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MULTIPLE SOLUTIONS OF NONLINEAR FRACTIONAL ELLIPTIC EQUATIONS VIA MORSE THEORY

WEI QI, LIN ZHAO, XINGJIE YAN

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Abstract. This article concerns the existence and multiplicity of weak solutions of the nonlinear fractional elliptic problem. We extend some well known results of semilinear Laplacian equations to the nonlocal fractional setting. Using the variational methods based on the critical point theory, sub-supersolutions methods and Morse theory, we show that the problem has at least 6 nontrivial solutions.

1. Introduction

In the recent years, there has been a considerable interest to study the partial differential equations involving nonlocal operators. Nonlocal operators often appear in complex systems, such as anomalous diffusion and geophysical flows [1, 4, 14, 15], a thin obstacle problem [20], finance [9] and stratified materials [17] etc. A special and important nonlocal operator is the fractional Laplacian operator arising in non-Gaussian stochastic systems. For a stochastic differential system with a s-stable Lévy motion (a non-Gaussian stochastic process) L_t^s for $s \in (0,1)$

$$dX_t = b(X_t)dt + dL_t^s, \quad X_0 = x,$$

the corresponding Fokker-Planck equation contains the fractional Laplacian operator $(-\Delta)^s$. When the drift term b in the above stochastic differential system depends on the probability distribution of the system state, the Fokker-Planck equation becomes a nonlinear and nonlocal partial differential equation [1]. There are many works about the modeling techniques, well-posedness and regularity of solutions for the nonlocal partial differential equations with the fractional Laplacian operator $(-\Delta)^s$, $s \in (0,1)$, see [3, 4, 7, 8, 10, 14, 16, 19, 20].

Motivated by an evident and increasing interest in the current literature on fractional elliptic problems, in this paper, we are interested in the multiplicity of solution of the problem

$$\begin{gathered} (-\Delta)^s u = f(u), \quad x \in \Omega, \\ u = 0, \quad x \in \mathcal{C}\Omega, \end{gathered} \tag{1.1}$$

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where $\mathcal{C}\Omega = \mathbb{R}^n \backslash \Omega$. Note that the boundary condition is given in $\mathbb{R}^n \backslash \Omega$, not simply on $\partial\Omega$, as it appears in the classical case of the Laplacian equation, consistently with the nonlocal character of the fractional Laplacian operator.

The hypotheses on the nonlinearity $f(t)$ of problem (1.1) are as follows

- (H1) $f(t)$ is $\mathcal{C}^1$ and $f(0) = 0$;
- (H2) for all $t \in \mathbb{R}$, $|f'(t)| \le C(1 + |t|^{r-2})$, with $C > 0$ a constant and $2 < r < 2^*$, $2^* = \frac{2n}{n-2s}$, $n \ge 3$;
- (H3) there exists an integer $m \ge 2$ such that $\alpha = f'(\infty) \in (\lambda_m, \lambda_{m+1})$, where $f'(\infty) := \lim_{|t|\to\infty} \frac{f(t)}{t}$;
- (H4) $\lambda_1 < \beta = \lim_{|t|\to 0} \frac{f(t)}{t} < \lambda_2$;
- (H5) there exist ξ_- and ξ_+ such that $\xi_- < 0 < \xi_+$ and $f(\xi_+) \le 0 \le f(\xi_-)$.

Note that these equations have a variational structure, hence the variational method may be a powerful tool for dealing with such problems. A lot of efforts have been done in studying the problem (1.1), for instance, in [18] it shows that there exists non-trivial solutions of the problem (1.1) by the Mountain Pass Theorem and Linking Theorem. The existence and multiplicity of the problem (1.1) was studied in [21], and it shows that there are at least six solutions when the nonlinear term with the concave-convex property. In [6], the existence of positive ground state was presented by using the minimax arguments with a general Berestycki-Lions type nonlinear term. In [11], it shows that the problem (1.1) has three or four non-trivial solutions with the subcritical or critical nonlinear term respectively by using Morse theory method.

In this paper, using variational methods based on the critical point theory, the sub-supersolutions methods and Morse theory, we obtain at least six nontrivial solutions of the problem (1.1) with a more general nonlinear term. We state our main result as follows.

Theorem 1.1. *Assume that* (H1)–(H5) *hold. Then problem* (1.1) *with* $s \in (0, 1)$ *and* $n > 2s$ *has at least six nontrivial solutions:* u_1 *and* u_3 *have the Morse index 0,* u_2 *and* u_4 *have the Morse index 1, and* u_5 *has the Morse index* $d \ge 2$.

It is worth mentioning that if we choose suitable fractional Sobolev space and the fractional Laplacian operator is replaced by the more general nonlocal operator $\mathcal{L}_K$,

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^n,$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ is a function such that

$$mK \in L^1(\mathbb{R}^n), \quad m(x) = \min\{|x|^2, 1\}.$$

There exists $\theta > 0$ and $s \in (0, 1)$ such that $K(x) \ge \theta |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$ and for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$K(x) = K(-x),$$

then the result in this article also holds.

The rest of this article is organized as follows. In Section 2, we present some necessary preliminary notations and results. Section 3 is devoted to the proof of our main result.

2. Preliminaries

In this section, we present some notation and useful results. The classical fractional Sobolev space is

$$H^s(\Omega)=\Big\{u\in L^2(\Omega):\frac{|u(x)-u(y)|}{|x-y|^{\frac{n+2s}{2}}}\in L^2(\Omega\times\Omega)\Big\}$$

with the Gagliardo norm

$$\|u\|_{H^s(\Omega)}=|u|_2+\Big(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{\frac{1}{2}}.$$

For the details about fractional Sobolev spaces, we refer to [16]. Define

$$H_0^s(\Omega)=\{g\in H^s(\mathbb{R}^n):g=0\text{ a.e. in }\mathcal{C}\Omega\}.$$

The space $H_0^s(\Omega)$ is not empty and is a Hilbert space with the scalar product.

$$\langle u,v\rangle_{H_0^s(\Omega)}=\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy.$$

Note that the above integral can be extended to $\mathbb{R}^n$, since the function belongs to $H_0^s(\Omega)$.

Define $\delta:\overline{\Omega}\to\mathbb{R}_+$ by $\delta(x)=\operatorname{dist}(x,\mathbb{R}^N\setminus\Omega)$, $x\in\overline{\Omega}$, and the space

$$C_\delta^0(\overline{\Omega})=\Big\{u\in C^0(\overline{\Omega})\Big|\frac{u}{\delta^s}\text{admits a continuous extension to }\overline{\Omega}\Big\}$$

with the norm $\|u\|_{0,\delta}=\|\frac{u}{\delta^s}\|_\infty$.

The Banach space $C(\overline{\Omega})$ is an ordered Banach space with the positive cone

$$C_+=\{u\in C(\overline{\Omega}):u(x)\ge 0\text{ for }x\in\overline{\Omega}\}.$$

The cone has a nonempty interior, given by

$$\operatorname{int}C_+=\{u\in C_+:u(x)>0\text{ for }x\in\Omega\}.$$

We recall the definition of the fractional Laplacian operator.

Definition 2.1. For $u\in C_0^\infty(\mathbb{R}^n)$ and $s\in(0,1)$, we define

$$(-\Delta)^s u=C(n,s)\,\mathrm{P.\,V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy,$$

where the principle value (P.V.) is taken as the limit of the integral over $\mathbb{R}^n\backslash B_\epsilon(x)$ as $\epsilon\to 0$, with $B_\epsilon(x)$ the ball of radius ϵ centered at x, and

$$C(n,s)=\frac{2s}{2^{1-2s}\pi^{\frac{n}{2}}}\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(1-s)}.$$

Here $\Gamma(\lambda)=\int_0^\infty t^{\lambda-1}e^{-t}dt$ defined for $\lambda>0$. For more information of the fractional Laplacian operator, we refer to [7, 19].

Next we give some properties of the nonlocal operator $(-\Delta)^s$, which can explain why $(-\Delta)^s$ possesses the elliptic property. For the sake of convenience, we assume $C(n,s)=1$.

Lemma 2.2. *The operator $(-\Delta)^s$ admits the following properties:*

(i) *If u is a constant, then $(-\Delta)^s u=0$.*

(ii) *Let $x \in \mathbb{R}^n$. For any maximal point x_0 such that satisfies $u(x_0) \geq u(x)$, we have $(-\Delta)^s u(x_0) \geq 0$. Similarly, if $x^{'}$ is a minimal point such that $u(x^{'}) \leq u(x)$, then $(-\Delta)^s u(x^{'}) \leq 0$.*

(iii) *$(-\Delta)^s u$ is a positive semidefine operator, i.e., $((-\Delta)^s u, u) \geq 0$ in the sense of $L^2(\mathbb{R}^n)$ inner product.*

Proof. By the definition of nonlocal operator $(-\Delta)^s$, we see that (i) is trivial. If x_0 is the maximum point of u in $\mathbb{R}^n$, i.e., $u(x_0) \geq u(x)$, $x \in \mathbb{R}^n$, then by definition it holds

$$(-\Delta)^s u(x_0) = C(n,s)\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x_0)-u(y)}{|x-y|^{n+2s}}dy \geq 0, \tag{2.1}$$

which implies property (ii) because a similar argument can be used for the minimal point. By the inner product we have

$$((-\Delta)^s u, u) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \geq 0, \tag{2.2}$$

which shows property (iii) holds. The proof is complete. □

Definition 2.3. We say that $u \in H_0^s(\Omega)$ is a weak supersolution of the problem (1.1) if the following inequality holds for all $v \in H_0^s(\Omega)$:

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy \geq \int_\Omega fv\,dx.$$

The definition of a weak subsolution can be given in an analogous manner.

The energy functional for problem (1.1) is

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega F(u(x))dx,$$

where

$$F(x,t) = \int_0^t f(x,s)ds, \quad x \in \Omega,\ t \in \mathbb{R}.$$

It is well known that the function J is well defined and Frechét differentiable in $H_0^s(\Omega)$, and

$$\langle J^{'}(u), \varphi\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega f(u(x))\varphi(x)dx,$$

for any $\varphi \in H_0^s(\Omega)$. Thus, the critical points of J are solutions of problem (1.1).

To obtain our main result, the following eigenvalue problem plays a crucial role.

$$\begin{gathered}(-\Delta)^s u = \lambda u, \quad x \in \Omega,\\ u = 0, \quad x \in \mathcal{C}\Omega.\end{gathered} \tag{2.3}$$

The weak formulation of the problem (2.1) is

$$\begin{gathered}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\,dx\,dy = \lambda\int_\Omega u(x)\varphi(x)dx,\\ u \in H_0^s(\Omega), \quad \varphi \in H_0^s(\Omega).\end{gathered} \tag{2.4}$$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2.1), if there exists a non-trivial solution $u \in H_0^s(\Omega)$ of problem (2.1) for some $\lambda > 0$. Any solution will be called an eigenfunction corresponding to the eigenvalue λ.

Lemma 2.4 ([19]). *Let $s \in (0,1)$, $n > 2s$, and Ω be an open bounded set of $\mathbb{R}^n$. Then we have*

(1) *the set of the eigenvalues of (2.1) consists of a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with*
$$0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_j \le \dots, \quad \lambda_j \to +\infty \quad \text{as } j \to \infty. \tag{2.5}$$
(2) *The sequence $\{e_n\}_{n\in\mathbb{N}}$ of eigenfunctions corresponding to λ_n is an orthonormal basis of $L^2(\Omega)$ and an orthonormal basis of $H_0^s(\Omega)$.*

By Lemma 2.4, we have the direct sum decomposition
$$H_0^s(\Omega) = H_k \oplus H_k^{\perp},$$
where $H_k = \operatorname{span}\{e_1, \dots, e_k\}$ and $H_k^{\perp}$ denotes orthogonal completement of H_k.

Lemma 2.5. *For all $u \in H_k$, $k \in \mathbb{N}$, we have*
$$\|u\|^2_{H_0^s(\Omega)} \le \lambda_k |u|_2^2.$$

This is so because $u \in H_k$, we have $u = \sum_{i=1}^k u_i e_i$ with $u_i \in \mathbb{R}$, $i = 1, \dots, k$. Then
$$\|u\|^2_{H_0^s(\Omega)} = \sum_{i=1}^k u_i^2 \|e_i\|^2_{H_0^s(\Omega)} = \sum_{i=1}^k \lambda_i u_i^2 \le \lambda_k \sum_{i=1}^k u_i^2 = \lambda_k |u|_2^2. \tag{2.6}$$

Lemma 2.6. *For all $u \in H_k^{\perp}$, $k \in \mathbb{N}$, we have*
$$\|u\|^2_{H_0^s(\Omega)} \ge \lambda_{k+1} |u|_2^2. \tag{2.7}$$

For $u \in H_k^{\perp}$, $u = \sum_{i=k}^{\infty} u_i e_i$, it is straightforward to check
$$\|u\|^2_{H_0^s(\Omega)} = \sum_{i=k}^{\infty} u_i^2 \|e_i\|^2_{H_0^s(\Omega)} = \sum_{i=k}^{\infty} \lambda_i u_i^2 \ge \lambda_{k+1} \sum_{i=k}^{\infty} u_i^2 = \lambda_{k+1} |u|_2^2.$$

Lemma 2.7 ([16]). *Let $s \in (0,1)$, $n > 2s$, and Ω be an open bounded set of $\mathbb{R}^n$. The embedding $H_0^s(\Omega) \hookrightarrow L^q(\Omega)$, $q \le 2^* = \frac{2n}{n-2s}$, is continuous. Moreover, the embedding $H_0^s(\Omega) \hookrightarrow L^q(\Omega)$, $q < 2^*$, is compact.*

Lemma 2.8 (Weak maximum principle [11]). *If $u \in H_0^s(\Omega)$ is a weak supersolution of (1.1) with $f = 0$, then $u \ge 0$ a.e. in Ω and u admits a lower semi-continuous representative in Ω.*

Lemma 2.9 (Strong maximum principle [11]). *If $u \in H_0^s(\Omega) \setminus \{0\}$ is a weak supersolution of (1.1) with $f = 0$, then $u > 0$ in Ω.*

Lemma 2.10 (Local minimizer [11]). *Let Ω be a bounded $\mathcal{C}^{1,1}$ domain, f satisfies* (H1)–(H5), *and $u_0 \in H_0^s(\Omega)$. Then, the following two assertions are equivalent:*

(i) *there exists $\rho > 0$ such that $J(u_0 + v) \ge J(u_0)$ for all $v \in H_0^s(\Omega) \cap C_\delta^0(\overline{\Omega})$, $\|v\|_{0,\delta} \le \rho$;*
(ii) *there exists $\varepsilon > 0$ such that $J(u_0+v) \ge J(u_0)$ for all $v \in H_0^s(\Omega)$, $\|v\|_{H_0^s(\Omega)} \le \varepsilon$.*

Let us recall some basic facts about variational methods and Morse theory [5, 13]. Suppose that (X, Y) is a pair of topological spaces with $Y \subset X$. We call $Y \subset X$ a topological pair. Let $Y_2 \subset Y_1 \subset X$ and $k \ge 0$ is an integer. We denote the kth singular homology group by $H_k(Y_1, Y_2)$ for the pair (Y_1, Y_2) with integer coefficients.

The critical groups of φ at an isolated critical point $u_0 \in X$ with $\varphi(u_0) = c$ are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u_0\}),\ \forall k \geq 0,$$

where U is a neighborhood of u_0 such that $K \cap \varphi^c \cap U = \{u_0\}$ with $K = \{u \in X | \varphi'(u) = 0\}$ and $\varphi^c = \{u \in X : \varphi(u) \leq c\}$. If φ satisfies the $(P - S)$ condition and the critical values of φ are bounded from below by some $a > -\infty$, then the critical groups of φ at infinity are given by [2]:

$$C_k(\varphi, \infty) = H_k(X, \varphi^a), \quad k \geq 0,$$

By the deformation lemma, $H_k(X, \varphi^a)$ does not depend on the choice of a. We have the Morse inequality

$$\sum_{k=0}^{\infty} M_k t^k = \sum_{k=0}^{\infty} \beta_k t^k + (1 + t)Q(t),$$

where Q is a formal series with non-negative coefficients in $\mathbb{N}$,

$$M_k = \sum_{\varphi'(u)=0} \mathrm{rank} C_k(\varphi, u), \quad \beta_k = \mathrm{rank} C_k(\varphi, \infty).$$

Lemma 2.11 ([5]). *Suppose that $\varphi \in \mathcal{C}^1(X, \mathbb{R})$ satisfies the $P - S$ condition, and φ has only finitely many critical points. If for some $k \in \mathbb{N}$, we have $C_k(\varphi, \infty) \neq 0$, then φ has a critical point u with $C_k(\varphi, u) \neq 0$.*

Lemma 2.12 ([12]). *Suppose that $\varphi \in \mathcal{C}^2(X, \mathbb{R})$, and u_0 is an isolated critical point with the finite Morse index μ and nullity ν. If $\varphi''(u_0)$ is a Fredholm operator, then $C_k(\varphi, u_0) = 0$ for $k \notin [\mu, \mu + \nu]$. Moreover, it holds*

(1) $C_\mu(\varphi, u_0) \neq 0$ implies

$$C_k(\varphi, u_0) = \begin{cases} G & k = \mu, \\ 0 & k \neq \mu. \end{cases} \tag{2.8}$$

(2) $C_{\mu+\nu}(\varphi, u_0) \neq 0$ implies

$$C_k(\varphi, u_0) = \begin{cases} G & k = \mu + \nu, \\ 0 & k \neq \mu + \nu. \end{cases} \tag{2.9}$$

3. Main result

We prove a technical lemma before giving the proof of the main theorem.

Lemma 3.1. *If* (H1) *and* (H3) *hold, then J satisfies the P-S condition.*

Proof. We only need to show that $|J(u_n)| \leq C$ and $J'(u_n) \to 0$ as $n \to \infty$ implies that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence. Since $H_0^s(\Omega) = H_m \bigoplus H_m^\perp$, there exists

$$v_n \in H_m \quad \text{and} \quad \omega_n \in H_m^\perp$$

such that $u_n = v_n + \omega_n$, where m is given by (H3). From $J'(u_n) \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\langle J'(u_n), h\rangle \leq \|h\|$, for $n \geq N$ and $h \in H_0^1(\Omega)$. Set $h = -v_n + \omega_n$. Then we have

$$\langle J'(u_n), h\rangle = \langle J'(v_n + \omega_n), -v_n + \omega_n\rangle \leq \| - v_n + \omega_n\|_{H_0^s} = \|v_n + \omega_n\|_{H_0^s},$$

and

$$\langle J'(v_n + \omega_n), -v_n + \omega_n\rangle = -\|v_n\|_{H_0^s}^2 + \|\omega_n\|_{H_0^s}^2 - \int_\Omega f(v_n + \omega_n)(-v_n + \omega_n)dx.$$

This implies

$$-\|v_n\|_{H_0^s}^2 + \|\omega_n\|_{H_0^s}^2 \le \|v_n+\omega_n\|_{H_0^s} + \int_\Omega f(v_n+\omega_n)(-v_n+\omega_n)dx. \tag{3.1}$$

By (H3), we may find a function g satisfying $f(t)=\alpha t+g(t)$ and $\frac{g(t)}{t}\to 0$ as $|t|\to\infty$, where $\alpha\in(\lambda_m,\lambda_{m+1})$. In view of this fact and (3.1), we have

$$\begin{aligned}
&\|\omega_n\|_{H_0^s}^2 - \alpha\int_\Omega |\omega_n|^2dx - \|v_n\|_{H_0^s}^2 + \alpha\int_\Omega |v_n|^2dx \\
&\le \|v_n+\omega_n\|_{H_0^s} + \int_\Omega g(v_n+\omega_n)(-v_n+\omega_n)dx.
\end{aligned}$$

By (2.4) and (2.5), it follows that

$$\begin{aligned}
&(1-\frac{\alpha}{\lambda_{m+1}})\|\omega_n\|_{H_0^s}^2 + (\frac{\alpha}{\lambda_m}-1)\|v_n\|_{H_0^s}^2 \\
&\le \|v_n+\omega_n\|_{H_0^s} + \int_\Omega g(v_n+\omega_n)(-v_n+\omega_n)dx.
\end{aligned} \tag{3.2}$$

Next, we deal with term $\int_\Omega g(v_n+\omega_n)(-v_n+\omega_n)dx$ of (3.2). Since the continuity of g and $\frac{g(t)}{t}\to 0$ as $|t|\to\infty$, we have

$$\begin{gathered}
|\frac{g(t)}{t}|<\varepsilon \quad \text{if } |t|>M, \\
|g(t)|\le K \quad \text{if } |t|\le M,
\end{gathered}$$

for some constants $M,K>0$. By the Sobolev embedding $H_0^s(\Omega)\hookrightarrow L^p(\Omega)$, $1\le p<\frac{2n}{n-2s}$ (see lemma 2.7), we get

$$\begin{aligned}
&\int_\Omega g(v_n+\omega_n)(-v_n+\omega_n)dx \\
&\le \int_{|u_n|>M} |g(v_n+\omega_n)||-v_n+\omega_n|dx + \int_{|u_n|\le M} |g(v_n+\omega_n)||-v_n+\omega_n|dx \\
&\le \varepsilon\int_{|u_n|>M} |v_n+\omega_n||-v_n+\omega_n|dx + K\int_{|u_n|\le M} |-v_n+\omega_n|dx \\
&\le \varepsilon C_2^2\|v_n+\omega_n\|_{L^2(\Omega)}\|-v_n+\omega_n\|_{L^2(\Omega)} + K\int_\Omega |-v_n+\omega_n|dx \\
&\le \varepsilon C_2^2\|v_n+\omega_n\|_{H_0^s}\|-v_n+\omega_n\|_{H_0^s} + KC_1\|-v_n+\omega_n\|_{H_0^s} \\
&= \varepsilon C_2^2\|v_n+\omega_n\|_{H_0^s}^2 + KC_1\|v_n+\omega_n\|_{H_0^s}.
\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$(1-\frac{\alpha}{\lambda_{m+1}}-\varepsilon C_2^2)\|\omega_n\|_{H_0^s}^2 + (\frac{\alpha}{\lambda_m}-1-\varepsilon C_2^2)\|v_n\|_{H_0^s}^2 \le (1+KC_1)\|v_n+\omega_n\|_{H_0^s}. \tag{3.4}$$

In view of $\alpha\in(\lambda_m,\lambda_{m+1})$, we may set ε small enough such that

$$1-\frac{\alpha}{\lambda_{m+1}}-\varepsilon C_2^2>0 \quad \text{and} \quad \frac{\alpha}{\lambda_m}-1-\varepsilon C_2^2>0.$$

It follows that $\{v_n\}$ and $\{\omega_n\}$ are bounded sequences. Thus, $\{u_n\}$ is a bounded sequence. $\square$

Proof of Theorem 1.1. We divide the proof into three steps. We only deal with $u \in \operatorname{int} C_+$. The case of $u \in -\operatorname{int} C_+$ is similar.

Step 1. Find a positive local minimizer. From (H4), we have $f(t) = \beta t + o(t)$ as $t \to 0$ with $t > 0$ and $\beta \in (\lambda_1, \lambda_2)$. So it holds

$$\begin{gathered}(-\Delta)^s u = f(u) = \beta u + o(u) > \lambda_1 u + o(u) > \lambda_1 \varepsilon e_1, \quad x \in \Omega, \\ u = 0, \quad x \in \mathcal{C}\Omega,\end{gathered} \tag{3.5}$$

where $e_1 > 0$ is the first eigenfunction of the problem

$$\begin{gathered}(-\Delta)^s u = \lambda u, \quad x \in \Omega, \\ u = 0, \quad x \in \mathcal{C}\Omega,\end{gathered}$$

with $\max_{\overline{\Omega}} e_1 = 1$ for ε small enough. Then $\underline{u} = \varepsilon e_1$ is a strict positive subsolution of the problem (1.1). We consider the following truncation of f,

$$\hat{f}(t) = \begin{cases} f(\underline{u}) & \text{if } t < \underline{u}, \\ f(t) & \text{if } \underline{u} \le t \le \xi_+, \\ f(\xi_+) & \text{if } t > \xi_+, \end{cases} \tag{3.6}$$

and $\hat{F}(u) = \int_0^u \hat{f}(t)dt$, and ξ_+ is given by (H5). Then the $\mathcal{C}^1$ function $\widehat{J} : H_0^s \to \mathbb{R}$ is defined by

$$\begin{aligned}\widehat{J}(u) &= \frac{1}{2} \int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy - \int_\Omega \hat{F}(u) dx \\ &= \frac{1}{2} \int_Q |u(x) - u(y)|^2 K(x - y)\, dx\, dy - \int_\Omega \hat{F}(u) dx.\end{aligned}$$

We claim that if u is a critical point of $\widehat{J}$, then $u \in \operatorname{int} C_+$ and $\underline{u} < u < \xi_+$ for all $x \in \Omega$. In fact, since $u \in H_0^s(\Omega)$ is a critical point of $\widehat{J}$, we have $\widehat{J}'(u) = 0$, i.e.

$$\int_\Omega (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x - y)\, dx\, dy = \int_\Omega \hat{f}(u)\psi dx, \tag{3.7}$$

with $\psi \in H_0^s$ and $\psi \ge 0$. Set $\psi = (\underline{u} - u)^+$ in (3.7), then

$$\begin{aligned}&\int_Q (u(x) - u(y))[(\underline{u} - u)^+(x) - (\underline{u} - u)^+(y)] K(x - y)\, dx\, dy \\ &= \int_\Omega \hat{f}(u)(\underline{u} - u)^+ dx \\ &= \int_{\{\underline{u} > u\}} f(\underline{u})(\underline{u} - u)^+ dx.\end{aligned}$$

From the subsolution of (1.1), we have

$$\begin{aligned}&\int_Q (u(x) - u(y))[(\underline{u} - u)^+(x) - (\underline{u} - u)^+(y)] K(x - y)\, dx\, dy \\ &\le \int_\Omega f(\underline{u})(\underline{u} - u)^+ dx \\ &= \int_{\{\underline{u} > u\}} f(\underline{u})(\underline{u} - u)^+ dx.\end{aligned}$$

Moreover, we get

$$\begin{aligned}
&\int_Q (\underline{u}(x)-\underline{u}(y))[(\underline{u}-u)^+(x)-(\underline{u}-u)^+(y)]K(x-y)\,dx\,dy\\
&\le \int_Q (u(x)-u(y))[(\underline{u}-u)^+(x)-(\underline{u}-u)^+(y)]K(x-y)\,dx\,dy\\
&\le \int_{\{\underline{u}>u\}} (u(x)-u(y))[\underline{u}(x)-u(x)-\underline{u}(y)+u(y)]K(x-y)\,dx\,dy.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{\{\underline{u}>u\}} (\underline{u}(x)-\underline{u}(y))[(\underline{u}(x)-u(x)-\underline{u}(y)+u(y)]K(x-y)\,dx\,dy\\
&\le \int_{\{\underline{u}>u\}} (u(x)-u(y))[\underline{u}(x)-u(x)-\underline{u}(y)+u(y)]K(x-y)\,dx\,dy,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\{\underline{u}>u\}} [(\underline{u}(x)-\underline{u}(y))^2-(\underline{u}(x)-\underline{u}(y))(u(x)-u(y))]K(x-y)\,dx\,dy\\
&\le \int_{\{\underline{u}>u\}} [(u(x)-u(y))(\underline{u}(x)-\underline{u}(y))-(u(x)-u(y))^2]K(x-y)\,dx\,dy.
\end{aligned}$$

From the above inequalities, we obtain

$$\begin{aligned}
&\int_{\{\underline{u}>u\}} \big[(\underline{u}(x)-\underline{u}(y))^2-2(u(x)-u(y))(\underline{u}(x)-\underline{u}(y))\\
&\quad +(u(x)-u(y))^2\big]K(x-y)\,dx\,dy\le 0.
\end{aligned}$$

This implies

$$\int_{\{\underline{u}>u\}} |(\underline{u}(x)-\underline{u}(y)-u(x)+u(y)|^2K(x-y)\,dx\,dy\le 0.$$

Then we have $m\{x\in\Omega:\underline{u}(x)>u(x)\}=0$ i.e. $u\ge\underline{u}>0$, a.e. Ω.

By the regularity of elliptic equations and the strong maximum principle, we obtain $u\in \mathrm{int}C_+$ and $\underline{u}<u$ for all $x\in\Omega$. Similarly, set $\psi=(u-\xi_+)^+$ in (3.7). From $f(\xi_+)<0$ and $(u-\xi_+)^+\ge 0$, we deduce that

$$m\{x\in\Omega: u>\xi_+\}=0,$$

i.e. $u\le\xi_+$ a.e. Ω.

Again, by the regularity of elliptic equations and the strong maximum principle or Lemma 2.8, we have $u(x)<\xi_+$ for all $x\in\Omega$. It means that ξ_+ is a strict supersolution of (1.1). Set $\bar{u}=\xi_+$. By the definition of $\hat{f}$, u is also a critical point of J. Then $\underline{u}$ and $\bar{u}$ are a pair of positive strict sub-supersolutions of the problem (1.1). By Lemmas 2.11 and 2.12, we have

$$C_k(J,u_1)=\begin{cases} G & k=0,\\ 0 & k\ne 0,\end{cases} \tag{3.8}$$

(or see [11]). Obviously, as in the previous proof, $\xi_+>u_1>\underline{u}>0$ holds for all $x\in\Omega$ and $u_1\in\mathrm{int}C_+$.

Step 2. Find a positive mountain pass point. The truncation of f is

$$\widetilde{f}(t) = \begin{cases} f(\underline{u}) & \text{if } t < \underline{u}, \\ f(t) & \text{if } \underline{u} \le t, \end{cases} \tag{3.9}$$

and $\widetilde{F}(u) = \int_0^u \widetilde{f}(t)dt$. The functional $\widetilde{J} : H_0^s \to \mathbb{R}$, is defined by

$$\widetilde{J}(u) = \frac{1}{2}\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \int_\Omega \widetilde{F}(u)dx.$$

Obviously, the local minimizer u_1 of $\widehat{J}$ is also a local minimizer of $\widetilde{J}$. In view of (H3) and mountain pass lemma (see [5]), we have a mountain pass critical point u_2 of $\widetilde{J}$. As in the previous proof, we obtain

$$u_2 \in \operatorname{int} C_+ \quad \text{and} \quad \underline{u} < u_2(x)$$

for all $x \in \Omega$. u_2 is a mountain pass critical point of J:

$$C_k(J, u_2) = \begin{cases} G & k = 1, \\ 0 & k \neq 1. \end{cases} \tag{3.10}$$

For the case of $u \in -\operatorname{int} C_+$, we can find $\underline{u} = \xi_-$ and $\bar{u} = -\varepsilon\varphi_1$ are a pair of positive strict sub-supersolutions of the problem (1.1), a local minimizer $u_3 \in -\operatorname{int} C_+$ and a mountain pass critical point $u_4 \in -\operatorname{int} C_+$.

Step 3. For computing the critical groups at 0 and ∞, we get the six nontrivial solutions by the Morse inequality. From (H4), we have

$$C_k(J, 0) = \begin{cases} G & k = 1, \\ 0 & k \neq 1. \end{cases}$$

In fact, since $\lambda_1 < \beta < \lambda_2$, the negative space of the operator

$$\mathrm{d}^2 J(0) = (-\Delta)^{-s}((-\Delta)^s - f'(0)I) = (-\Delta)^{-s}(\lambda I - \beta I)$$

has 1 dimension and 0 is isolated critical point.

Denote $d := \dim H_m = \dim(\bigoplus_{k=1}^m V_k)$. Then from (H3) we have

$$C_k(J, \infty) = \begin{cases} G & k = d \ge 2, \\ 0 & k \neq d. \end{cases}$$

The proof is similar to $C_k(J, 0)$. By Lemmas 2.11 and 2.12, there exists a critical point u_5 with $C_d(J, u_5) \neq 0$ and

$$C_k(J, u_5) = \begin{cases} G & k = d \ge 2, \\ 0 & k \neq d. \end{cases}$$

Since

$$C_k(J, u_1) = C_k(J, u_3) = \begin{cases} G & k = 0, \\ 0 & k \neq 0, \end{cases}$$

and

$$C_k(J, 0) = C_k(J, u_2) = C_k(J, u_4) = \begin{cases} G & k = 1, \\ 0 & k \neq 1, \end{cases}$$

we obtain $u_5 \neq 0, u_1, u_2, u_3, u_4$. Suppose that there are only six solutions 0, u_1, u_2, u_3, u_4 and u_5, then a contradiction would occur due to the Morse inequality:

$$(-1)^0 \times 2 + (-1)^1 \times 3 + (-1)^d \times 1 = (-1)^d.$$

Namely, it yields a contradiction $-1 = 0$. This implies that there exists other critical point u_6, different from 0, u_1, u_2, u_3, u_4 and u_5. Consequently, problem (1.1) has six nontrivial solutions u_1, u_2, u_3, u_4, u_5 and u_6. $\square$

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Wei Qi
School of physical science and technology, Xuzhou, Jiangsu 221008, China
E-mail address: qiw04@cumt.edu.cn

Lin Zhao (corresponding author)
Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, China
E-mail address: zhaolinmath@yeah.net

Xingjie Yan
Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, China
E-mail address: yanxj04@163.com