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INDIRECT METHOD OF EXPONENTIAL CONVERGENCE ESTIMATION FOR NEURAL NETWORK WITH DISCRETE AND DISTRIBUTED DELAYS

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Abstract. The purpose of this research is to develop method of calculation of exponential decay rate for neural network model based on differential equations with discrete and distributed delays. The method results in quasipolynomial inequality allowing us to investigate qualitative behavior of model in dependence on parameters. In such way it was shown direct dependency in changes of exponential decay rate and minimal threshold of distributed time delay. An example of two-neuron network with four delays is given and numerical simulations are performed to illustrate the obtained results. It was shown numerically that distributed delays combined with discrete delays narrow the interval of parameters admitting exponential convergence.

1. Introduction

This work concerns the neural network modeling and stability investigation with help of differential equations with delays. Differential equations are found to be of central importance in many disciplines such as control theory, neural networks, epidemiology, etc. [4]. In analyzing the behavior of real populations, delay differential equations are regarded as effective tools.

Recently there were obtained a series of results that consider discrete delays in neural network models [5, 7, 14, 15, 16].

When considering results of exponential estimation of neural networks dealing with distributed delays we should mention the following works.

Most of papers are concerned with application of Lyapunov-Krasovskii functionals resulting in construction of corresponding liner matrix inequalities (LMIs). So, in [3] by employing a Lyapunov-Krasovskii functional, the LMI approach is exploited to establish sufficient conditions for the neural networks to be globally exponentially stable, which are offered to be solved by using the Matlab LMI toolbox.

In [1] they study the delay-dependent exponential stability for uncertain neural networks with discrete and distributed time-varying delays. By decomposing the

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delay interval into multiple equidistant subintervals and multiple nonuniform subintervals, a suitable augmented Lyapunov-Krasovskii functionals are constructed on these intervals. A set of sufficient conditions leading to LMIs are obtained.

In spite of its universal character the approaches based on LMIs do not offer clear answer in theoretical reasoning if we would like to get clear evidences for dependencies of decay rates and model parameters.

However there were attempts to develop Lyapunov-Krasovskii functional approach allowing to get conditions different from LMIs. So, in [9] by constructing several Lyapunov functionals, some sufficient criteria for the existence of a unique equilibrium and global exponential stability of the network are derived. These results are fairly general and can be easily verified because of usage of easily verified inequalities (not LMIs).

Fewer results were obtained for neural network models with distributed delays without application of Lyapunov-Krasovskii functionals approach

In [17] they concern the exponential convergence of bidirectional associative memory (BAM) neural networks with unbounded distributed delays. Sufficient conditions are derived by exploiting the exponentially fading memory property of delay kernel functions. The method is based on comparison principle of delay differential equations and does not need the construction of any Lyapunov functions also.

In [2] for a family of non-autonomous differential equations with continuously distributed delays there were given sufficient conditions for the global exponential stability including integral inequality of quazipolynomial type to search exponential rate in the form of continuous functions. The approach that was offered doesn't include Lyapunov-Krasovskii functional and is sort of indirect one. But in spite of this approach generality a solution of inequality mentioned above is not a trivial problem.

That's why the purpose of this work is to offer a method of obtaining estimates for exponential decays for neural networks with discrete and distributed delays resulting in solution of scalar nonlinear inequality. Such general approach was stated in [11] and applied in case of discrete delays. The method comes from the work [12] where it was applied for compartmental systems.

In Section 2 we describe model of neural network with discrete and distributed delays studied in the paper. In Section 3 we present method of exponential estimate construction and demonstrate its application when analysing dependence of exponential decay rate and time delay. In Section 4 we apply Theorem 3.1 for two-neuron model with four delays. In this paper we use the following notation:

- the norm of a vector-function $|\phi(\bullet)|^\tau = \sup_{\theta\in[-\tau,0], i=\overline{1,n}} |\phi_i(\theta)|$, where functions $\phi \in \mathbb{C}^1[-\tau,0]$ are continuously differentiable on $[-\tau,0]$;
- an arbitrary matrix norm $\|M\|$ for matrix $M \in \mathbb{R}^{n\times n}$;
- Euclidean norm $\|x\|$ for vector $x \in \mathbb{R}^n$.

2. Problem Statement

We consider neural network described by system with mixed delays

$$\dot{x}(t) = -Ax(t) + \sum_{m=1}^{r} W_{1,m} g(x(t-\tau_m(t))) + \sum_{m=1}^{r} W_{2,m} \int_{t-\tau_m(t)}^{t-h_m(t)} g(x(\theta))d\theta \quad (2.1)$$

$x(t) \in \mathbb{R}^n$ is the state vector. $A = diag(a_1, a_2, \dots, a_n)$ is a diagonal matrix with positive entries $a_i > 0$, $W_{1,m} = (w_{ij}^{1,m})_{n\times n}$, $W_{2,m} = (w_{ij}^{2,m})_{n\times n}$ $m = \overline{1,r}$ are the connection weight matrices, $g(x(t)) = [g_1(x(t)), g_2(x(t)), \dots, g_n(x(t))]^\top \in \mathbb{R}^n$ denotes the neuron activation functions which are bounded monotonically nondecreasing with $g_j(0) = 0$ and satisfy the condition

$$0 \le \frac{g_j(\xi_1) - g_j(\xi_2)}{\xi_1 - \xi_2} \le l_j \tag{2.2}$$

$\xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \ne \xi_2$, $j = 1, 2, \dots, n$. In (2.1) the symbol $\int g(x(\theta))d\theta$ means

$$[\int g_1(x(\theta))d\theta, \int g_2(x(\theta))d\theta, \dots, \int g_n(x(\theta))d\theta]^\top \in \mathbb{R}^n.$$

According to the customary, in the system (2.1) we call the second term with discrete time-varying delays and the third term with distributed time-varying delays. The bounded functions $\tau_m(t)$ represent mixed delays of system with $0 \le \tau_m(t) \le \tau_M$, $\dot{\tau_m}(t) \le \tau_D < 1$, $m = \overline{1,r}$. The bounded functions $h_m(t)$ represent minimal threshold for distributed delays of system with $h_{\min} \le h_m(t) \le \tau_m(t)$, $m = \overline{1,r}$, $t > 0$. Delays $h_m(t)$ and $\tau_m(t)$ have physical meaning as "controllable memory" of the network if neurons effects on network output only during some time interval. Here we consider the case if we have discrete delays as "maximal" thresholds for distributed delays. Indeed reasonings of this work can be extended to the case if we have entirely other "maximal" thresholds.

The initial conditions associated with system (2.1) are of the form

$$x_i(s) = \phi_i(s), \quad s \in [-\tau_M, 0], \tag{2.3}$$

where $\phi_i(s)$ is a continuous real-valued function for $s \in [-\tau_M, 0]$. Then, the solution of system (2.1) exists for all $t \ge 0$ and is unique [4] under assumption (2.2).

3. Main Result

Theorem 3.1. *Let system* (2.1) *be such that*

- *matrix A satisfies the inequality $\|e^{-At}\| \le ke^{-\alpha t}$ for $t \ge 0$ and some $k \ge 1$, $\alpha > 0$; Note that in case of diagonal matrix A with positive entries α can be chosen as $\alpha := \min_{1\le i\le n}\{a_i\}$;*
- *there exists a solution $\lambda > 0$ of the quasipolynomial inequality*

$$\frac{e^{-\lambda\tau_M}}{k}(\alpha - \lambda) \ge \sup_{t\ge 0}\Big(\sum_{m=0}^{r}\big(\|W_{1,m}\| + \|W_{2,m}\|(\tau_m(t) - h_m(t))\big)l_m\Big). \tag{3.1}$$

Then the estimate $\|x(t)\| \le k|\phi(\theta)|^{\tau_M}e^{-\lambda t}$ holds for the solution of system (2.1) *for any $t \ge 0$, where $\lambda > 0$ is a number satisfying inequality* (3.1).

Note that assumption (3.1) for positive λ implies $\lambda < \alpha$ obviously.

Proof of Theorem 3.1. For the solution $x(t)$ of the system (2.1) by the Cauchy formula the equality holds

$$\begin{aligned} x(t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-s)}\Big(\sum_{m=1}^{r} W_{1,m}g(x(s-\tau_m(s))) \\ + \sum_{m=1}^{r} W_{2,m}\int_{s-\tau_m(s)}^{s-h_m(s)} g(x(\theta))d\theta\Big)ds \end{aligned} \tag{3.2}$$

Denote

$$
\begin{aligned}
y(t) &= \dot{x}(t) + Ax(t) \\
&= \sum_{m=1}^{r} W_{1,m} g(x(t-\tau_m(t))) + \sum_{m=1}^{r} W_{2,m} \int_{t-\tau_m(t)}^{t-h_m(t)} g(x(\theta))d\theta
\end{aligned} \tag{3.3}
$$

Then

$$
\begin{aligned}
\|x(t)\| &\le k\|\phi(0)\|e^{-\alpha t} + \int_0^t k e^{-\alpha(t-s)}\|y(s)\|ds \\
&\le k|\phi(\theta)|^{\tau_M} e^{-\alpha t} + \int_0^t k e^{-\alpha(t-s)}\|y(s)\|ds
\end{aligned} \tag{3.4}
$$

It is necessary to estimate $\|x(t)\|$, i.e., to find $\lambda > 0$ such that

$$
\|x(t)\| \le k|\phi(\theta)|^{\tau_M} e^{-\lambda t}\,. \tag{3.5}
$$

Denote

$$
X(t) = k|\phi(\theta)|^{\tau_M} e^{-\lambda t}
$$

and let $Y(t)$ be an unknown function such that

$$
\|y(t)\| \le Y(t)
$$

for all $[-\tau_M, \infty)$. Select function $Y(t)$ so that

$$
X(t) = k|\phi(\theta)|^{\tau_M} e^{-\alpha t} + \int_0^t k e^{-\alpha(t-s)} Y(s)ds. \tag{3.6}
$$

Equality (3.6) does not guarantee that the equality $\|y(t)\| \le Y(t)$ holds if $\|x(t)\| \le X(t)$.

Let us show that the function $Y(s) = |\phi(\theta)|^{\tau_M}(\alpha-\lambda)e^{-\lambda s}$ is a solution of (3.6). Indeed, we have

$$
\begin{aligned}
& k|\phi(\theta)|^{\tau_M} e^{-\lambda t} \\
&= k|\phi(\theta)|^{\tau_M} e^{-\alpha t} + \int_0^t k e^{-\alpha(t-s)} |\phi(\theta)|^{\tau_M}(\alpha-\lambda)e^{-\lambda s}ds \\
&= k|\phi(\theta)|^{\tau_M} e^{-\alpha t} + k|\phi(\theta)|^{\tau_M}(\alpha-\lambda)e^{-\alpha t}\int_0^t e^{(\alpha-\lambda s)s}ds \\
&= k|\phi(\theta)|^{\tau_M} e^{-\alpha t} + k|\phi(\theta)|^{\tau_M}\frac{(\alpha-\lambda)e^{-\lambda t}}{\alpha-\lambda} - k|\phi(\theta)|^{\tau_M}\frac{(\alpha-\lambda)e^{-\alpha t}}{\alpha-\lambda} \\
&= k|\phi(\theta)|^{\tau_M} e^{-\lambda t} =: X(t)
\end{aligned}
$$

for all $t \in [0, \infty)$.

Further, it is necessary to find $\lambda > 0$ such that $\|x(t)\| \le X(t)$, $\|y(t)\| \le Y(t)$, $t \in [-\tau_M, \infty)$.

Let us first consider an interval $t \in [-\tau_M, 0]$. The relation $\|x(t)\| = \|\phi(t)\| \le k|\phi(\theta)|^{\tau_M} e^{-\lambda t} = X(t)$ holds if $k \ge 1$ (since $e^{\lambda t} \ge 1$ for $t \in [-\tau_M, 0]$ for all $\lambda > 0$). On this interval, let us derive a similar inequality for $\|y(t)\|$. Since

$$
y(t) = \sum_{m=1}^{r} W_{1,m} g(x(t-\tau_m(t))) + \sum_{m=1}^{r} W_{2,m} \int_{t-\tau_m(t)}^{t-h_m(t)} g(x(\theta))d\theta,
$$

we should have the value of $x(t)$ on the interval $[-2\tau_M, -\tau_M]$.

For the sake of determinacy, let $x(t) = \phi(-\tau_M)$ for any $t \in [-2\tau_M, -\tau_M]$. Then, taking into account that $g_j(\bullet)$, $j = \overline{1,n}$ are nondecreasing and denoting

$$(g_1(|\phi(\theta)|^{\tau_M}), g_2(|\phi(\theta)|^{\tau_M}), \dots, g_n(|\phi(\theta)|^{\tau_M}))^\top =: g(|\phi(\theta)|^{\tau_M})$$

we obtain

$$\begin{aligned}
\|y(t)\| =& \| \sum_{m=1}^{r} W_{1,m} g(x(t-\tau_m(t))) + \sum_{m=1}^{r} W_{2,m} \int_{t-\tau_m(t)}^{t-h_m(t)} g(x(\theta))d\theta \| \\
&\le \sum_{m=1}^{r} \|W_{1,m} g(x(t-\tau_m(t)))\| + \sum_{m=1}^{r} \|W_{2,m} \int_{t-\tau_m(t)}^{t-h_m(t)} g(x(\theta))d\theta\| \\
&\le \sum_{m=1}^{r} \|W_{1,m}\| \|g(|\phi(\bullet)|^{\tau_M})\| + \sum_{m=1}^{r} \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(|\phi(\bullet)|^{\tau_M})\| d\theta \\
&\le \sum_{m=1}^{r} \|W_{1,m}\| \|g(|\phi(\bullet)|^{\tau_M})\| + \sum_{m=1}^{r} \|W_{2,m}\| (\tau_M - h_{\min}) \|g(|\phi(\bullet)|^{\tau_M})\| \\
&\le \sum_{m=1}^{r} \|W_{1,m}\| \|g(|\phi(\bullet)|^{\tau_M})\| + \sum_{m=1}^{r} \|W_{2,m}\| (\tau_M - h_{\min}) \|g(|\phi(\bullet)|^{\tau_M})\| \\
&= \sum_{m=1}^{r} (\|W_{1,m}\| + \|W_{2,m}\| (\tau_M - h_{\min})) \|g(|\phi(\bullet)|^{\tau_M})\| .
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{m=1}^{r} (\|W_{1,m}\| + \|W_{2,m}\| (\tau_M - h_{\min})) \|g(|\phi(\bullet)|^{\tau_M})\| \\
&\le \sum_{m=1}^{r} (\|W_{1,m}\| + \|W_{2,m}\| (\tau_M - h_{\min})) \|g(|\phi(\bullet)|^{\tau_M})\| e^{-\lambda t} .
\end{aligned}$$

The above inequality holds for $t \in [-\tau_M, 0]$ and for all $\lambda > 0$. Therefore, to derive the inequality $\|y(t)\| \le Y(t)$, it is necessary to choose $\lambda > 0$ such that

$$\sum_{m=1}^{r} (\|W_{1,m}\| + \|W_{2,m}\| (\tau_M - h_{\min})) \|g(|\phi(\bullet)|^{\tau_M})\| \le (\alpha - \lambda) |\phi(\theta)|^{\tau_M} \tag{3.7}$$

Then

$$\begin{aligned}
\|y(t)\| \le& \sum_{m=1}^{r} (\|W_{1,m}\| + \|W_{2,m}\| (\tau_M - h_{\min})) \|g(|\phi(\bullet)|^{\tau_M})\| e^{-\lambda t} \\
\le& (\alpha - \lambda) |\phi(\theta)|^{\tau_M} e^{-\lambda t} = Y(t).
\end{aligned}$$

For the further reasoning, let us introduce the notation

$$\rho_1(t) = \|x(t)\| - X(t), \quad \rho_2(t) = \|y(t)\| - Y(t), \quad t \in [0, \infty).$$

It was shown that on the interval $t \in [-\tau_M, 0]$ we have $\rho_1(t) \le 0$ and $\rho_2(t) \le 0$. Let us now find $\lambda > 0$ such that $\|x(t)\| \le X(t)$ or $\rho_1(t) \le 0$ for $t \ge 0$. Let us estimate

$\rho_1(t)$ by subtracting (3.6) from (3.4),

$$\begin{aligned}\rho_1(t) \le & k|\phi(\theta)|^{\tau_M} e^{-\alpha t} + \int_0^t k e^{-\alpha(t-s)} \|y(s)\| ds \\ & - k|\phi(\theta)|^{\tau_M} e^{-\alpha t} - \int_0^t k e^{-\alpha(t-s)} Y(s) ds \\ = & k \int_0^t k e^{-\alpha(t-s)} (\|y(s)\| - Y(s)) ds = k \int_0^t e^{-\alpha(t-s)} \rho_2(s) ds\end{aligned} \tag{3.8}$$

Considering (3.8), we can estimate $\rho_2(s)$:

$$\begin{aligned}\rho_2(t) = & \|y(t)\| - Y(t) \\ = & \|\sum_{m=1}^r W_{1,m} g(x(t-\tau_m(t))) + \sum_{m=1}^r W_{2,m} \int_{t-\tau_m(t)}^{t-h_m(t)} g(x(\theta)) d\theta \| - Y(t) \\ \le & \sum_{m=1}^r \|W_{1,m}\| \|g(x(t-\tau_m(t)))\| + \sum_{m=1}^r \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - Y(t)\end{aligned}$$

Some identical transformations yield

$$\begin{aligned}Y(t) = & |\phi(\theta)|^{\tau_M} (\alpha - \lambda) e^{-\lambda t} = \frac{e^{-\lambda \tau_M}}{k} k e^{\lambda \tau_M} |\phi(\theta)|^{\tau_M} (\alpha - \lambda) e^{-\lambda t} \\ = & \frac{e^{-\lambda \tau_M}}{k} k |\phi(\theta)|^{\tau_M} e^{-\lambda(t-\tau_M)} (\alpha - \lambda) = \frac{e^{-\lambda \tau_M}}{k} (\alpha - \lambda) X(t - \tau_M).\end{aligned}$$

Then

$$\begin{aligned}& \sum_{m=1}^r \|W_{1,m}\| \|g(x(t-\tau_m(t)))\| + \sum_{m=1}^r \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - Y(t) \\ & = \sum_{m=1}^r \|W_{1,m}\| \|g(x(t-\tau_m(t)))\| \\ & + \sum_{m=1}^r \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - \frac{e^{-\lambda \tau_M}}{k} (\alpha - \lambda) X(t - \tau_M)\end{aligned}$$

Since $\sum_{m=1}^r \|W_{1,m}\| \|g(x(t-\tau_m(t)))\| \ge 0$, $\sum_{m=1}^r \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta \ge 0$ and $\frac{e^{-\lambda \tau_M}}{k} (\alpha - \lambda) X(t - \tau_M) \ge 0$ (assuming (3.1)), their difference only increases if we assume that $\lambda > 0$ satisfies (3.1). We obtain

$$\begin{aligned}& \sum_{m=1}^r \|W_{1,m}\| \|g(x(t-\tau_m(t)))\| \\ & + \sum_{m=1}^r \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - \frac{e^{-\lambda \tau_M}}{k} (\alpha - \lambda) X(t - \tau_M) \\ & \le \sum_{m=1}^r \|W_{1,m}\| l_m \|x(t-\tau_m(t))\| - \Big(\sum_{m=1}^r \|W_{1,m}\| l_m \Big) X(t - \tau_M) \\ & \quad + \sum_{m=1}^r \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - \Big(\sum_{m=1}^r \|W_{2,m}\| l_m \Big) \int_{t-\tau_m(t)}^{t-h_m(t)} X(t - \tau_M) d\theta.\end{aligned}$$

Since $X(t)$ is monotonically decreasing,

$$X(t-\tau_M) \geq X(t-\tau_m(t)), \quad m=\overline{1,r}.$$

Therefore, taking into account (2.2),

$$\begin{aligned}
&\sum_{m=1}^{r} \|W_{1,m}\| l_m \|x(t-\tau_m(t))\| - \Big(\sum_{m=1}^{r} \|W_{1,m}\| l_m\Big) X(t-\tau_M)\\
&+\sum_{m=1}^{r} \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - \Big(\sum_{m=1}^{r} \|W_{2,m}\| l_m\Big) \int_{t-\tau_m(t)}^{t-h_m(t)} X(t-\tau_M) d\theta\\
&\leq \sum_{m=1}^{r} \|W_{1,m}\| l_m \|x(t-\tau_m(t))\| - \sum_{m=1}^{r} \|W_{1,m}\| l_m X(t-\tau_m(t))\\
&\quad +\sum_{m=1}^{r} \|W_{2,m}\| \int_{t-\tau_m(t)}^{t-h_m(t)} \|g(x(\theta))\| d\theta - \sum_{m=1}^{r} \|W_{2,m}\| l_m \int_{t-\tau_m(t)}^{t-h_m(t)} X(t-\tau_m(t)) d\theta\\
&= \sum_{m=1}^{r} \|W_{1,m}\| l_m \rho_1(t-\tau_m(t)) + \sum_{m=1}^{r} \|W_{2,m}\| l_m \int_{t-\tau_m(t)}^{t-h_m(t)} \rho_1(\theta) d\theta,
\end{aligned}$$

i.e., we have

$$\begin{aligned}
\rho_2(t) \leq& \sum_{m=1}^{r} \|W_{1,m}\| l_m \rho_1(t-\tau_m(t))\\
&+\sum_{m=1}^{r} \|W_{2,m}\| l_m \int_{t-\tau_m(t)}^{t-h_m(t)} \rho_1(\theta) d\theta, \quad t \geq 0.
\end{aligned} \tag{3.9}$$

Since the integral is monotonic, substituting estimate (3.9) into (3.8) yields

$$\begin{aligned}
\rho_1(t) &\leq k \int_0^t e^{-\alpha(t-s)} \rho_2(s) ds\\
&\leq k \int_0^t e^{-\alpha(t-s)} \Big(\sum_{m=1}^{r} \|W_{1,m}\| l_m \rho_1(s-\tau_m(s))\\
&\quad +\sum_{m=1}^{r} \|W_{2,m}\| l_m \int_{s-\tau_m(s)}^{s-h_m(s)} \rho_1(\theta) d\theta \Big) ds,
\end{aligned} \tag{3.10}$$

Consider inequality (3.10) on the interval $t \in [0, h_{\min}]$. Since $\rho_1 \leq 0$ for $t \in [-\tau_M, 0]$, we obtain based on (3.10) that $\rho_1(t) \leq 0$ for all $t \in [0, h_{\min}]$.

Let us consider $t \in [h_{\min}, 2h_{\min}]$. Since $\rho_1(t) \leq 0$ for all $t \in [0, h_{\min}]$, from (3.10) $\rho_1(t) \leq 0$ for all $t \in [h_{\min}, 2h_{\min}]$. Whence we may conclude that $\rho_1 \leq 0$, $t \in [0, \infty)$. This completes the proof. □

Remark 3.2. Theorem 3.1 can be proved even for the case if we have functions different from $\tau_m(t)$ describing distributed delays in model (2.1).

Corollary 3.3. *In practice instead of* (3.1) *we may use "rougher" quasipolynomial inequality*

$$\frac{e^{-\lambda\tau_M}}{k}(\alpha-\lambda) \geq \sum_{m=0}^{r} \left(\|W_{1,m}\| + \|W_{2,m}\|(\tau_M - h_{\min})\right) l_m. \tag{3.11}$$

Remark 3.4. The positive solution λ of quasipolynomial inequalities (3.1) or (3.11) exists only if $\alpha > \lambda$.

Theorem 3.1 gives us a clear estimate for lower memory threshold allowing exponential convergence due to (3.11). Analysing inequality (3.11) we can see general relations between estimates of model characteristics.

Corollary 3.5. *The value of $h_{\min}$ admitting local exponential stability with decay rate because* (3.11) *can be estimated from inequality*

$$\begin{aligned} h_{\min} \geq & \Big(\sum_{m=0}^{r} \|W_{2,n}\| l_m \Big)^{-1} \\ & \times \Big(\sum_{m=0}^{r} (\|W_{1,m}\| + \|W_{2,m}\| \tau_M) l_m - \frac{e^{-\lambda \tau_M}}{k} (\alpha - \lambda) \Big) \end{aligned} \tag{3.12}$$

The above corollary follows directly from (3.11).

Corollary 3.6. *Under the assumption of Theorem 3.1 there exists direct dependency between $h_{\min}$ and λ. That is, when increasing in model* (2.1) *the value of $h_{\min}$ we increase the estimate of exponential decay rate λ and vice versa.*

Proof. The corollary follows immediately when considering dependency

$$\begin{aligned} h_{\min}(\lambda) := & \Big(\sum_{m=0}^{r} \|W_{2,n}\| l_m \Big)^{-1} \\ & \times \Big(\sum_{m=0}^{r} (\|W_{1,m}\| + \|W_{2,m}\| \tau_M) l_m - \frac{e^{-\lambda \tau_M}}{k} (\alpha - \lambda) \Big) \end{aligned}$$

and calculating its derivative

$$\frac{dh_{\min}}{d\lambda} = \Big(\sum_{m=0}^{r} \|W_{2,n}\| l_m \Big)^{-1} \frac{e^{-\lambda \tau_M}}{k} [\tau_m (\alpha - \lambda) + 1] \geq 0 \,.$$

□

Corollary 3.7. *For arbitrary $m \in \overline{1,r}$ exponential decay rate estimate λ calculated based on the Theorem 3.1 is symmetric with respect to $W_{i,m}$, $i = 1, 2$, i.e.*

$$\lambda(W_{i,m}) = \lambda(-W_{i,m})$$

Moreover, the estimate depends exceptially on the matrix norm $\|W_{i,m}\|$, $i = 1, 2$.

The above corollary follows immediately from inequality (2.2) including matrix norms $\|W_{i,m}\|$.

4. Illustrative Example

For the numerical experiment, simple example is presented here to illustrate the usefulness of our main result. The model comes from [6, p. 808], where they considered the simple two-neuron network with four delays ($n = 2$, $r = 4$) for some constant rates b and c:

$$\begin{gathered}
A=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \quad W_{11}=\begin{pmatrix}b & 0\\ 0 & 0\end{pmatrix}, \quad W_{12}=\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\\
W_{13}=\begin{pmatrix}0 & 0\\ b & 0\end{pmatrix}, \quad W_{14}=\begin{pmatrix}cc0 & 0\\ 0 & b\end{pmatrix}, \quad W_{21}=\begin{pmatrix}c & 0\\ 0 & 0\end{pmatrix},\\
W_{22}=\begin{pmatrix}0 & c\\ 0 & 0\end{pmatrix}, \quad W_{23}=\begin{pmatrix}0 & 0\\ c & 0\end{pmatrix}, \quad W_{24}=\begin{pmatrix}0 & 0\\ 0 & c\end{pmatrix}\\
g_1(x)=g_2(x)=\tanh(x) \quad \text{for } x\in\mathbb{R}^2,\\
\tau_1=\frac{13}{12}\pi, \quad \tau_2=\frac{11}{12}\pi, \quad \tau_3=\frac{7}{12}\pi, \quad \tau_4=\frac{5}{12}\pi,\\
h_1=h_2=h_3=h_4=\frac{1}{12}\pi
\end{gathered} \tag{4.1}$$

Considering the initial conditions $x_1(t)\equiv 0.001$, $x_2(t)\equiv 0.004$, $t\in[-\tau_M,0]$ and applying Theorem 3.1 we can calculate the value of exponential decay λ. It can be readily solved by using the numerically efficient R package.

In [11] model (4.1) was studied when we do not have distributed delays, i.e., $c=0$. In this case Table 1 shows the dependence of λ on the value of b.

Table 1. Dependence of value of b and $\lambda>0$ calculated for the example without distributed delays

b	-0.25	-0.2	-0.1	-0.05	0.1	0.2	0.25
λ	0	0.0503686	0.2026738	0.3474646	0.2026738	0.0503686	0

If we have distributed delays with parameter $c=0.005$, then the resulting values of λ are presented in the Table 2.

Table 2. Dependence of value of b and $\lambda>0$ calculated from (3.11) for Example 1 at $c=0.005$. Symbol "-" means absence of positive solutions of (3.11).

b	-0.25	-0.2	-0.1	-0.05	0.1	0.2	0.25
λ	-	0.03337481	0.171189	0.2914205	0.171189	0.03337481	-

For the reasons given we conclude that distributed delays combined with discrete delays narrow the interval of parameters b admitting exponential convergence.

As a supplement, Figure 1 shows the time response of state variables $x_1(t)$, $x_2(t)$ in this example with $b=-0.1$ and initial vector $(0.001,0.004)^\top$. Figure 2 shows exponential estimate constructed in this model at $b=-0.1$.

The dependence of $h_{\min}$ on λ due to (3.12) is presented on the Table 3

As it was shown in [6, Theorem 2.1] that the equilibrium $(0,0)$ of system (4.1) with discrete delays only is delay-independently locally asymptotically stable if $b\in(-0.5,0.5)$. Here from Table 1 we can see that for network with both discrete and distributed delays, positive estimate of exponential decay rate based on Theorem 3.1 can be calculated for $b\in[-0.2,0.2]$. That is in this case the equilibrium $(0,0)$ of system (4.1) is delay-dependently locally exponentially stable

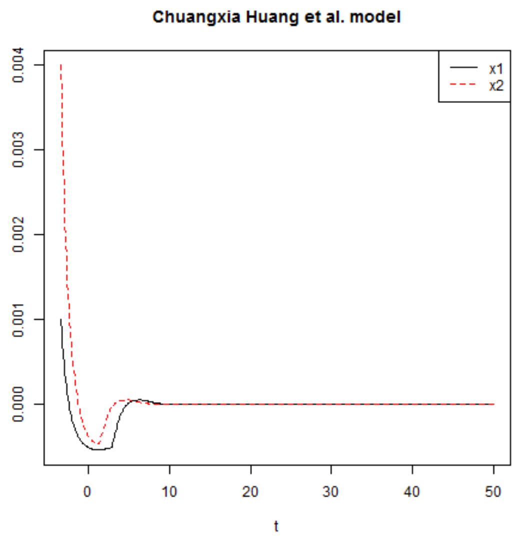

FIGURE 1. State trajectories in example 1 with $b = -0.1$ and $c = 0.005$

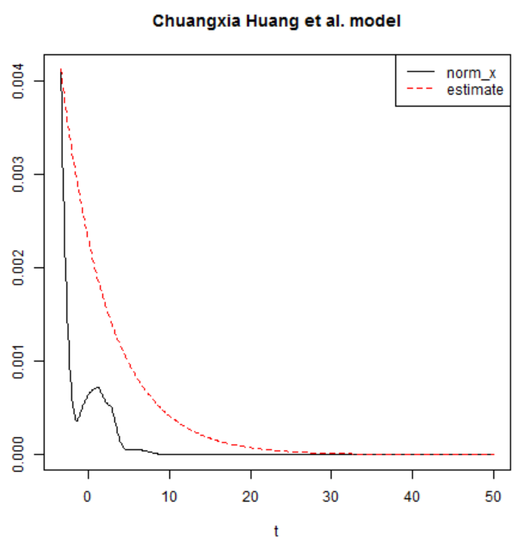

FIGURE 2. Exponential estimate and norm of the solution of Example 1 with $b = -0.1$ and $c = 0.005$

Conclusions. Investigation of exponential stability for neural network models require decay estimates that can be obtained from clear dependences (not LMIs). Earlier we have done some attepts to construct exponential estimates for linear

Table 3. Dependence of value of h_{min} and $\lambda > 0$ calculated from (3.12) for Example 1 at $c = 0.005$.

$h_{\min}$	0.2616517	0.26168	0.2627265
λ	0.03337481	0.171189	0.2914205

systems with delay. In [8, 10, 13] such clear estimates are obtained for Lyapunov-Krasovskii functionals satisfying to some difference-differential inequalities. As a rule they try to apply such techniques for real application like neural networks models. Unfortunately, it requires decay rates that can be calculated as a result of clear dependencies between model parameters. It stimulated development of indirect method.

The term "indirect method" in title of this work is used in order to contrast with methods of obtaining exponential estimates based on application of Lyapunov functions (or "direct" method)

As compared with Lyapunov-Krasovskii functional approach method offered here does not have such flexible possibilities for optimization of estimates and estimates obtained with help of developed approach are likely more rough and less accurate.

The "price" of this inaccuracy and roughness is comparatively clear form of expression for decay rate (as compared with multidimensional LMIs). This expression is quasipolynomial inequality which is well-known in stability analysis of delay differential equations.

Such simplicity of expressions is of importance in practical application like neural networks for obtaining analytical results. Namely, it allows to study dependencies of neural network exponential stability and changes in model parameters

It should be noted that estimates obtained here are compatible in some special cases with the results of application of comparison principle.

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