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BIFURCATION ANALYSIS OF ELLIPTIC EQUATIONS DESCRIBED BY NONHOMOGENEOUS DIFFERENTIAL OPERATORS

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Abstract. In this article, we are concerned with a class of nonlinear partial differential elliptic equations with Dirichlet boundary data. The key feature of this paper consists in competition effects of two generalized differential operators, which extend the standard operators with variable exponent. This class of problems is motivated by phenomena arising in non-Newtonian fluids or image reconstruction, which deal with operators and nonlinearities with variable exponents. We establish an existence property in the framework of small perturbations of the reaction term with indefinite potential. The mathematical analysis developed in this paper is based on the theory of anisotropic function spaces. Our analysis combines variational arguments with energy estimates.

1. Introduction

Partial differential equations driven by nonhomogeneous differential operators have been a very productive and rich research field in the last few decades because of the multiple relevant applications in various fields. We mainly refer to nonlinear stationary problems with associated energy that changes pointwise its growth properties and ellipticity. Problems with this structure have been comprehensively analyzed. We refer, e.g., to the seminal works of Halsey [12] and Zhikov [26, 27], in close connection with the qualitative and quantitative mathematical analysis of some classes of anisotropic materials and their applications to fields like homogenization and nonlinear elasticity.

In the framework of materials with non-homogeneous structure, the standard abstract analytic approach relying on the classical theory of L^p and $W^{k,p}$ function spaces (Lebesgue and Sobolev) is not satisfactory. We refer to electro-rheological fluids (also called "smart fluids") as well as to image processing, which should enable that the exponent p is varying; see Chen, Levine and Rao [8] and Ruzicka [24]. For instance, we refer to the Winslow effect of some fluids (like lithium polymetachrylate) in which the viscosity in a certain magnetic or electric range is inversely proportional to the field strength. This corresponds to non-Newtonian

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electro-rheological fluids, which are mathematically understood by means of nonlinear equations with one or more variable exponents.

Such a study corresponds to the abstract setting of Lebesgue and Sobolev function spaces $L^{p(x)}$ and $W^{1,p(x)}$. Here, p is a nonconstant smooth real-valued function with given properties. The abstract theory of function spaces with variable exponent was studied by Diening, Hästo, Harjulehto and Ruzicka [11] while the recent book by Rădulescu and Repovš [22] is devoted to the careful mathematical analysis of some models of nonlinear problems with one or more variable exponents; see also Harjulehto, Hästö, Le and Nuortio [13] and Rădulescu [20]. We also refer to Alsaedi *et al.* [1, 2], Mingione *et al.* [5, 9, 10], Pucci *et al.* [3, 19], Repovš *et al.* [7, 23] for related results.

Recently, Kim and Kim [14] introduced an extended class of non-homogeneous differential operators. The main feature of their work is in relationship with the thorough mathematical understanding of nonlinear models with lack of uniform convexity. More precisely, Kim and Kim [14] studied some classes of the boundary-value problems

$$\begin{gathered} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u) = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$.

The reaction term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfills a Carathéodory-type hypothesis and the function $\phi(x,t)$ behaves as $|t|^{p(x)-2}$ with $p : \overline{\Omega} \to (1,\infty)$ continuous. In the case where $\phi(x,t) = |t|^{p(x)-2}$, then the operator involved in problem (1.1) reduces to the $p(x)$-Laplace operator.

In many papers (see, e.g., [18, Hypothesis (A4), p. 2629]), the functional Φ induced by the principal part of problem (1.1) is assumed to be uniformly convex. This means that there exists $k > 0$ such that for each $(x,\xi,\psi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^N$,

$$\Phi\Big(x, \frac{\xi+\psi}{2}\Big) \le \frac{1}{2}\,\Phi(x,\xi) + \frac{1}{2}\,\Phi(x,\psi) - k\,|\xi-\psi|^{p(x)}.$$

However, since the function $\Psi(x,s) = s^p$ is not uniformly convex for $s \in (0,\infty)$ for $1 < p < 2$, this condition is not applicable to all p-Laplacian problems. A feature of the abstract setting developed in [14] is that the main results are obtained without any uniform convexity assumption. Related properties can be found in the recent paper of Baraket, Chebbi, Chorfi and Rădulescu [4].

We study some nonlinear phenomena driven by non-homogeneous differential operators. Our main purpose in this paper is to establish some qualitative properties of solutions in the framework of small perturbations.

2. Terminology and preliminary results

We suppose that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Define

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p > 1 \text{in } \overline{\Omega}\}.$$

For $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \sup_{x\in\Omega} p(x); \quad p^- = \inf_{x\in\Omega} p(x).$$

We define the Banach space

$$L^{p(x)}(\Omega) = \{u : u \text{ is measurable and } \int_\Omega |u|^{p(x)}\,dx < \infty\}$$

with the associated Luxemburg norm

$$|u|_{p(x)} = \inf \{\mu > 0 : \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\, dx \le 1\}.$$

According to [22], $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$.

The usual continuous embedding property of Lebesgue function spaces extends to variable exponent spaces. More precisely, if Ω has finite measure and p_1, p_2 are two functions satisfying $p_1 \le p_2$ in Ω then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. Then for all $u \in L^{p(x)}(\Omega)$ and all $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality holds:

$$\Big|\int_\Omega uv\, dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)}\,. \tag{2.1}$$

The *modular* of $L^{p(x)}(\Omega)$ has a crucial role in arguments dealing with variable exponent Lebesgue spaces. This *modular* is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If u, $(u_n) \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following properties are true:

$$\text{if } |u|_{p(x)} > 1 \text{ then } |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}, \tag{2.2}$$

$$\text{if } |u|_{p(x)} < 1, \text{ then } |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}, \tag{2.3}$$

$$|u_n - u|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

Let

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

This Banach space is usually equipped with the norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\|_{p(x)} = \inf \Big\{\mu > 0 : \int_\Omega \Big(|\frac{\nabla u}{\mu}|^{p(x)} + |\frac{u}{\mu}|^{p(x)}\Big)\, dx \le 1\Big\}.$$

Zhikov [27] showed that smooth functions are not always dense in $W^{1,p(x)}(\Omega)$. This property is in relationship with the *Lavrentiev phenomenon*. Roughly speaking, this phenomenon asserts that there are problems with variational structure such that the infimum over the family of smooth functions is bigger than the infimum over the set of all functions satisfying the same boundary conditions. We refer to [22, pp. 12-13] for more details.

Let $W_0^{1,p(x)}(\Omega)$ denote the closure with respect to $\|u\|_{p(x)}$ of the family of all $W^{1,p(x)}$-functions with compact support. In the case where smooth functions are dense, we can use as alternative approach the closure of the function space $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. We also point out that Poincaré's inequality enables to define, equivalently, the space $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The vector space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a reflexive and separable Banach space. Moreover, if Ω has finite measure and p_1, p_2 are two functions satisfying $p_1 \le p_2$ in Ω then there is a continuous embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$.

Let

$$\varrho_{p(x)}(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,dx. \tag{2.5}$$

Assume that (u_n), $u \in W_0^{1,p(x)}(\Omega)$. Then the following properties are true:

$$\|u\| > 1 \;\Rightarrow\; \|u\|^{p^-} \le \varrho_{p(x)}(u) \le \|u\|^{p^+}\,, \tag{2.6}$$

$$\|u\| < 1 \;\Rightarrow\; \|u\|^{p^+} \le \varrho_{p(x)}(u) \le \|u\|^{p^-}\,, \tag{2.7}$$

$$\|u_n - u\| \to 0 \;\Leftrightarrow\; \varrho_{p(x)}(u_n - u) \to 0\,. \tag{2.8}$$

Set

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \ge N. \end{cases}$$

We recall that if p and q belong to $C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for every $x \in \overline{\Omega}$ then the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

If the function p is constant, then variable exponent Lebesgue and Sobolev spaces reduce to the standard Lebesgue and Sobolev spaces.

From [22], some curious properties are valid in this framework, such as:

(i) If p is a smooth function, then the following coarea formula

$$\int_\Omega |w(x)|^p dx = p\int_0^\infty t^{p-1}\,|\{x \in \Omega;\; |w(x)| > t\}|\,dt$$

is no longer valid for variable exponent spaces.

(ii) Suppose that p is a nonconstant smooth (continuous) function in a ball B. Then there exists $w \in L^{p(x)}(B)$ such that $w(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$, provided that the norm of h is sufficiently small.

3. Main result

Assume that $p_1, p_2 \in C_+(\overline{\Omega})$ and let $\phi, \psi : \Omega \times [0,\infty) \to [0,\infty)$ be functions that satisfy the following growth assumptions:

(H1) the functions $\phi(\cdot,\xi)$, $\psi(\cdot,\xi)$ are measurable in the domain Ω for every $\xi \ge 0$ and the mappings $\phi(x,\cdot)$, $\psi(x,\cdot)$ are locally absolutely continuous in $[0,\infty)$ for almost all $x \in \Omega$;

(H2) there are $a_1 \in L^{p_1'}(\Omega)$, $a_2 \in L^{p_2'}(\Omega)$ and $b > 0$ such that

$$|\phi(x,|v|)v| \le a_1(x) + b|v|^{p_1(x)-1}, \quad |\psi(x,|v|)v| \le a_2(x) + b|v|^{p_2(x)-1}$$

for almost all $x \in \Omega$ and for every $v \in \mathbb{R}^N$;

(H3) there exists a positive real number c such that

$$\phi(x,\xi) \ge c\xi^{p_1(x)-2}, \quad \phi(x,\xi) + \xi\frac{\partial\phi}{\partial\xi}(x,\xi) \ge c\xi^{p_1(x)-2},$$

$$\psi(x,\xi) \ge c\xi^{p_2(x)-2}, \quad \psi(x,\xi) + \xi\frac{\partial\psi}{\partial\xi}(x,\xi) \ge c\xi^{p_2(x)-2}$$

for almost all $x \in \Omega$ and for all $\xi > 0$.

An interesting consequence of these assumptions is that ϕ and ψ satisfy a Simon-type inequality. More precisely, if we denote

$$\Omega_1 := \{x \in \Omega : \ 1 < p(x) < 2\} \quad \text{and} \quad \Omega_2 := \{x \in \Omega; \ p(x) \geq 2\},$$

then

$$\begin{aligned} &\langle \phi(x, |u|)u - \phi(x, |v|)v, u - v\rangle \\ &\geq \begin{cases} c(|u| + |v|)^{p(x)-2}|u - v|^2 & \text{if } x \in \Omega_1 \text{ and } (u, v) \neq (0, 0) \\ 4^{1-p^+}c|u - v|^{p(x)} & \text{if } x \in \Omega_2 \end{cases} \end{aligned} \tag{3.1}$$

is valid for all $u, v \in \mathbb{R}^N$, where c is the constant given in hypothesis (H3). This inequality is used in [14] to show that $A' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is both a nonlinear monotone operator and a (S_+) mapping. We refer to Simon [25] for the initial version of inequality (3.1) in the framework of the p-Laplace operator.

Consider the problem

$$\begin{gathered} -\operatorname{div}(\phi(x, |\nabla u|)\nabla u) - \operatorname{div}(\psi(x, |\nabla u|)\nabla u) \\ = \lambda a(x)|u|^{r(x)-2}u - b(x)|u|^{s(x)-2}u, \quad x \in \Omega \\ u = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.2}$$

This problem extends in a general setting results that are valid for standard operators with variable exponent, such as the $p(x)$-Laplace operator, the mean curvature equation with variable exponent, or the nonhomogeneous capillarity equation.

We assume that λ is a positive parameter and $r, s \in C_+(\overline{\Omega})$. We study problem (3.2) under the following hypotheses:

- (H4) $a \in L^{q_1(x)}(\Omega)$ and there exists $\omega \Subset \Omega$, $|\omega| > 0$ such that $a > 0$ in ω; $b \in L^{q_2(x)}(\Omega)$, $b > 0$ almost everywhere in Ω;
- (H5) we have $\max\{r(x), s(x)\} < \max\{p_1(x), p_2(x)\} \leq N < \min\{q_1(x), q_2(x)\}$ for all $x \in \overline{\Omega}$;
- (H6) we have $\inf_{x\in\omega} r(x) < \inf_{x\in\omega}(p_1 \wedge p_2 \wedge s)(x)$.

When p_1, p_2 are the exponents introduced in (H2) and (H3), we set

$$p(x) := \max\{p_1(x), p_2(x)\} \quad \text{for all } x \in \overline{\Omega}.$$

Throughout this paper, we say that u is a (weak) solution of problem (3.2) if $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ and

$$\int_\Omega [\phi(x, |\nabla u|) + \psi(x, |\nabla u|)]\nabla u \cdot \nabla v \, dx = \lambda \int_\Omega a(x)|u|^{r(x)-2}uv dx - \int_\Omega b(x)|u|^{s(x)-2}uv,$$

for all functions $v \in W_0^{1,p(x)}(\Omega)$.

Our main result of the present paper establishes that problem (3.2) has solutions in the case of small perturbation of the reaction term in the right-hand side of (3.2).

Theorem 3.1. *Assume that hypotheses* (H1)–(H6) *are fulfilled. Then there exists a positive real number* Λ *such that* (3.2) *has at least one solution for all* $\lambda \in (0, \Lambda)$.

4. Proof of Theorem 3.1

For $x \in \overline{\Omega}$ we set

$$\alpha_1(x) = \frac{q_1(x)r(x)}{q_1(x) - r(x)} \quad \text{and} \quad \alpha_2(x) = \frac{q_2(x)s(x)}{q_2(x) - s(x)}\,.$$

By hypothesis (H5), $\alpha_1(x)$ and $\alpha_2(x)$ are positive numbers. Assumption (H5) also yields that

$$\max\{\alpha_1(x), \alpha_2(x)\} < p^*(x) \quad \text{for } x \in \overline{\Omega}, \tag{4.1}$$

$$\max\{q_1'(x)\alpha_1(x), q_2'(x)\alpha_2(x)\} < p^*(x). \tag{4.2}$$

It follows that $W_0^{1,p(x)}(\Omega)$ is compactly embedded into the spaces $L^{\alpha_j(x)}(\Omega)$ and $L^{q_j'(x)\alpha_j(x)}(\Omega)$, $j = 1, 2$.

For functions ϕ and ψ satisfying (H1)-(H3), we define

$$A_0(x,t) := \int_0^t [\phi(x,s) + \psi(x,s)]s ds. \tag{4.3}$$

Consider the associated functional $A : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$A(u) := \int_\Omega A_0(x, |\nabla u|)dx, \tag{4.4}$$

where A_0 is introduced in (4.3).

By [14, Lemma 3.2] and since hypotheses (H1) and (H2) are fulfilled, we obtain that $A \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and its Gâteaux directional derivative is given by

$$A'(u)(v) = \int_\Omega [\phi(x, |\nabla u|) + \psi(x, |\nabla v|)]\nabla u \cdot \nabla v dx \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega). \tag{4.5}$$

Moreover, since conditions (H1)–(H3) are satisfied, [14, Lemma 3.4] implies that the nonlinear mapping $A : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is a strictly monotone operator. Moreover, this is a (S_+) mapping; namely, if

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega) \text{ as } n \to \infty \text{ and } \limsup_{n\to\infty} \langle A'(u_n) - A'(u), u_n - u\rangle \le 0,$$

then

$$u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \text{ as } n \to \infty.$$

It is straightforward that the nonlinear mapping A is weakly lower semicontinuous, see [14] for details and proofs.

Define the functionals $B, \mathcal{E} : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$B(u) = \lambda \int_\Omega \frac{a(x)}{r(x)} |u|^{r(x)}dx - \int_\Omega \frac{b(x)}{s(x)} |u|^{s(x)}dx,$$
$$\mathcal{E}(u) = A(u) - B(u).$$

We argue in what follows that B is well-defined in $W_0^{1,p(x)}(\Omega)$. Indeed, for all $u \in W_0^{1,p(x)}(\Omega)$ we have

$$\Big| \int_\Omega \frac{a(x)}{r(x)}|u|^{r(x)}dx\Big| \le \frac{1}{r^-} |a|_{q_1(x)}| \, |u|^{r(x)} \, |_{\alpha_1'(x)} \le \frac{1}{r^-} |a|_{q_1(x)}|u|^{k_1}_{r(x)\alpha_1'(x)}. \tag{4.6}$$

Here, k_1 is a positive real number not depending on u. Similarly, there exists $k_2 > 0$ such that for all $u \in W_0^{1,p(x)}(\Omega)$

$$\Big| \int_\Omega \frac{b(x)}{s(x)}|u|^{s(x)}dx\Big| \le \frac{1}{s^-} |b|_{q_2(x)}|u|^{k_2}_{s(x)\alpha_2'(x)}. \tag{4.7}$$

Relations (4.6) and (4.7) and the continuous embeddings of $W_0^{1,p(x)}(\Omega)$ into the spaces $L^{r(x)\alpha_1'(x)}(\Omega)$ and $L^{s(x)\alpha_2'(x)}(\Omega)$ imply that B is well-defined. Moreover, by standard computation we deduce that B is of class C^1 and for all $u,v\in W_0^{1,p(x)}(\Omega)$

$$B'(u)(v)=\lambda\int_\Omega a(x)|u|^{r(x)-2}uvdx-\int_\Omega b(x)|u|^{s(x)-2}uv\,dx.$$

Returning to (4.5), we conclude that $\mathcal{E}$ is of class C^1 on $W_0^{1,p(x)}(\Omega)$ and

$$\begin{aligned}\mathcal{E}'(u)(v)=&\int_\Omega[\phi(x,|\nabla u|)+\psi(x,|\nabla v|)]\nabla u\cdot\nabla v dx\\ &-\lambda\int_\Omega a(x)|u|^{r(x)-2}uvdx+\int_\Omega b(x)|u|^{s(x)-2}uv\,dx.\end{aligned}$$

These arguments also show that $u\in W_0^{1,p(x)}(\Omega)$ is a nontrivial critical point of the energy functional $\mathcal{E}$ if and only if u is a (weak) solution of problem (3.2).

Step 1. For all $\rho>0$ sufficiently small, we can find $\lambda^*,\eta>0$ such that $\mathcal{E}(u)\geq\eta$, provided that $\|u\|=\rho$ and $\lambda\in(0,\lambda^*)$.

Using (H3) and (2.3) we observe that for all $u\in W_0^{1,p(x)}(\Omega)$ with $\|u\|<1$

$$\int_\Omega\int_0^{|\nabla u|}\phi(x,s)s\,ds\,dx\geq c\int_\Omega\int_0^{|\nabla u|}s^{p_1(x)-1}\,ds\,dx\geq\frac{c}{p_1^+}\int_\Omega|\nabla u|^{p_1(x)}dx$$

and

$$\int_\Omega\int_0^{|\nabla u|}\psi(x,s)s\,ds\,dx\geq\frac{c}{p_2^+}\int_\Omega|\nabla u|^{p_2(x)}dx.$$

Thus, for all $u\in W_0^{1,p(x)}(\Omega)$ with $\|u\|<1$,

$$\begin{aligned}A(u)&\geq\frac{c}{p^+}\int_\Omega\Big(|\nabla u|^{p_1(x)}+|\nabla u|^{p_2(x)}\Big)dx\\ &\geq\frac{c}{p^+}\int_\Omega|\nabla u|^{p(x)}\geq\frac{c}{p^+}\,\|u\|^{p^+}.\end{aligned}\tag{4.8}$$

Next, by Hölder's inequality,

$$\int_\Omega\frac{a(x)}{r(x)}\,|u|^{r(x)}dx\leq\frac{1}{r^-}\int_\Omega a(x)|u|^{r(x)}dx\leq\frac{1}{r^-}|a|_{r(x)}|u|^{r^-}_{r(x)q_1'(x)}.$$

Assumption (H5) implies that $r(x)q_1'(x)<p^*(x)$, hence $W_0^{1,p(x)}(\Omega)$ is continuously embedded in $L^{r(x)q_1'(x)}(\Omega)$. Relation (2.7) implies the existence of some $C_1>0$ such that for all $u\in W_0^{1,p(x)}(\Omega)$ with sufficiently small norm we have

$$\int_\Omega\frac{a(x)}{r(x)}\,|u|^{r(x)}dx\leq\frac{C_1}{r^-}\,|a|_{r(x)}\,\|u\|^{r^-}.\tag{4.9}$$

Combining relations (4.8) and (4.9) we obtain that for every $u\in W_0^{1,p(x)}(\Omega)$ with sufficiently small norm we have

$$\begin{aligned}\mathcal{E}(u)&\geq\frac{c}{p^+}\,\|u\|^{p^+}-\frac{\lambda C_1}{r^-}|a|_{r(x)}\|u\|^{r^-}\\ &=C_2\,\|u\|^{p^+}-\lambda C_3\,\|u\|^{r^-}\\ &=\|u\|^{r^-}\big(C_2\|u\|^{p^+-r^-}-\lambda C_3\big).\end{aligned}$$

Then step 1 follows by using hypothesis (H5).

Step 2. There exist $w \in W_0^{1,p(x)}(\Omega)$ and $t_0 > 0$ such that $\mathcal{E}(tw) < 0$ for all $t \in (0, t_0)$.

Let ω be the subdomain of Ω defined in hypothesis (H4) and let $p_{1,\omega}^-$, $p_{2,\omega}^-$, s_ω^-, and r_ω^- denote the infima of p_1, p_2, s, and r in ω. Set

$$\delta := \min\{p_{1,\omega}^-, p_{2,\omega}^-, s_\omega^-\}.$$

By hypothesis (H6), there exists $\varepsilon_0 > 0$ such that

$$1 < r_\omega^- + \varepsilon_0 < \delta\,. \tag{4.10}$$

We fix $\omega_1 \subset\subset \omega$ such that

$$r_\omega^- - \varepsilon_0 \le r(x) \le r_\omega^- + \varepsilon_0.$$

We also fix $w \in C_0^\infty(\Omega)$ such that

$$\operatorname{supp}(w) \subset \omega_1 \quad \text{and} \quad 0 \le w \le 1 \quad \text{in } \omega_1.$$

Let $t \in (0,1)$. We have

$$\begin{aligned} A(tw) &= \int_\Omega A_0(x, t|\nabla w|)dx = \int_\Omega \int_0^{t|\nabla w|} [\phi + \psi] s\,ds\,dx \\ & \int_\omega \int_0^{t|\nabla w|} [\phi + \psi] s\,ds\,dx. \end{aligned}$$

Using hypothesis (H2) we obtain

$$\begin{aligned} A(tw) &\le \int_\omega \int_0^{t|\nabla w|} \big(|a_1(x)|s + bs^{p_1(x)}\big)\,ds\,dx \\ &\quad + \int_\omega \int_0^{t|\nabla w|} \big(|a_2(x)|s + bs^{p_2(x)}\big)\,ds\,dx \\ &\le \int_\omega \big(|a_1(x)|t|\nabla w| + bt^{p_1(x)}|\nabla w|^{p_1(x)}\big)dx \\ &\quad + \int_\omega \big(|a_2(x)|t|\nabla w| + bt^{p_2(x)}|\nabla w|^{p_2(x)}\big)dx \\ &\le bt^\delta \Big(\int_\omega |\nabla w|^{p_1(x)} + \int_\omega |\nabla w|^{p_2(x)} \Big) + C_6 t. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} B(tw) &= \lambda \int_\Omega \frac{a(x)}{r(x)}\, t^{r(x)} w^{r(x)} dx - \int_\Omega \frac{b(x)}{s(x)}\, t^{s(x)} w^{s(x)} dx \\ &= \lambda \int_\omega \frac{a(x)}{r(x)}\, t^{r(x)} w^{r(x)} dx - \int_\omega \frac{b(x)}{s(x)}\, t^{s(x)} w^{s(x)} dx \\ &\ge \lambda\, \frac{t^{r_\omega^- + \varepsilon_0}}{r^+} \int_\omega a(x) w^{r(x)} dx - \frac{t^{s_\omega^-}}{s^-} \int_\omega b(x) w^{s(x)} dx \\ &= \lambda C_7 t^{r_\omega^- + \varepsilon_0} - \frac{t^\delta}{s^-} \int_\omega b(x) w^{s(x)} dx. \end{aligned}$$

We conclude that

$$\begin{aligned} \mathcal{E}(tw) &= A(tw) - B(tw) \\ &\le t^\delta \Big[b \int_\omega \big(|\nabla w|^{p_1(x)} + |\nabla w|^{p_2(x)}\big)dx + \frac{1}{s^-} \int_\omega b(x) w^{s(x)} dx \Big] \\ &\quad + C_6 t - \lambda C_7 t^{r_\omega^- + \varepsilon_0}. \end{aligned}$$

Recalling the choice of ε_0 and the definition of δ (see relation (4.10)), we deduce that $\mathcal{E}(tw) < 0$ for all $t > 0$ sufficiently small.

Proof of Theorem 3.1 concluded. Combining steps 1 and 2, we deduce that there exist $\lambda^* > 0$ and $\rho > 0$ such that for every $\lambda \in (0, \lambda^*)$

$$\inf_{\|u\|=\rho} \mathcal{E}(u) > 0 \quad \text{and} \quad \inf_{\|u\|\leq\rho} \mathcal{E}(u) < 0.$$

Fix $\varepsilon > 0$ so that

$$\varepsilon < \inf_{\|u\|=\rho} \mathcal{E}(u) - \inf_{\|u\|\leq\rho} \mathcal{E}(u). \tag{4.11}$$

Consider the energy functional $\mathcal{E}$ restricted to the complete metric space $\overline{B(0,\rho)} \subset W_0^{1,p(x)}(\Omega)$. Applying Ekeland's variational principle, we find $u_\varepsilon \in W_0^{1,p(x)}(\Omega)$ with $\|u_\varepsilon\| \leq \rho$ such that

$$\inf_{\|u\|\leq\rho} \mathcal{E}(u) \leq \mathcal{E}(u_\varepsilon) \leq \inf_{\|u\|\leq\rho} \mathcal{E}(u) + \varepsilon, \tag{4.12}$$

$$\mathcal{E}(u) - \mathcal{E}(u_\varepsilon) + \varepsilon \|u - u_\varepsilon\| \geq 0 \quad \text{for all } u \neq u_\varepsilon. \tag{4.13}$$

The choice of ε given in (4.11) implies that $\|u_\varepsilon\| < \rho$, hence u_ε is an interior point of $B(0,\rho)$. Next, a standard argument based on relation (4.13) implies that $\|\mathcal{E}'(u_\varepsilon)\| \leq \varepsilon$.

In conclusion, we obtain a bounded sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ satisfying

$$\mathcal{E}(u_n) \to \inf_{\|u\|\leq\rho} \mathcal{E}(u) \quad \text{and} \quad \|\mathcal{E}'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

Thus, passing if necessary to a subsequence, we can assume that (u_n) is weakly convergent to $u \in W_0^{1,p(x)}(\Omega)$.

We claim that the sequence $(u_n) \subset W_0^{1,p(x)}(\Omega)$ is strongly convergent. The key argument for this purpose is that the nonlinear mapping $A' : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is an operator of type (S_+). For this purpose, we observe that the Hölder inequality yields

$$\begin{aligned}
&\Big| \int_\Omega a(x)|u_n|^{r(x)-2}u_n(u_n - u)dx \Big| \\
&\leq |a|_{q_1(x)} \big| |u_n|^{r(x)-2}u_n(u_n - u) \big|_{q_1'(x)} \\
&\leq |a|_{q_1(x)} \big| |u_n|^{r(x)-2}u_n \big|_{r(x)/[r(x)-1]} |u_n - u|_{\alpha_1(x)}.
\end{aligned} \tag{4.14}$$

Recall that $\alpha_1(x) < p^*(x)$. Thus, up to a subsequence, the convergence of (u_n) to u is strong in $L^{\alpha_1(x)}(\Omega)$. Returning to inequality (4.14), we obtain

$$\int_\Omega a(x)|u_n|^{r(x)-2}u_n(u_n - u)dx \to 0 \quad \text{as } n \to \infty. \tag{4.15}$$

A similar argument shows that

$$\int_\Omega b(x)|u_n|^{s(x)-2}u_n(u_n - u)dx \to 0 \quad \text{as } n \to \infty. \tag{4.16}$$

Relations (4.15) and (4.16) combined with the fact that $\|\mathcal{E}'(u_n)\| \to 0$ as $n \to \infty$ imply that

$$\mathcal{E}'(u_n)(u_n - u) - \mathcal{E}'(u)(u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{4.17}$$

But

$$\begin{aligned}
&\mathcal{E}'(u_n)(u_n-u)-\mathcal{E}'(u)(u_n-u)\\
&=\int_\Omega(\phi(x,|\nabla u_n|)+\psi(x,|\nabla u_n|))\nabla u_n\nabla(u_n-u)dx\\
&\quad-\int_\Omega(\phi(x,|\nabla u|)+\psi(x,|\nabla u|))\nabla u\nabla(u_n-u)dx\\
&\quad-\lambda\int_\Omega a(x)(|u_n|^{r(x)-2}u_n-|u|^{r(x)-2}u)(u_n-u)dx\\
&\quad+\int_\Omega b(x)(|u_n|^{s(x)-2}u_n-|u|^{s(x)-2}u)(u_n-u)dx.
\end{aligned} \tag{4.18}$$

Combining relations (4.15)–(4.18) we deduce that

$$\begin{aligned}
&\langle A'(u_n)-A'(u),u_n-u\rangle\\
&=\int_\Omega\left(\phi(x,|\nabla u_n|)\nabla u_n+\psi(x,|\nabla u_n|)\nabla u_n\right)\nabla(u_n-u)dx\\
&\quad-\int_\Omega\left(\phi(x,|\nabla u|)\nabla u+\psi(x,|\nabla u|)\nabla u\right)\nabla(u_n-u)dx\to 0\quad\text{as } n\to\infty.
\end{aligned} \tag{4.19}$$

Recall that the operator A' is a (S_+)-type mapping and $u_n \rightharpoonup u$. Thus, using relation (4.19), we deduce the strong convergence of (u_n) to u. It follows that $\mathcal{E}(u_n)\to\mathcal{E}(u)=\inf_{\|w\|\le\rho}\mathcal{E}(w)<0$, hence u is a nontrivial critical point of $\mathcal{E}$. $\square$

Perspectives. Problem (3.2) has been studied in the *subcritical case*, see relations (4.1) and (4.2). In our setting, these assumptions are crucial to establish that the bounded sequence of almost critical points of the energy functional $\mathcal{E}$ is, in fact, strongly convergent (passing eventually to a subsequence) in $W_0^{1,p(x)}(\Omega)$. We suggest to the reader the approach of a similar problem in the *almost critical* framework, namely subject to the following hypothesis: there are $x_0,x_1\in\Omega$ such that

$$\max\{\alpha_1(x_0),\alpha_2(x_0)\}=p^*(x_0);\quad \max\{\alpha_1(x),\alpha_2(x)\}<p^*(x)\quad\text{for all } x\in\overline{\Omega}\setminus\{x_0\} \tag{4.20}$$

and

$$\begin{gathered}
\max\{q_1'(x_1)\alpha_1(x_1),q_2'(x_1)\alpha_2(x_1)\}=p^*(x_1);\\
\max\{q_1'(x)\alpha_1(x),q_2'(x)\alpha_2(x)\}<p^*(x)\quad\text{for all } x\in\overline{\Omega}\setminus\{x_1\}
\end{gathered} \tag{4.21}$$

In our opinion, the result established in Theorem 3.1 remains valid if both hypotheses (4.20) and (4.21) are fulfilled.

Motivated by the results developed by Chen, Levine and Rao [8] in connection with models from image restoration, we consider that a rich field of investigation concerns the study of energy functionals of the type

$$W_0^{1,p(x)}(\Omega)\ni u\mapsto A(u)+\int_\Omega|u(x)-I(x)|^2dx,$$

where A is defined in (4.4), I is a given input corresponding to shades of gray in the domain Ω, and $1\le p_1(x),p_2(x)\le 2$. Cf. [8], the variable exponents p_1 and p_2 are close to 1 in regions where it is assumed to be edges, and close to 2 in the contrary case. In order to have information on the relative location of edges, this can be performed either by smoothing the input data or by looking for the region where the gradient is large. We refer to [22, pp. 5-6] for related results, including the staircase effect.

Another important step is to extend the approach corresponds to *double phase* problems, as introduced and developed by Mingione *et al.* [5, 9, 10]. In this framework the associated energy is either

$$w \mapsto \int_\Omega [|\nabla w|^{p_1(x)} + V|\nabla w|^{p_2(x)}]dx$$

or

$$w \mapsto \int_\Omega [|\nabla w|^{p_1(x)} + V|\nabla w|^{p_2(x)} \log(e+|x|)]dx,$$

where $p_1(x) \le p_2(x)$, $p_1 \ne p_2$, and $V(x) \ge 0$. Considering two materials having corresponding hardening exponents p_1 and p_2, the potential $V(x)$ characterizes the geometry of a composite of these materials. More precisely, if $V > 0$ then the associated $p_2(x)$-material is present in the composite. In the contrary case, the $p_1(x)$-material is the only that contributes to the structure of the composite.

Problems with this structure extend the pioneering contributions of Paolo Marcellini [16, 17] concerning variational functionals as $u \mapsto \int_\Omega F(x, \nabla u)dx$, where $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ fulfills asymmetrical growth properties of the type

$$|\eta|^p \lesssim F(x,\eta) \lesssim |\eta|^q, \quad \text{for all } (x,\eta) \in \Omega \times \mathbb{R}^N,$$

provided that $1 < p < q$.

We anticipate that the methods introduced in the present paper also work in a more general framework corresponding to Orlicz-Sobolev-Musielak function spaces (we refer to [22, Chaper 4] for a rigorous treatment of several models of stationary problems in Orlicz-Sobolev-Musielak spaces).

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