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MULTIPLICITY AND CONCENTRATION OF SOLUTIONS FOR FOURTH-ORDER ELLIPTIC EQUATIONS WITH MIXED NONLINEARITY

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Abstract. This article concerns the fourth-order elliptic equation

$$\Delta^2 u - \Delta u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$
$$u \in H^2(\mathbb{R}^N),$$

where $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V^{-1}(0)$ has nonempty interior. Under some mild assumptions, we establish the existence of two nontrivial solutions. Moreover, the concentration of these solutions is explored on the set $V^{-1}(0)$ as $\lambda \to \infty$. As an application, we give the similar results and concentration phenomenona for the above problem with concave and convex nonlinearities.

1. Introduction

This article concerns the fourth-order elliptic equation

$$\begin{gathered}\Delta^2 u - \Delta u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N),\end{gathered} \tag{1.1}$$

where $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $V \in C(\mathbb{R}^N)$, $f \in C(\mathbb{R}^N \times \mathbb{R})$, $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, $\lambda > 0$, $\mu > 0$ and $1 < p < 2$.

Problem (1.1) arises in the study of travelling waves in suspension bridge and the study of the static deflection of an elastic plate in a fluid, see [8, 10, 13]. There are many results for fourth-order elliptic equations, but most of them are focused on bounded domains, see [2, 3, 4, 5, 14, 18, 19, 20, 31, 30] and the references therein. Recently, the case of the whole space $\mathbb{R}^N$ was also considered in some works, see [11, 21, 22, 23, 24, 25, 26, 28, 29]. For the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. In order to overcome this difficulty, some authors assumed that the potential V satisfies certain coercive condition; that is,

(A1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^N} V(x) \geq a > 0$, where a is a positive constant;

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(A2) for any $b > 0$,meas$(V_b) < +\infty$, where meas denotes the Lebesgue measure and $V_b := \{x \in \mathbb{R}^N | V(x) \leq b\}$.

The authors in [21, 22, 25, 26] established the existence of infinitely many solutions under various hypotheses on the nonlinearity. Zhang et al. [28] studied the sign-changing solutions of problem (1.1) with Kirchhoff-type. When replacing (A2) by a more general assumption:

(A3) there is $b > 0$ such that meas$(V_b) < +\infty$,

the compactness of the embedding fails and this situation becomes more delicate. Recently, the authors in [11, 23] considered the following equation with a parameter under condition (A3),

$$\begin{gathered} \Delta^2 u - \Delta u + \lambda V(x) u = f(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{gathered}$$

With the aid of a parameter, they proved that the energy functional possess the property of being locally compact. Moreover, the authors of these article proved the existence of infinitely many high energy solutions for superlinear case. For somewhat related sublinear case and the existence of infinitely many small negative-energy solutions, see also [22, 23, 24]. For the singularly perturbed problem

$$\begin{gathered} \epsilon^4 \Delta^2 u + V(x) u = f(u), \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \end{gathered} \tag{1.2}$$

the authors [15, 16] considered when the potential V is positive and has global minimum. They obtained the existence of semi-classical solutions. Moreover, they also shown the concentration phenomenon of semi-classical solutions around global minimum of the potential V as $\epsilon \to 0$.

Motivated by the above papers, we will consider problem (1.1) with steep well potential, and study the existence of nontrivial solution and concentration results (as $\lambda \to \infty$). To deduce our statements, we need to make the following assumptions on potential V:

(A4) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$ on $\mathbb{R}^N$;

(A5) $\Omega = \operatorname{int} V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

This kind of hypotheses was first introduced by Bartsch and Wang [6] (see also [7]) in the study of a nonlinear Schrödinger equation and the potential $\lambda V(x)$ with V satisfying (A3)–(A5) is referred as the steep well potential. It is worth mentioning that the above papers always assumed the potential V is positive ($V > 0$). Compared with the case $V > 0$, our assumptions on V are rather weak, and perhaps more important. Generally speaking, there may exist some behaviours and phenomenons for the solutions of problem (1.1) under condition (A5), such as the concentration phenomenon of solutions. Very recently, in [27], the authors considered this case, and proved the existence and concentration of solutions when the nonlinearity is only sublinear. Besides, we are also interested in the case that the nonlinearity is a more general mixed nonlinearity involving a combination of superlinear ($f(x,u)$) and sublinear ($\xi(x)|u|^{p-2}u$, $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ and $1 < p < 2$) terms. To the best of our knowledge, few works concerning on this case up to now. Based on the above facts, the main purpose of this paper is to prove the existence of nontrivial solutions and to investigate the concentration phenomenon of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$. In order to state our results, we need the following assumptions for superlinear term $f(x,u)$:

- (A6) $f \in C(\mathbb{R}^N \times \mathbb{R})$ and $|f(x,u)| \leq c\big(1+|u|^{q-1}\big)$ for some $q \in (2, 2_*)$, where $2_* = \frac{2N}{N-4}$ if $N > 4$, $2_* = \infty$ if $N \leq 4$;
- (A7) $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \mathbb{R}^N$;
- (A8) there exists $\theta > 2$ such that $0 < \theta F(x,u) \leq uf(x,u)$ for every $x \in \mathbb{R}^N$ and $u \neq 0$, where $F(x,u) = \int_0^u f(x,t)dt$.

On the existence of solutions we have the following result.

Theorem 1.1. *Assume that the conditions* (A3)–(A8) *hold, and* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$ $(1 < p < 2)$*, then there exist two positive constants* Λ_0 *and* μ_0 *such that for every* $\lambda > \Lambda_0$ *and* $0 < \mu < \mu_0$*, problem* (1.1) *has at least two nontrivial solutions* u^i_λ $(i = 1, 2)$.

On the concentration of solutions we have the following result.

Theorem 1.2. *Let* u^i_λ, $(i = 1, 2)$ *be the solutions of problem* (1.1) *obtained in Theorem 1.1 and* $\mu \in (0, \mu_0)$*, then* $u^i_\lambda \to u^i_0$ *in* $H^2(\mathbb{R}^N)$ *as* $\lambda \to \infty$*, where* $u^i_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ *are nontrivial solutions of the equation*

$$\begin{gathered} \Delta^2 u - \Delta u = f(x,u) + \mu\xi(x)|u|^{p-2}u, \quad \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

A model of nonlinearity is

$$g(x,u) := |u|^{q-2}u + \mu\xi(x)|u|^{p-2}u \tag{1.4}$$

with $1 < p < 2 < q < 2_*$ and $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$. Clearly, $g(x,u)$ satisfies (A6)–(A8). Following [1], the nonlinear term $g(x,u)$ is called concave and convex nonlinear term. Therefore, our results can be applied to the concave and convex nonlinear term case. As a consequence, we have

Corollary 1.3. *Assume that the conditions* (A3)–(A5) *are satisfied and let the nonlinearity be of the form* (1.4)*, then there exist two positive constants* Λ_0 *and* μ_0 *such that for every* $\lambda > \Lambda_0$ *and* $0 < \mu < \mu_0$*, problem* (1.1) *has at least two nontrivial solutions* u^i_λ $(i = 1, 2)$.

Corollary 1.4. *Let* u^i_λ, $(i = 1, 2)$ *be the solutions of problem* (1.1) *obtained in Corollary 1.3 and* $\mu \in (0, \mu_0)$*, then* $u^i_\lambda \to u^i_0$ *in* $H^2(\mathbb{R}^N)$ *as* $\lambda \to \infty$*, where* $u^i_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ *are nontrivial solutions of the equation*

$$\begin{gathered} \Delta^2 u - \Delta u = |u|^{q-2}u + \mu\xi(x)|u|^{p-2}u, \quad \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.5}$$

Remark 1.5. Compared with the previous works, our results seem more general and complete, which is reflected in the following aspects. On the one hand, our assumptions on V are much weaker, and the existence and multiplicity of nontrivial solutions are obtained without any symmetric assumption. On the other hand, more importantly, we also explore the phenomenon of concentrations of these solutions as $\lambda \to \infty$, which seems to be rarely concerned in the previous studies.

The rest of this article is organized as follows. In Section 2, we establish the variational framework associated with problem (1.1), and we also give the proof of Theorem 1.1. In Section 3, we study the concentration of solutions and prove Theorem 1.2.

2. Variational setting and proof of Theorem 1.1

Below by $\|\cdot\|_s$ we denote the usual L^s-norm for $2 \le s \le 2_*$, c_i, C, C_i stand for different positive constants. Now, we establish the variational setting of problem (1.1). Let

$$E = \Big\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + V(x)u^2\right) dx < +\infty\Big\}$$

be equipped with the inner product

$$(u, v) = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + V(x)uv)\, dx, \quad u, v \in E,$$

and the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2) dx\Big)^{1/2}, \quad u \in E.$$

For $\lambda > 0$, we also need the inner product

$$(u, v)_\lambda = \int_{\mathbb{R}^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + \lambda V(x)uv)\, dx, \quad u, v \in E,$$

and the corresponding norm $\|u\|_\lambda^2 = (u, u)_\lambda$. It is clear that $\|u\| \le \|u\|_\lambda$, for $\lambda \ge 1$.

Set $E_\lambda = (E, \|\cdot\|_\lambda)$, then E_λ is a Hilbert space. By (A3)-(A4) and the statement of proof of [23, Lemma 2.1], we can demonstrate that there exists a positive constant γ_0 (independent of λ) such that

$$\|u\|_{H^2(\mathbb{R}^N)} \le \gamma_0 \|u\|_\lambda, \quad \text{for all } u \in E_\lambda.$$

Furthermore, the embedding $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [2, 2_*]$, and $E_\lambda \hookrightarrow L^s_{\rm loc}(\mathbb{R}^N)$ is compact for $s \in [2, 2_*)$, i.e., there are constants $\gamma_s, \gamma_0 > 0$ such that

$$\|u\|_s \le \gamma_s \|u\|_{H^2(\mathbb{R}^N)} \le \gamma_s \gamma_0 \|u\|_\lambda, \quad \text{for all } u \in E_\lambda,\ 2 \le s \le 2_*. \tag{2.1}$$

Let

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + |\nabla u|^2 + \lambda V(x)u^2\right) dx - \Psi(u), \tag{2.2}$$

where

$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u) dx + \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|u|^p dx.$$

By a standard argument and Hölder inequality, it is easy to verify that $\Phi_\lambda \in C^1(E_\lambda, \mathbb{R})$ and

$$\langle \Phi_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} [\Delta u \Delta v + \nabla u \cdot \nabla v + \lambda V(x)uv]\, dx - \langle \Psi'(u), v\rangle, \tag{2.3}$$

for all $u, v \in E_\lambda$, where

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^N} f(x, u)v dx + \mu \int_{\mathbb{R}^N} \xi(x)|u|^{p-2}uv dx.$$

We say that $I \in C^1(X, \mathbb{R})$ satisfies (PS) condition if any sequence $\{u_n\}$ such that $I(u_n) \to d$, $I'(u_n) \to 0$ has a convergent subsequence. To prove our result, we need the following Mountain Pass Theorem.

Theorem 2.1 ([17, Theorem 2.2])**.** *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfying (PS) condition. Suppose $I(0) = 0$ and*

(1) *there are constants $\rho, \eta > 0$ such that $I_{\partial B_\rho(0)} \ge \eta$,*

(2) *there is an constant* $e \in X \setminus \bar{B}_\rho(0)$ *such that* $I(e) \leq 0$, *then* I *possesses a critical value* $\beta \geq \eta$.

Lemma 2.2. *Assume that* (A6), (A7) *are satisfied, and* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$. *Then there exist three positive constants* μ_0, ρ *and* η *such that* $\Phi_\lambda(u)|_{\|u\|_\lambda=\rho} \geq \eta > 0$ *for all* $\mu \in (0, \mu_0)$.

Proof. For any $\varepsilon > 0$, it follows from conditions (A6) and (A7) that there exist $C_\varepsilon > 0$ such that

$$F(x,u) \leq \frac{\varepsilon}{2}|u|^2 + \frac{C_\varepsilon}{q}|u|^q, \quad \text{for all } u \in E_\lambda. \tag{2.4}$$

Thus, from (2.1), (2.4) and the Sobolev inequality, we have that for all $u \in E_\lambda$,

$$\int_{\mathbb{R}^N} F(x,u)dx \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^N} u^2 dx + \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N} |u|^q dx \leq \frac{\gamma_2^2\gamma_0^2\varepsilon}{2}\|u\|_\lambda^2 + \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}\|u\|_\lambda^q,$$

which implies that

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,u)dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|u|^p dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\gamma_2^2\gamma_0^2\varepsilon}{2}\|u\|_\lambda^2 - \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}\|u\|_\lambda^q - \frac{\mu\gamma_2^p\gamma_0^p}{p}\|\xi\|_{\frac{2}{2-p}}\|u\|_\lambda^p \\ &= \|u\|_\lambda^p\Big[\frac{1}{2}\left(1-\gamma_2^2\gamma_0^2\varepsilon\right)\|u\|_\lambda^{2-p} - \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}\|u\|_\lambda^{q-p} - \frac{\mu\gamma_2^p\gamma_0^p}{p}\|\xi\|_{\frac{2}{2-p}}\Big]. \end{aligned} \tag{2.5}$$

Take $\varepsilon = \frac{1}{2\gamma_2^2\gamma_0^2}$ and define

$$g(t) = \frac{1}{4}t^{2-p} - \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}t^{q-p}, \quad \text{for } t \geq 0.$$

It is easy to prove that there exists $\rho > 0$ such that

$$\max_{t\geq 0} g(t) = g(\rho) = \frac{q-2}{4(q-p)}\left[\frac{(2-p)q}{4C_\varepsilon\gamma_q^q\gamma_0^q(q-p)}\right]^{\frac{2-p}{q-2}}.$$

Then it follows from (2.5) that there exist positive constants μ_0 and η such that $\Phi_\lambda(u)|_{\|u\|_\lambda=\rho} \geq \eta$ for all $\mu \in (0,\mu_0)$. □

Lemma 2.3. *Assume that* (A6)–(A8) *are satisfied, and* $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$. *Let* ρ *be as in Lemma 2.2. Then there exists* $e \in E_\lambda$ *with* $\|e\|_\lambda > \rho$ *such that* $\Phi_\lambda(e) < 0$ *for all* $\mu \geq 0$.

Proof. By (2.4) and (A8), there exists $c > 0$ such that

$$F(x,u) \geq c\left(|u|^\theta - |u|^2\right), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, for $t > 0$, $u \in E_\lambda$, we have

$$\begin{aligned} \Phi_\lambda(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,tu)dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|tu|^p dx \\ &\leq \frac{t^2}{2}\|u\|_\lambda^2 - ct^\theta\int_{\mathbb{R}^N} |u|^\theta dx + ct^2\int_{\mathbb{R}^N} |u|^2 dx - \frac{\mu}{p}t^p\int_{\mathbb{R}^N} \xi(x)|u|^p dx, \end{aligned}$$

which implies that $\Phi_\lambda(tu) \to -\infty$ as $t \to \infty$. Therefore, there exist $t_0 > 0$ and $e := t_0 u$ with $\|e\|_\lambda > \rho$ such that $\Phi_\lambda(e) < 0$. This completes the proof. □

To find the critical points of Φ_λ, we shall show that Φ_λ satisfies the (PS) condition, i.e. any (PS) sequence $\{u_n\}$ has a convergent subsequence in E_λ. Since there is no compactness of the Sobolev embedding, the situation is more difficult. To overcome this difficulty, we need the following convergence results.

Lemma 2.4. *Suppose that $u_n \rightharpoonup u_0$ in E_λ. Then, passing to a subsequence*

$$\Phi_\lambda(u_n) = \Phi_\lambda(u_n - u_0) + \Phi_\lambda(u_0) + o(1), \tag{2.6}$$

$$\Phi'_\lambda(u_n) = \Phi'_\lambda(u_n - u_0) + \Phi'_\lambda(u_0) + o(1) \quad \textit{as } n \to \infty. \tag{2.7}$$

Particularly, if $\{u_n\}$ is a (PS) sequence such that $\Phi_\lambda(u_n) \to d$ for some $d \in \mathbb{R}$, then

$$\Phi_\lambda(u_n - u_0) \to d - \Phi_\lambda(u_0) \quad \textit{and} \quad \Phi'_\lambda(u_n - u_0) \to 0 \tag{2.8}$$

after passing to a subsequence.

Proof. Since $u_n \rightharpoonup u_0$ in E_λ, we have

$$(u_n, u_0)_\lambda \to (u_0, u_0)_\lambda, \quad \text{as } n \to \infty.$$

which yields

$$\begin{aligned} \|u_n\|_\lambda^2 &= (u_n - u_0, u_n - u_0)_\lambda + (u_0, u_n)_\lambda + (u_n - u_0, u_0)_\lambda \\ &= \|u_n - u_0\|_\lambda^2 + \|u_0\|_\lambda^2 + o(1). \end{aligned}$$

It is clear that

$$(u_n, \phi)_\lambda = (u_n - u_0, \phi)_\lambda + (u_0, \phi)_\lambda \quad \text{for all } \phi \in E_\lambda.$$

Hence, to obtain (2.6) and (2.7), it sufficient to check that

$$\int_{\mathbb{R}^N} [F(x, u_n) - F(x, u_n - u_0) - F(x, u_0)]\, dx = o(1), \tag{2.9}$$

$$\int_{\mathbb{R}^N} \xi(x) [|u_n|^p - |u_n - u_0|^p - |u_0|^p]\, dx = o(1), \tag{2.10}$$

$$\int_{\mathbb{R}^N} (f(x, u_n) - f(x, u_n - u_0) - f(x, u_0))\, \phi dx = o(1) \quad \forall \phi \in E_\lambda, \tag{2.11}$$

$$\int_{\mathbb{R}^N} \xi(x) \left(|u_n|^{p-2} u_n - |u_n - u_0|^{p-2}(u_n - u_0) - |u_0|^{p-2} u_0\right) \phi dx = o(1) \\ \text{for all } \phi \in E_\lambda. \tag{2.12}$$

Here, we only prove $(2.9) and (2.10)$, the verification of (2.11) and (2.12) is similar. Take $\omega_n := u_n - u_0$, we have $\omega_n \rightharpoonup 0$ in E_λ and $\omega_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$. It follows from (A6) and (A7) that

$$|f(x, u)| \le \varepsilon |u| + C_\varepsilon |u|^{q-1} \quad \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.13}$$

$$|F(x, u)| \le \int_0^1 |f(x, tu)||u| dt \le \varepsilon |u|^2 + C_\varepsilon |u|^q,\ \forall (x, u) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.14}$$

Then

$$\begin{aligned} |F(x, \omega_n + u_0) - F(x, \omega_n)| &\le \int_0^1 |f(x, \omega_n + \zeta u_0)||u_0| d\zeta \\ &\le \int_0^1 \left(\varepsilon |\omega_n + \zeta u_0||u_0| + C_\varepsilon |\omega_n + \zeta u_0|^{q-1} |u_0|\right) d\zeta \\ &\le c_1 \left(\varepsilon |\omega_n||u_0| + \varepsilon |u_0|^2 + C_\varepsilon |\omega_n|^{q-1} |u_0| + C_\varepsilon |u_0|^q\right). \end{aligned}$$

By Young's inequality, we have

$$|F(x,\omega_n+u_0)-F(x,\omega_n)| \le c_2\left(\varepsilon|\omega_n|^2+\varepsilon|u_0|^2+\varepsilon|\omega_n|^q+C_\varepsilon|u_0|^q\right),$$

so that, using (2.14), we obtain

$$|F(x,\omega_n+u_0)-F(x,\omega_n)-F(x,u_0)| \le c_3\left(\varepsilon|\omega_n|^2+\varepsilon|u_0|^2+\varepsilon|\omega_n|^q+C_\varepsilon|u_0|^q\right),$$

for $n\in\mathbb{N}$. Let

$$H_n(x):=\max\left\{|F(x,\omega_n+u_0)-F(x,\omega_n)-F(x,u_0)|-c_3\varepsilon\left(|\omega_n|^2+|\omega_n|^q\right),0\right\}.$$

It follows that

$$0\le H_n(x)\le c_3\left(\varepsilon|u_0|^2+C_\varepsilon|u_0|^q\right)\in L^1(\mathbb{R}^N).$$

Thus, using Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^N} H_n(x)dx\to 0,\quad \text{as } n\to\infty. \tag{2.15}$$

From the definition of $H_n(x)$, we have

$$|F(x,\omega_n+u_0)-F(x,\omega_n)-F(x,u_0)| \le c_3\varepsilon\left(|\omega_n|^2+|\omega_n|^q\right)+H_n(x),$$

for all $n\in\mathbb{N}$. which, together with (2.15) and (2.1), we obtain

$$\int_{\mathbb{R}^N}|F(x,\omega_n+u_0)-F(x,\omega_n)-F(x,u_0)|dx \le c_3\varepsilon\left(\|\omega_n\|_2^2+\|\omega_n\|_q^q\right)+\varepsilon\le c_4\varepsilon,$$

for n sufficiently large, hence

$$\int_{\mathbb{R}^N}\left[F(x,u_n)-F(x,u_n-u_0)-F(x,u_0)\right]dx=o(1)$$

that is, (2.9) holds.

Observe that $\xi\in L^{\frac{2}{2-p}}(\mathbb{R}^N,\mathbb{R}^+)$, thus, for any $\epsilon>0$ we can choose $R_\epsilon>0$ such that

$$\Big(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}}|\xi(x)|^{\frac{2}{2-p}}dx\Big)^{\frac{2-p}{2}}<\epsilon. \tag{2.16}$$

By Sobolev's embedding theorem, $u_n\rightharpoonup u_0$ in E_λ implies $u_n\to u_0$ in $L^2_{\rm loc}(\mathbb{R}^N)$, and hence,

$$\lim_{n\to\infty}\int_{B_{R_\epsilon}}|u_n-u_0|^2dx=0. \tag{2.17}$$

By (2.17), there exists $N_0\in\mathbb{N}$ such that

$$\int_{B_{R_\epsilon}}|u_n-u_0|^2dx<\epsilon^2,\quad \text{for } n\ge N_0. \tag{2.18}$$

Hence, by (2.1), (2.18) and the Hölder inequality, for any $n\ge N_0$, we have

$$\begin{aligned}
&\frac{\mu}{p}\int_{B_{R_\epsilon}}\xi(x)|u_n-u_0|^p dx\\
&\le \frac{\mu}{p}\Big(\int_{B_{R_\epsilon}}|\xi(x)|^{\frac{2}{2-p}}dx\Big)^{\frac{2-p}{2}}\Big(\int_{B_{R_\epsilon}}|u_n-u_0|^2dx\Big)^{p/2}\\
&\le \frac{\mu}{p}\epsilon^p\|\xi(x)\|_{\frac{2}{2-p}}.
\end{aligned} \tag{2.19}$$

On the other hand, by (2.1) and (2.16), we have

$$\begin{aligned} &\frac{\mu}{p}\int_{\mathbb{R}^N\setminus B_{R_\epsilon}}\xi(x)|u_n-u_0|^p dx\\ &\le \frac{\mu}{p}\Big(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}}|\xi(x)|^{\frac{2}{2-p}}dx\Big)^{\frac{2-p}{2}}\Big(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}}|u_n-u_0|^2dx\Big)^{p/2}\\ &\le \frac{\mu}{p}\epsilon\left(\|u_n\|_2^p+\|u_0\|_2^p\right)\\ &\le \frac{\mu}{p}\epsilon\gamma_2^p\gamma_0^p\left(\|u_n\|_\lambda^p+\|u_0\|_\lambda^p\right)\\ &\le \frac{\mu}{p}\epsilon\gamma_2^p\gamma_0^p\left(c_5^p+\|u_0\|_\lambda^p\right). \end{aligned} \tag{2.20}$$

Since ϵ is arbitrary, combining (2.19) with (2.20), we have

$$\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u_n-u_0|^p dx=o(1), \tag{2.21}$$

$$\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)\left(|u_n|^p-|u_0|^p\right)dx\le\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u_n-u_0|^p dx.$$

Therefore

$$\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)\left(|u_n|^p-|u_n-u_0|^p-|u_0|^p\right)=o(1),$$

that is, (2.10) holds.

Now, we consider the case $\{u_n\}$ is a (PS) sequence such that $\Phi_\lambda(u_n)\to d$ and $\Phi'_\lambda(u_n)\to 0$. It follows from (2.6) and (2.7) that

$$\Phi_\lambda(u_n-u_0)=d-\Phi_\lambda(u_0)+o(1),\quad \Phi'_\lambda(u_n-u_0)=-\Phi'_\lambda(u_0)+o(1), \tag{2.22}$$

we show that $\Phi'_\lambda(u_0)=0$. For every $\psi\in C_0^\infty(\mathbb{R}^N)$, it follows from (2.13) and the fact that $u_n\to u_0$ in $L^s_{\rm loc}(\mathbb{R}^N)$ that

$$\int_{\mathbb{R}^N}\left(f(x,u_n)-f(x,u_0)\right)\psi dx=\int_{\operatorname{supp}\psi}\left(f(x,u_n)-f(x,u_0)\right)\psi dx=o(1)$$

and

$$\begin{aligned} &\mu\int_{\mathbb{R}^N}\xi(x)\left(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0\right)\psi dx\\ &=\mu\int_{\operatorname{supp}\psi}\xi(x)\left(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0\right)\psi dx=o(1) \end{aligned}$$

which implies

$$\langle\Phi'_\lambda(u_0),\psi\rangle=\lim_{n\to\infty}\langle\Phi'_\lambda(u_n),\psi\rangle=0.$$

Hence, $\Phi'_\lambda(u_0)=0$, which together with the second equation of (2.22) shows that $\Phi'_\lambda(u_n-u_0)\to 0$ as $n\to\infty$. Consequently, (2.8) holds and the proof is complete. □

Lemma 2.5. *Let* (A3)–(A5), (A6)–(A8) *be satisfied, there exists* $\Lambda_0>0$*, any (PS) sequence of* Φ_λ *has a convergent subsequence for all* $\lambda\ge\Lambda_0$.

Proof. We adapt an argument in [9]. Let $\{u_n\}$ be a sequence such that $\Phi_\lambda(u_n)\to d$ and $\Phi'_\lambda(u_n)\to 0$ for some $d\in\mathbb{R}$; thus

$$1+d+\|u_n\|_\lambda\ge\Phi_\lambda(u_n)-\frac{1}{\theta}\langle\Phi'_\lambda(u_n),u_n\rangle$$

$$
\begin{aligned}
&= (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \big[\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\big] dx \\
&\quad + \int_{\mathbb{R}^N} (\frac{1}{\theta} - \frac{1}{p})\mu \xi(x)|u_n|^p dx,
\end{aligned}
$$

hence

$$
\begin{aligned}
&1 + d + \|u_n\|_\lambda + (\frac{1}{p} - \frac{1}{\theta})\mu \int_{\mathbb{R}^N} \xi(x)|u_n|^p dx \\
&\geq (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \Big[\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\Big] dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
(\frac{1}{p} - \frac{1}{\theta})\mu \int_{\mathbb{R}^N} \xi(x)|u_n|^p dx &\leq (\frac{1}{p} - \frac{1}{\theta})\mu \Big(\int_{\mathbb{R}^N} |\xi(x)|^{\frac{2}{2-p}} dx\Big)^{\frac{2-p}{2}} \Big(\int_{\mathbb{R}^N} |u_n|^2 dx\Big)^{p/2} \\
&= (\frac{1}{p} - \frac{1}{\theta})\mu \|\xi\|_{\frac{2}{2-p}} \|u_n\|_2^p \\
&\leq (\frac{1}{p} - \frac{1}{\theta})\mu \gamma_2^p \gamma_0^p \|\xi\|_{\frac{2}{2-p}} \|u_n\|_\lambda^p.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&1 + d + \|u_n\|_\lambda + (\frac{1}{p} - \frac{1}{\theta})\mu \gamma_2^p \gamma_0^p \|\xi\|_{\frac{2}{2-p}} \|u_n\|_\lambda^p \\
&\geq (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \big[\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\big] dx \\
&\geq (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_\lambda^2.
\end{aligned}
$$

This proves that $\{u_n\}$ is bounded in E_λ. Then, passing to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in E_λ. Taking $\omega_n := u_n - u_0$, we have

$$
\begin{aligned}
\|\omega_n\|_2^2 &\leq \frac{1}{\lambda b} \int_{\{x \in \mathbb{R}^N : V(x) > b\}} \lambda V(x) \omega_n^2 dx + \int_{V_b} \omega_n^2 dx \\
&\leq \frac{1}{\lambda b} \|\omega_n\|_\lambda^2 + o(1),
\end{aligned}
\tag{2.23}
$$

since $\omega_n \rightharpoonup 0$ in E_λ and $V(x) < b$ on a set of finite measure. Combining this with (2.1) and the Hölder inequality, we obtain for $2 < \sigma < q < 2_*$

$$
\begin{aligned}
\|\omega_n\|_\sigma^\sigma &\leq \|\omega_n\|_2^{\frac{2(q-\sigma)}{q-2}} \|\omega_n\|_q^{\frac{q(\sigma-2)}{q-2}} \\
&\leq \big(\frac{1}{\lambda b}\big)^{\frac{q-\sigma}{q-2}} \|\omega_n\|_\lambda^{\frac{2(q-\sigma)}{q-2}} \big(\gamma_q \gamma_0 \|\omega_n\|_\lambda\big)^{\frac{q(\sigma-2)}{q-2}} + o(1) \\
&\leq (\gamma_q \gamma_0)^{\frac{q(\sigma-2)}{q-2}} \big(\frac{1}{\lambda b}\big)^{\frac{q-\sigma}{q-2}} \|\omega_n\|_\lambda^\sigma + o(1).
\end{aligned}
\tag{2.24}
$$

For convenience, let $\mathcal{F}(x, u) = \frac{1}{2} f(x, u)u - F(x, u)$. It follows from Lemma 2.4 and (2.21) that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \mathcal{F}(x, \omega_n) dx \\
&= \Phi_\lambda(\omega_n) - \frac{1}{2}\langle \Phi_\lambda'(\omega_n), \omega_n \rangle - \big(\frac{1}{2} - \frac{1}{p}\big)\mu \int_{\mathbb{R}^N} \xi(x)|\omega_n|^p dx \to d - \Phi_\lambda(u_0).
\end{aligned}
\tag{2.25}
$$

Therefore, there exists $M > 0$ such that

$$\Big|\int_{\mathbb{R}^N} \mathcal{F}(x,\omega_n)dx\Big| \le M. \tag{2.26}$$

Now we note that $\frac{q}{q-2} > \max\{1, \frac{N}{4}\}$ because $q \in (2, 2_*)$. Fix $\tau \in \big(\max\{1, \frac{N}{4}\}, \frac{q}{q-2}\big)$, from (2.13), we know if $|u| \ge 1$, then $|f(x,u)| \le c_6|u|^{q-1}$. Choose R_1 so large that $\frac{1}{\theta} \le \frac{1}{2} - \frac{c_6^{\tau-1}}{|u|^{q-(q-2)\tau}}$, whenever $|u| \ge R_1$. Then, for $|u|$ large enough, we have

$$\begin{aligned} 0 \le F(x,u) \le \frac{1}{\theta}uf(x,u) &\le \big[\frac{1}{2} - \frac{c_6^{\tau-1}}{|u|^{q-(q-2)\tau}}\big]uf(x,u) \\ &\le \big[\frac{1}{2} - \frac{|f(x,u)|^{\tau-1}}{|u|^{\tau+1}}\big]uf(x,u), \end{aligned}$$

which implies that, for $|u|$ sufficiently large

$$\frac{|f(x,u)|^\tau}{|u|^\tau} \le \frac{1}{2}uf(x,u) - F(x,u) = \mathcal{F}(x,u). \tag{2.27}$$

Combining this with (2.24), (2.26) with $\sigma = \frac{2\tau}{\tau-1} \in (2, 2_*)$ and the Hölder inequality, we obtain for large n,

$$\begin{aligned} &\int_{|\omega_n|\ge R_1} f(x,\omega_n)\omega_n dx \\ &\le \Big(\int_{|\omega_n|\ge R_1} \Big|\frac{f(x,\omega_n)}{\omega_n}\Big|^\tau dx\Big)^{1\tau}\Big(\int_{|\omega_n|\ge R_1} |\omega_n|^\sigma dx\Big)^{2/\sigma} \\ &\le \Big(\int_{|\omega_n|\ge R_1} \mathcal{F}(x,\omega_n)dx\Big)^{1\tau}\|\omega_n\|_\sigma^2 \\ &\le M^{1\tau}(\gamma_q\gamma_0)^{\frac{2q(\sigma-2)}{(q-2)\sigma}}\big(\frac{1}{\lambda b}\big)^{\frac{2(q-\sigma)}{(q-2)\sigma}}\|\omega_n\|_\lambda^2 + o(1) \\ &= c_7\big(\frac{1}{\lambda b}\big)^{\theta_1}\|\omega_n\|_\lambda^2 + o(1). \end{aligned} \tag{2.28}$$

where $c_7 = M^{1\tau}(\gamma_q\gamma_0)^{\frac{2q(\sigma-2)}{(q-2)\sigma}} > 0$, $\theta_1 = \frac{2(q-s)}{s(q-2)} > 0$. In addition, using (2.13) and (2.24), we have

$$\begin{aligned} \int_{|\omega_n|\le R_1} f(x,\omega_n)\omega_n dx &\le \int_{|\omega_n|\le R_1} \Big(\epsilon + C_\epsilon R_1^{q-2}\Big)\omega_n^2 dx \\ &\le \frac{C_\epsilon R_1^{q-2}}{\lambda b}\|\omega_n\|_\lambda^2 + o(1) \\ &= \frac{c_8}{\lambda b}\|\omega_n\|_\lambda^2 + o(1), \end{aligned} \tag{2.29}$$

where $c_8 = C_\epsilon R_1^{q-2}$. Consequently, combining (2.21), (2.28) with (2.29), we obtain

$$\begin{aligned} o(1) &= \langle \Phi'_\lambda(\omega_n), \omega_n\rangle \\ &= \|\omega_n\|_\lambda^2 - \int_{\mathbb{R}^N} f(x,\omega_n)\omega_n dx - \mu\int_{\mathbb{R}^N} \xi(x)|\omega_n|^p dx \\ &\ge \big[1 - \frac{c_8}{\lambda b} - c_7\big(\frac{1}{\lambda b}\big)^{\theta_1}\big]\|\omega_n\|_\lambda^2 + o(1). \end{aligned}$$

Choosing $\Lambda_0 > 0$ large enough such that the term in the brackets above is positive when $\lambda > \Lambda_0$, we obtain $\omega_n \to 0$ in E_λ, thus $u_n \to u_0$ in E_λ. This completes the proof. □

Define

$$d_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{0\le t\le 1} \Phi_\lambda(\gamma(t))$$

where $\Gamma_\lambda = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e\}$.

Proof of Theorem 1.1. By Theorem 2.1, and Lemmas 2.2 and 2.3, we obtain that, for each $\lambda \ge \Lambda_0$, $0 < \mu < \mu_0$, there exists (PS) sequence $\{u_n\} \subset E_\lambda$ for Φ_λ on E_λ. Then, by Lemma 2.5, we can conclude that there exist a subsequence $\{u_n\} \subset E_\lambda$ and $u^1_\lambda \in E_\lambda$ such that $u_n \to u^1_\lambda$ in E_λ. Moreover, $\Phi_\lambda(u^1_\lambda) = d_\lambda \ge \eta > 0$.

The second solution of problem (1.1) will be constructed through the local minimization. Since $\xi \in L^{\frac{2}{2-p}}(\mathbb{R}^N, \mathbb{R}^+)$, we can choose a function $\phi \in E_\lambda$ such that

$$\int_{\mathbb{R}^N} \xi(x)|\phi|^p dx > 0.$$

Thus, by (A8) we have

$$\begin{aligned} \Phi_\lambda(l\phi) &= \frac{l^2}{2}\|\phi\|^2_\lambda - \int_{\mathbb{R}^N} F(x, l\phi)dx - \frac{\mu l^p}{p}\int_{\mathbb{R}^N} \xi(x)|\phi|^p dx \\ &\le \frac{l^2}{2}\|\phi\|^2_\lambda - \frac{\mu l^p}{p}\int_{\mathbb{R}^N} \xi(x)|\phi|^p dx < 0, \end{aligned} \tag{2.30}$$

for $l > 0$ small enough. Hence, there exists $\rho_1 > 0$ such that $\beta := \inf\{\Phi_\lambda(u) : u \in \bar{B}_{\rho_1}\} < 0$. By the Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \bar{B}_{\rho_1}$ such that $\Phi_\lambda(u_n) \to \beta$ and $\Phi'_\lambda(u_n) \to 0$ as $n \to \infty$. Hence, Lemma 2.5 implies that there exists a nontrivial solution u^2_λ of problem (1.1) satisfying

$$\Phi_\lambda(u^2_\lambda) < 0 \quad \text{and} \quad \|u^2_\lambda\|_\lambda < \rho_1.$$

Moreover, (2.30) implies that there exists $l_0 > 0$ and $\kappa < 0$ are independent of λ such that $\Phi_\lambda(l_0\phi) = \kappa$ and $\|l_0\phi\|_\lambda < \rho_1$. Therefore, we can conclude that

$$\Phi_\lambda(u^2_\lambda) \le \kappa < 0 < \eta < d_\lambda = \Phi_\lambda(u^1_\lambda) \quad \text{for all } \lambda > \Lambda_0 \text{ and } 0 < \mu < \mu_0.$$

This completes the proof. □

3. Concentration of solutions

Here we study the concentration of solutions and give the proof of Theorem 1.2. Define

$$d_0 = \inf_{\gamma\in\widetilde{\Gamma}_\lambda} \max_{0\le t\le 1} \Phi_\lambda|_{H^2(\Omega)\cap H^1_0(\Omega)}(\gamma(t))$$

where

$$\widetilde{\Gamma}_\lambda = \{\gamma \in C([0,1], H^2(\Omega) \cap H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = e\},$$

and $\Phi_\lambda|_{H^2(\Omega)\cap H^1_0(\Omega)}$ is a restriction of Φ_λ on $H^2(\Omega) \cap H^1_0(\Omega)$. Note that

$$\Phi_\lambda|_{H^2(\Omega)\cap H^1_0(\Omega)}(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 + |\nabla u|^2)dx - \int_\Omega F(x,u)dx - \mu\int_\Omega \xi(x)|u|^p dx$$

and d_0 independent of λ. From the above arguments, we conclude that functional $\Phi_\lambda|_{H^2(\Omega)\cap H^1_0(\Omega)}$ has a mountain pass type solution $\tilde{u}$ such that $\Phi_\lambda|_{H^2(\Omega)\cap H^1_0(\Omega)}(\tilde{u}) =$

d_0. Since $(H^2(\Omega)\cap H_0^1(\Omega))\subset E_\lambda$ for all $\lambda>0$, it is easy to see that $0<\eta<d_\lambda<d_0$ for all $\lambda\geq\Lambda_0$ and $0<\mu<\mu_0$. Take $C_0>d_0$, thus

$$0<\eta<d_\lambda<d_0<C_0,\quad \text{for all } \lambda\geq\Lambda_0 \text{ and } 0<\mu<\mu_0.$$

Proof of Theorem 1.2. We follow the arguments in [7]. For any sequence $\lambda_n\to\infty$, let $u_n^i:=u_{\lambda_n}^i$ be the critical points of Φ_{λ_n} obtained in Theorem 1.1 for $i=1,2$. Since

$$\Phi_{\lambda_n}(u_n^2)\leq\kappa<0<\eta<d_{\lambda_n}=\Phi_{\lambda_n}(u_n^1) \tag{3.1}$$

and

$$\begin{aligned}
&\Phi_{\lambda_n}(u_n^i)-\frac{1}{\theta}\langle\Phi'_{\lambda_n}(u_n^i),u_n^i\rangle\\
&=\big(\frac12-\frac1\theta\big)\|u_n^i\|_{\lambda_n}^2+\int_{\mathbb{R}^N}\big(\frac1\theta f(x,u_n^i)u_n^i-F(x,u_n^i)\big)dx\\
&\quad-\big(\frac\mu p-\frac\mu\theta\big)\int_{\mathbb{R}^N}\xi(x)|u_n^i|^pdx\\
&=\big(\frac12-\frac1\theta\big)\|u_n^i\|_{\lambda_n}^2-\big(\frac\mu p-\frac\mu\theta\big)\int_{\mathbb{R}^N}\xi(x)|u_n^i|^pdx,
\end{aligned}$$

it follows that

$$\|u_n^i\|_{\lambda_n}\leq c_0, \tag{3.2}$$

where the constant c_0 is independent of λ_n. Therefore, we assume that $u_n^i\rightharpoonup u_0^i$ in E_{λ_n} and $u_n^i\to u_0^i$ in $L^q_{\rm loc}(\mathbb{R}^N)$ for $2\leq q<2_*$. From Fatou's lemma, we have

$$\int_{\mathbb{R}^N}V(x)|u_0^i|^2dx\leq\liminf_{n\to\infty}\int_{\mathbb{R}^N}V(x)|u_n^i|^2dx\leq\liminf_{n\to\infty}\frac{\|u_n^i\|_{\lambda_n}^2}{\lambda_n}=0,$$

which implies that $u_0^i=0$ a.e. in $\mathbb{R}^N\setminus V^{-1}(0)$ and $u_0^i\in H^2(\Omega)\cap H_0^1(\Omega)$ by (A5). Now for any $\varphi\in C_0^\infty(\Omega)$, since $\langle\Phi'_{\lambda_n}(u_n^i),\varphi\rangle=0$, it is easy to verify that

$$\int_\Omega\big(\Delta u_0^i\Delta\varphi+\nabla u_0^i\cdot\nabla\varphi\big)\,dx-\int_\Omega f(x,u_0^i)\varphi dx-\mu\int_{\mathbb{R}^N}\xi(x)|u_0^i|^{p-2}u_0^i\varphi dx=0,$$

which implies that u_0^i is a weak solution of problem (1.3) by the density of $C_0^\infty(\Omega)$ in $H^2(\Omega)\cap H_0^1(\Omega)$.

Now we prove that $u_n^i\to u_0^i$ in $L^q(\mathbb{R}^N)$ for $2\leq q<2_*$. Otherwise, by Lions vanishing lemma [12, 19], there exist $\delta>0,R_0>0$ and $x_n\in\mathbb{R}^N$ such that

$$\int_{B_{R_0}(x_n)}|u_n^{(i)}-u_0^i|^2dx\geq\delta.$$

Since $u_n^i\to u_0^i$ in $L^2_{\rm loc}(\mathbb{R}^N)$, $|x_n|\to\infty$. Hence $\operatorname{meas}(B_{R_0}(x_n)\cap V_b)\to0$. By Hölder's inequality, we have

$$\begin{aligned}
&\int_{B_{R_0}(x_n)\cap V_b}|u_n^i-u_0^i|^2dx\\
&\leq(\operatorname{meas}(B_{R_0}(x_n)\cap V_b))^{\frac{2_*-2}{2_*}}\Big(\int_{\mathbb{R}^N}|u_n^i-u_0^i|^{2_*}\Big)^{2/2_*}\to0.
\end{aligned}$$

Consequently,

$$\|u_n^i\|_{\lambda_n}^2\geq\lambda_nb\int_{B_{R_0}(x_n)\cap\{x\in\mathbb{R}^N:V(x)\geq b\}}|u_n^i|^2dx$$

$$
\begin{aligned}
&= \lambda_n b \int_{B_{R_0}(x_n)\cap\{x\in\mathbb{R}^N: V(x)\geq b\}} |u_n^i - u_0^i|^2 dx \\
&= \lambda_n b\Big(\int_{B_{R_0}(x_n)} |u_n^i - u_0^i|^2 dx - \int_{B_{R_0}(x_n)\cap V_b} |u_n^i - u_0^i|^2 dx + o(1)\Big) \\
&\to \infty,
\end{aligned}
$$

which contradicts (3.2).

Next, we show that $u_n^i \to u_0^i$ in $H^2(\mathbb{R}^N)$. From $\langle \Phi'_{\lambda_n}(u_n^i), u_n^i\rangle = \langle \Phi'_{\lambda_n}(u_n^i), u_0^i\rangle = 0$ and the fact that $u_n^i \to u_0^i$ in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2_*$, we have

$$
\lim_{n\to\infty} \|u_n^i\|_{\lambda_n}^2 = \lim_{n\to\infty} (u_n^i, u_0^i)_{\lambda_n} = \lim_{n\to\infty} (u_n^i, u_0^i) = \|u_0^i\|^2,
$$

therefore

$$
\limsup_{n\to\infty} \|u_n^i\|^2 \leq \|u_0^i\|^2.
$$

On the other hand, the weak lower semi-continuity of norm yields

$$
\|u_0^i\|^2 \leq \liminf_{n\to\infty} \|u_n^i\|^2 \leq \limsup_{n\to\infty} \|u_n^i\|^2 \leq \lim_{n\to\infty} \|u_n^i\|_{\lambda_n}^2,
$$

thus, $u_n^i \to u_0^i$ in E_λ, and so

$$
u_n^i \to u_0^i \quad \text{in } H^2(\mathbb{R}^N).
$$

Using (3.1) and the constants κ, η are independent of λ_n, we have

$$
\frac{1}{2}\int_\Omega \big(|\Delta u_0^1|^2 + |\nabla u_0^1|^2\big)\, dx - \int_\Omega F(x, u_0^1) dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|u_0^1|^p dx \geq \eta > 0
$$

and

$$
\frac{1}{2}\int_\Omega \big(|\Delta u_0^2|^2 + |\nabla u_0^2|^2\big)\, dx - \int_\Omega F(x, u_0^2) dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)|u_0^2|^p dx \leq \kappa < 0,
$$

which implies that $u_0^i \neq 0$ and $u_0^1 \neq u_0^2$. This completes the proof. □

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