}{2}}|^2 dx \leq \frac{k}{4(k-1)}\|u_0\|^k_{L^k(\Omega)} \quad \textit{for all } t \in (0,T^*). \tag{2.4}$$

Proof. Testing $(1.3)_1$ against ku^{k-1}, substituting $(1.3)_2$ into the resulting equality, and invoking Young's inequality yields

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^k dx + \frac{4(k-1)}{k}\int_\Omega |\nabla u^{\frac{k}{2}}|^2 dx \\
&= -\chi k(k-1)\int_\Omega u^{m+k-2}\nabla u\cdot\nabla v dx \\
&= -\frac{\chi k(k-1)}{m+k-1}\int_\Omega \nabla u^{m+k-1}\cdot\nabla v dx \\
&= \frac{\chi k(k-1)}{m+k-1}\int_\Omega u^{m+k-1}\Delta v dx \\
&= \frac{\chi k(k-1)}{m+k-1}\int_\Omega u^{m+k-1} v dx - \frac{\chi k(k-1)}{m+k-1}\int_\Omega u^{m+k+\alpha-1} dx \\
&\le -\frac{\alpha\chi k(k-1)}{(m+k-1)(m+k+\alpha-1)}\int_\Omega u^{m+k+\alpha-1} dx \\
&\quad + \frac{\alpha\chi k(k-1)}{(m+k-1)(m+k+\alpha-1)}\int_\Omega v^{\frac{m+k+\alpha-1}{\alpha}} dx
\end{aligned} \tag{2.5}$$

for all $t \in (0, T^*)$. Next multiplying the second equation of (1.3) by $v^{\frac{m+k-1}{\alpha}}$, integrating by parts, and using Young's inequality yields

$$\begin{aligned}
&\int_\Omega v^{\frac{m+k+\alpha-1}{\alpha}} dx + \frac{4\alpha(m+k-1)}{(m+k+\alpha-1)^2}\int_\Omega |\nabla v^{\frac{m+k+\alpha-1}{2\alpha}}|^2 dx \\
&= \int_\Omega u^\alpha v^{\frac{m+k-1}{\alpha}} dx \\
&\le \frac{\alpha}{m+k+\alpha-1}\int_\Omega u^{m+k+\alpha-1} dx + \frac{m+k-1}{m+k+\alpha-1}\int_\Omega v^{\frac{m+k+\alpha-1}{\alpha}} dx
\end{aligned}$$

for all $t \in (0, T^*)$. Thus, we have

$$\begin{aligned}
&\frac{\alpha}{m+k+\alpha-1}\int_\Omega v^{\frac{m+k+\alpha-1}{\alpha}} dx + \frac{4\alpha(m+k-1)}{(m+k+\alpha-1)^2}\int_\Omega |\nabla v^{\frac{m+k+\alpha-1}{2\alpha}}|^2 dx \\
&\le \frac{\alpha}{m+k+\alpha-1}\int_\Omega u^{m+k+\alpha-1} dx
\end{aligned} \tag{2.6}$$

for all $t \in (0, T^*)$. Combining (2.5) and (2.6), we have

$$\frac{d}{dt}\int_\Omega u^k dx + \frac{4(k-1)}{k}\int_\Omega |\nabla u^{\frac{k}{2}}|^2 dx + \frac{4\alpha\chi k(k-1)}{(m+k+\alpha-1)^2}\int_\Omega |\nabla v^{\frac{m+k+\alpha-1}{2\alpha}}|^2 dx \le 0 \tag{2.7}$$

for all $t \in (0, T^*)$, which, integrating with respect to t over $(0, t)$, immediately leads to (2.3), (2.4). This completes the proof. □

We are now in a position to prove the boundedness result.

Lemma 2.4. *Let $\chi > 0$, $m > 0$, $0 < \alpha \le 1$. $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with smooth boundary. Then for any nonnegative $u_0 \in C^0(\bar{\Omega})$($u_0 \not\equiv 0$), there exists a positive constant C such that the solution of system (1.3) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (0, T^*), \tag{2.8}$$

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all } t \in (0, T^*). \tag{2.9}$$

Proof. Integrating the second equation of (1.3) with respect to space, we get

$$\int_\Omega v dx = \int_\Omega u^\alpha dx.$$

By Hölder's inequality and (2.2), if $0 < \alpha \le 1$, we have

$$\|v(\cdot,t)\|_{L^1(\Omega)} \le |\Omega|^{1-\alpha}\|u_0\|_{L^1(\Omega)}.$$

Invoking (2.3), if $\alpha > 1$, we also have $\|v(\cdot,t)\|_{L^1(\Omega)} \le \|u_0\|^\alpha_{L^\alpha(\Omega)}$. That is, for any $\alpha > 0$, it holds that

$$\|v(\cdot,t)\|_{L^1(\Omega)} \le |\Omega|^{1-\min\{1,\alpha\}}\|u_0\|^{\max\{\alpha,1\}}_{L^{\max\{\alpha,1\}}(\Omega)} \quad \text{for all } t \in (0,T^*).$$

Moreover, in view of (2.3) and (2.6), one may easily derive

$$\|v(\cdot,t)\|_{L^{\frac{m+k+\alpha-1}{\alpha}}(\Omega)} \le \|u(\cdot,t)\|^\alpha_{L^{m+k+\alpha-1}(\Omega)} \le \|u_0\|^\alpha_{L^{m+k+\alpha-1}(\Omega)}$$

for any $k \ge 2$. By passing to the limit as $k \to \infty$, yields

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \|u(\cdot,t)\|^\alpha_{L^\infty(\Omega)} \le \|u_0\|^\alpha_{L^\infty(\Omega)}.$$

Furthermore, one may invoke the Agmon-Douglis-Nirenberg L^k estimates [1, 2] on linear elliptic equations with the (zero) Neumann boundary condition to obtain

$$\|v(\cdot,t)\|_{W^{2,k}(\Omega)} \le C_1\|u^\alpha(\cdot,t)\|_{L^k(\Omega)} \le C_2 \quad \text{for all } t \in (0,\infty)$$

with some positive constants C_1, C_2. This, in conjunction with the Sobolev embedding [7]: $W^{2,k}(\Omega) \hookrightarrow C^1_B(\Omega) := \{u \in C^1(\Omega) | Du \in L^\infty(\Omega)\}$ if $k > n$, yields

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t \in (0,\infty).$$

We thus complete the proof of (2.8) and (2.9). □

Lemma 2.4 and the extensibility criterion (2.1) yields directly the existence a global solution.

Corollary 2.5. *Let $\chi > 0$, $m > 0$, $\alpha > 0$. $\Omega \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with smooth boundary. Then for any nonnegative $u_0 \in C^0(\bar\Omega)$ ($u_0 \not\equiv 0$), there exists a pair of nonnegative bounded functions $(u,v) \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))$ solving* (1.3) *classically.*

3. Large time behavior

In this section, we mainly focus on the large time behavior of the global classical bounded solution of (1.3). We first note that ∇u and ∇v converge to zero in the following sense:

Lemma 3.1. *Under the same assumptions as Corollary 2.5, the solution of* (1.3) *satisfies*

$$\int_0^\infty \int_\Omega |\nabla u|^2 \,dx\,dt \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{3.1}$$

If we further assume $\alpha > m$ or $\alpha \ge 1$, then we also have

$$\int_0^\infty \int_\Omega |\nabla v|^2 \,dx\,dt \le \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}. \tag{3.2}$$

Proof. Since $T^* = \infty$, (3.1) results from (2.4) with $k = 2$ directly. To establish (3.2), we divide it into two steps.

Step 1. In the case $\alpha \geq 1$, we first test $(1.3)_2$ against $-\Delta v$, then apply the integration by parts and Young's inequality to obtain

$$\begin{aligned} \int_\Omega |\Delta v|^2 dx + \int_\Omega |\nabla v|^2 dx &= \int_\Omega \nabla u^\alpha \cdot \nabla v dx \\ &\leq \frac{1}{2}\int_\Omega |\nabla u^\alpha|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx \\ &= \frac{\alpha^2}{2}\int_\Omega u^{2(\alpha-1)}|\nabla u|^2 dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx \end{aligned}$$

Integrating with respect to t over $(0,\infty)$ and invoking (3.1) and (2.8), we deduce

$$\begin{aligned} 2\int_0^\infty \int_\Omega |\Delta v|^2\,dx\,dt + \int_0^\infty \int_\Omega |\nabla v|^2\,dx\,dt &\leq \alpha^2 \|u\|_{L^\infty(\Omega)}^{2(\alpha-1)} \int_0^\infty \int_\Omega |\nabla u|^2\,dx\,dt \\ &\leq \frac{\alpha^2}{2}\|u_0\|_{L^\infty(\Omega)}^{2(\alpha-1)}\|u_0\|_{L^2(\Omega)}^2, \end{aligned}$$

which implies (3.2).

Step 2. In the case of $\alpha > m$, we can take $k = 1+\alpha-m$ in (2.7), then integrate with respect to t over $(0,\infty)$ to deduce

$$\frac{\chi(1+\alpha-m)(\alpha-m)}{\alpha}\int_0^\infty \int_\Omega |\nabla v|^2\,dx\,dt \leq \int_\Omega u_0^{1+\alpha-m},$$

which also implies (3.2). We thus complete the proof. □

Inspired by an argument developed in [22], we next give a weak stabilization property for u.

Lemma 3.2. *Let the assumptions in Corollary 2.5 hold. Then the solution of* (1.3) *satisfies*

$$\int_0^\infty \|u(\cdot,t) - \bar{u}_0\|_{(W^{n,2}(\Omega))^*}^2 dt \leq C \tag{3.3}$$

for some positive constant C, where $\bar{u}_0$ is as defined in (1.5)*, $(W^{n,2}(\Omega))^*$ is the dual space of $W^{n,2}(\Omega)$.*

Proof. We first assert that

$$\int_0^\infty \|u(\cdot,t) - \bar{u}_0\|_{L^{\frac{n}{n-1}}(\Omega)}^2 dt \leq C \tag{3.4}$$

for some positive constant C, which along with the fact that $L^{\frac{n}{n-1}}(\Omega) \hookrightarrow (W^{n,2}(\Omega))^*$ yields (3.3). In fact, invoking Sobolev's inequality and Poincaré's inequality, we have

$$\|u(\cdot,t) - \bar{u}_0\|_{L^{\frac{n}{n-1}}(\Omega)} \leq C_1\|\nabla u(\cdot,t)\|_{L^1(\Omega)} \quad \text{for all } t > 0.$$

Integrating with respect to t over $(0,\infty)$ and invoking Hölder's inequality and (3.1), we have

$$\begin{aligned} \int_0^\infty \|u(\cdot,t) - \bar{u}_0\|_{L^{\frac{n}{n-1}}(\Omega)}^2 dt &\leq C_1^2\int_0^\infty \|\nabla u(\cdot,t)\|_{L^1(\Omega)}^2 dt \\ &\leq C_1^2|\Omega|\int_0^\infty \int_\Omega |\nabla u(\cdot,t)|^2 dt \end{aligned}$$

$$\leq \frac{1}{2} C_1^2 |\Omega| \|u_0\|_{L^2(\Omega)}^2,$$

which implies (3.4) with $C := \frac{1}{2} C_1^2 |\Omega| \|u_0\|_{L^2(\Omega)}^2 > 0$. This completes the proof. □

The following decay property of u_t shows that u_t decays at least in some weak sense as the time goes to infinity, which will be used to improve the stabilization property of u in the sequel.

Lemma 3.3. *In addition to the assumptions in Corollary 2.5, we further assume $\alpha > m$ or $\alpha \geq 1$, then the solution of (1.3) satisfies*

$$\int_0^\infty \|u_t(\cdot, t)\|_{(W^{n,2}(\Omega))^*}^2 dt \leq C \tag{3.5}$$

for some positive constant C.

Proof. Take $\varphi \in W^{n,2}(\Omega)$ and test $(1.3)_1$ against φ to get

$$\begin{aligned} \int_\Omega u_t \varphi dx &= \int_\Omega \Delta u \varphi dx + \int_\Omega \nabla \cdot (\chi u^m \nabla v) \varphi dx \\ &= -\int_\Omega \nabla u \nabla \varphi dx - \int_\Omega \chi u^m \nabla v \cdot \nabla \varphi dx \end{aligned} \tag{3.6}$$

for all $t > 0$. Next we will estimate each term on the right hand side. For the first term, by Hölder's inequality, we have

$$\Big| -\int_\Omega \nabla u \nabla \varphi dx \Big| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi\|_{W^{n,2}(\Omega)}. \tag{3.7}$$

For the second term, by Hölder's inequality and (2.8), we have

$$\begin{aligned} \Big| -\int_\Omega \chi u^m \nabla v \cdot \nabla \varphi dx \Big| &\leq \chi \|u(\cdot, t)\|_{L^\infty(\Omega)}^m \|\nabla v\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \\ &\leq C_2 \|\nabla v\|_{L^2(\Omega)} \|\nabla \varphi\|_{W^{n,2}(\Omega)} \end{aligned} \tag{3.8}$$

with $C_2 := \chi \|u(\cdot, t)\|_{L^\infty(\Omega)}^m > 0$. We thus obtain

$$\begin{aligned} \|u_t(\cdot, t)\|_{(W^{n,2}(\Omega))^*}^2 &= \sup_{\varphi \in W^{n,2}(\Omega), \|\varphi\|_{W^{n,2}(\Omega)} \leq 1} \Big| \int_\Omega u_t \varphi dx \Big|^2 \\ &\leq 2 \|\nabla u\|_{L^2(\Omega)}^2 + 2C_2^2 \|\nabla v\|_{L^2(\Omega)}^2 \end{aligned} \tag{3.9}$$

for all $t > 0$. Then(3.5) may result from an integration (3.9) over $t \in (0, \infty)$ in conjunction with (3.1) and (3.2) directly. □

We next state a regularity estimate of the solution.

Lemma 3.4. *Let the assumptions in Corollary 2.5 hold, and further assume $\alpha > m$ or $\alpha \geq 1$. Then there exist a positive constant C and $\gamma \in (0, 1)$ such that the solution of (1.3) satisfies*

$$\|u_t(\cdot, t)\|_{C^\gamma(\bar{\Omega})} \leq C \quad \text{for all } t \geq 1. \tag{3.10}$$

Proof. The proof is similar to that of [20, Lemma 4.3]. We just outline the idea here. We first invoke (2.8) and (2.9) to obtain

$$\|\chi u^m \nabla v\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0$$

with some positive constant C. Then applying the operator A^θ with some $\theta \in (0, \frac{1}{2})$ to the Duhamel formula for u in the form

$$u(\cdot, t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\nabla \cdot (\chi u^m \nabla v)(\cdot, s)ds, \quad t > 0,$$

where A^θ denotes the fractional power of the realization of $-\Delta + 1$ in $L^q(\Omega)$ with $q > 1$ large enough satisfying $2\theta - \frac{n}{q} > 0$ under homogeneous Neumann boundary conditions, yields

$$\|A^\theta u(\cdot, t)\|_{L^q(\Omega)} \leq C \quad \text{for all } t > 0 \tag{3.11}$$

with a positive constant C. This, along with the fact that $D(A^\theta) \hookrightarrow C^\gamma(\bar{\Omega})$ for all $\gamma \in (0, 2\theta - \frac{n}{q})$ [8], yields (3.10). $\square$

Now we are ready to prove the stabilization property of u and also v.

Lemma 3.5. *Let the assumptions in Corollary 2.5 hold, and further assume $\alpha > m$ or $\alpha \geq 1$. Then the solution of* (1.3) *satisfies*

$$\|u(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty, \tag{3.12}$$

$$\|v(\cdot, t) - \bar{u}_0^\alpha\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty, \tag{3.13}$$

where $\bar{u}_0$ is as defined in (1.5).

Proof. The proof of the stabilization property (3.12) of u is similar to that of [20, Lemma 4.4], we omit it here. To achieve the stabilization property (3.13) of v, we set $w(x, t) := v(x, t) - \bar{u}_0^\alpha$, then it satisfies

$$\begin{gathered} 0 = \Delta w - w + u^\alpha - \bar{u}_0^\alpha, \quad x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0. \end{gathered} \tag{3.14}$$

Applying the elliptic maximum principle [7] to (3.14), we obtain

$$\|v(\cdot, t) - \bar{u}_0\|_{L^\infty(\Omega)} = \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u^\alpha(\cdot, t) - \bar{u}_0^\alpha\|_{L^\infty(\Omega)} \quad \text{for all } t > 0,$$

which in conjunction with (3.12) yields (3.13) directly. $\square$

Now we can prove our main result by collecting what we have found so far. Indeed, Theorem 1.1 follows from Corollary 2.5 and Lemma 3.5.

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