Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 212, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

STURM-LIOUVILLE OPERATOR WITH PARAMETER-DEPENDENT BOUNDARY CONDITIONS ON TIME SCALES

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Communicated by Jerome A. Goldstein

Abstract. In this study, we consider a boundary-value problem generated by a Sturm-Liouville dynamic equation on a time scale with boundary conditions depending on a spectral parameter. We introduce the operator formulation of the problem and give some properties of eigenvalues and eigenfunctions. We also formulate the number of eigenvalues when the time scale is finite.

1. Introduction and Preliminaries

Time scale theory was introduced by Hilger in 1988. He gave a new derivation in order to unify continuous and discrete analysis [19]. From then on this approach has received a lot of attention and has applied quickly to various area in mathematics. Sturm-Liouville theory on time scales was studied first by Erbe and Hilger [13] in 1993. Some important results on the properties of eigenvalues and eigenfunctions of the classical Sturm-Liouville problem on time scales were given in various publications (see e.g. [1, 3, 4, 10, 11, 12, 14, 17, 18, 20, 21, 22, 27, 28, 29] and the references therein).

In classical analysis, Sturm-Liouville problems with boundary conditions which depend on the parameter were studied extensively. These kinds of problems appear in physics, mechanics and engineering. There is a vast literature about this subject. Particularly, an operator-theoretic formulation of the problem with the spectral parameter contained linearly in the boundary conditions has been given in [15, 16, 30]. Oscillation and comparison results and some other properties of the eigenvalues have been obtained in [6, 7, 23]. Basis properties and eigenfunction expansions have been considered in [24, 25, 26]. For Sturm-Liouville problem with eigenparameter-dependent boundary conditions on arbitrary time scale we refer to the study [2] and the references therein.

In this article, Sturm-Liouville dynamic equation with boundary conditions depending on the spectral parameter on a time scale is studied. We define an operator which is appropriate to this boundary-value problem. We prove that all eigenvalues

2010 *Mathematics Subject Classification.* 34B05, 34L05, 39A05.
Key words and phrases. Time scale; Sturm-Liouville operator; parameter-dependent boundary conditions.

Submitted May 18, 2017. Published September 11, 2017.

are real, algebraically simple and two eigenfunctions, corresponding to the different eigenvalues are orthogonal. We also obtain a formula to give the number of eigenvalues of the problem which is constructed on a finite time scale.

Before presenting our main results, we recall the some important concepts of the time scale theory. For further knowledge, the reader is referred to [5, 8, 9].

If $\mathbb{T}$ is a closed subset of $\mathbb{R}$ it is called as a time scale. The jump operators σ, ρ and graininess operator μ on $\mathbb{T}$ are defined as follows:

$$\begin{gathered}
\sigma:\mathbb{T}\to\mathbb{T},\quad \sigma(t)=\inf\{s\in\mathbb{T}: s>t\}\quad \text{if } t\neq\sup\mathbb{T},\\
\rho:\mathbb{T}\to\mathbb{T},\ \rho(t)=\sup\{s\in\mathbb{T}: s<t\}\quad \text{if } t\neq\inf\mathbb{T},\\
\sigma(\sup\mathbb{T})=\sup\mathbb{T},\quad \rho(\inf\mathbb{T})=\inf\mathbb{T},\\
\mu:\mathbb{T}\to[0,\infty)\quad \mu(t)=\sigma(t)-t.
\end{gathered}$$

A point of $\mathbb{T}$ is called as left-dense, left-scattered, right-dense, right-scattered and isolated if $\rho(t)=t$, $\rho(t)<t$, $\sigma(t)=t$, $\sigma(t)>t$ and $\rho(t)<t<\sigma(t)$, respectively.

A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous on $\mathbb{T}$ if it is continuous at all right-dense points and has left-sided limits at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$ or C_{rd}. If f is continuous on $\mathbb{T}$ it is also rd-continuous. Put

$$\mathbb{T}^k:=\begin{cases}\mathbb{T}-\{\sup\mathbb{T}\}, & \sup\mathbb{T}<\infty \text{ is left-scattered}\\ \mathbb{T}, & \text{otherwise,}\end{cases}$$

$\mathbb{T}^{k^2}:=(\mathbb{T}^k)^k$.

Let $t\in\mathbb{T}^k$. Suppose that for given any $\varepsilon>0$, there exists a neighborhood $U=(t-\delta,t+\delta)\cap\mathbb{T}$ such that

$$|[f(\sigma(t))-f(s)]-f^{\Delta}(t)[\sigma(t)-s]|\leq\varepsilon|\sigma(t)-s|$$

for all $s\in U$. Then f is called Δ-differentiable at $t\in\mathbb{T}^k$. We call $f^{\Delta}(t)$ the Δ-derivative of f at t. A function $F:\mathbb{T}\to\mathbb{R}$ defined as $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}^k$ is called an antiderivative of f on $\mathbb{T}$. In this case, the Cauchy integral of f is defined by

$$\int_a^b f(t)\triangle t=F(b)-F(a) \text{ for } a,b\in\mathbb{T}.$$

We collect some necessary relations in the following lemma. Their proofs can be found in [8, Chapter 1].

Lemma 1.1. *Let $f:\mathbb{T}\to\mathbb{R}$, $g:\mathbb{T}\to\mathbb{R}$ be two functions and $t\in\mathbb{T}^k$.*

- (i) *if $f^{\Delta}(t)$ exists, then f is continuous at t;*
- (ii) *if t is right-scattered and f is continuous at t, then f is Δ-differentiable at t and $f^{\Delta}(t)=\frac{f^{\sigma}(t)-f(t)}{\sigma(t)-t}$, where $f^{\sigma}(t)=f(\sigma(t))$;*
- (iii) *if $f^{\Delta}(t)$ exists, then $f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t)$; if $f^{\Delta}\equiv0$, then f is constant;*
- (iv) *if $f^{\Delta}(t)$ and $g^{\Delta}(t)$ exist, then $(f\pm g)^{\Delta}(t)=f^{\Delta}(t)\pm g^{\Delta}(t)$, $(fg)^{\Delta}(t)=(f^{\Delta}g+f^{\sigma}g^{\Delta})(t)$ and if $(gg^{\sigma})(t)\neq0$, then $(\frac{f}{g})^{\Delta}(t)=(\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}})(t)$;*
- (v) *if $f\in C_{rd}(\mathbb{T})$, then it has an antiderivative on $\mathbb{T}$.*

Now, let us recall the spaces L_p and H_1. For detailed knowledge related to Lebesgue measure, Lebesgue integration and generalized derivation on the time scale we refer to [10].

$L_p(\mathbb{T}) = \{f : |f|^p$ is integrable on $\mathbb{T}$ in Lebesgue sense$\}$ is a Banach space with the norm $\|f\|_{L_p} = \left(\int_a^b |f(t)|^p \triangle t\right)^{1/p}$. Moreover, the set $C^1_{rd}(\mathbb{T}) = \{f : f\Delta$-differentiable on $\mathbb{T}$ and $f^\Delta \in C_{rd}(\mathbb{T}^k)\}$ is a normed space with the norm $\|f\|_1 := \|f\|_{L_2} + \|f^\Delta\|_{L_2}$. Finally, the Sobolev space H_1 is defined to be the completion of $C^1_{rd}(\mathbb{T})$ with respect to the norm $\|\cdot\|_1$. It is proven in [10] that if $f \in H_1$, then there exists the generalized derivative f^{Δ_g} of f in $L_2(\mathbb{T})$ and the following properties are valid.

(i) If $f \in H_1$, then the function f^{Δ_g} is unique in Lebesgue sense.
(ii) If $f \in C^1_{rd}(\mathbb{T})$, then $f^{\Delta_g} = f^\Delta$.
(iii) item (ii)-(iv) in Lemma 1.1 are valid with f^{Δ_g} instead of f^Δ.
(iv) $\int_a^b f^{\Delta_g}(t)\triangle t = f(b) - f(a)$ for $a, b \in \mathbb{T}$.

2. Main Results

Throughout this paper we assume that $\mathbb{T}$ is a bounded time scale. Let us consider the boundary-value problem

$$\ell y := -y^{\Delta\Delta}(t) + q(t)y^\sigma(t) = \lambda y^\sigma(t), \quad t \in \mathbb{T}^{k^2} \tag{2.1}$$

$$U(y) := (a_1\lambda + a_0)y^\Delta(\alpha) - (b_1\lambda + b_0)y(\alpha) = 0 \tag{2.2}$$

$$V(y) := (c_1\lambda + c_0)y^\Delta(\beta) - (d_1\lambda + d_0)y(\beta) = 0, \tag{2.3}$$

where $y^{\Delta\Delta} = (y^\Delta)^{\Delta_g}$, $q(t)$ is real valued continuous function on $\mathbb{T}$, $a_i, b_i, c_i, d_i \in \mathbb{R}$, $i = 0, 1$, $\alpha = \inf \mathbb{T}$, $\beta = \rho(\sup \mathbb{T})$ and λ is a spectral parameter. Additionally, we assume that $\alpha \neq \rho(\beta)$, $K_1 := a_1b_0 - a_0b_1 > 0$ and $K_2 := c_0d_1 - c_1d_0 > 0$.

Definition 2.1. The values of the parameter for which equation (2.1) has nonzero solutions satisfying (2.2) and (2.3), are called eigenvalues and the corresponding nontrivial solutions are called eigenfunctions.

Let the inner product in the Hilbert Space $H = L_2(\mathbb{T}^k) \oplus \mathbb{C}^2$ be defined by

$$\langle Y, Z\rangle := \int_\alpha^\beta y(t)\overline{z(t)}\Delta t + \frac{1}{K_1}Y_\alpha\overline{Z_\alpha} + \frac{1}{K_2}Y_\beta\overline{Z_\beta}$$

for

$$Y = (y(t), Y_\alpha, Y_\beta), \quad Z = (z(t), Z_\alpha, Z_\beta) \in H.$$

Define an operator L with the domain $D(L) = \{Y \in H : y(t) \in C^1_{rd}(\mathbb{T}),\ y^\Delta(t) \in H_1(\mathbb{T}^k),\ Y_\alpha = a_1y^\Delta(\alpha) - b_1y(\alpha),\ Y_\beta = c_1y^\Delta(\beta) - d_1y(\beta)\}$ such that

$$\begin{gathered} L(Y) = (z(t), Z_\alpha, Z_\beta), \\ z(t) = -y^{\Delta\Delta}(t) + q(t)y^\sigma(t), \\ Z_\alpha = b_0y(\alpha) - a_0y^\Delta(\alpha), \\ Z_\beta = d_0y(\beta) - c_0y^\Delta(\beta) \end{gathered}$$

It is obvious that the eigenvalue problem $L(Y) = \lambda Y^\sigma$ coincide with the problem (2.1)-(2.3).

Theorem 2.2. *The relation $\langle LY, Z^\sigma\rangle = \langle Y^\sigma, LZ\rangle$ holds for all $Y, Z \in D(L)$.*

Proof. Let $Y, Z \in D(L)$ be given as

$$\begin{aligned}
Y &= (y(t), a_1 y^{\Delta}(\alpha) - b_1 y(\alpha), c_1 y^{\Delta}(\beta) - d_1 y(\beta)),\\
Z &= (z(t), a_1 z^{\Delta}(\alpha) - b_1 z(\alpha), c_1 z^{\Delta}(\beta) - d_1 z(\beta)).
\end{aligned}$$

It can be calculated by two partial integrations that

$$\begin{aligned}
\langle LY, Z^{\sigma}\rangle - \langle Y^{\sigma}, LZ\rangle &= \int_{\alpha}^{\beta} y^{\sigma}(t)\overline{z}^{\Delta\Delta}(t)\Delta t - \int_{\alpha}^{\beta} y^{\Delta\Delta}(t)\overline{z^{\sigma}}(t)\Delta t\\
&\quad + \frac{1}{K_1}[b_0 y(\alpha) - a_0 y^{\Delta}(\alpha)][a_1 \overline{z}^{\Delta}(\alpha) - b_1 \overline{z}(\alpha)]\\
&\quad - \frac{1}{K_1}[a_1 y^{\Delta}(\alpha) - b_1 y(\alpha)][b_0 \overline{z}(\alpha) - a_0 \overline{z}^{\Delta}(\alpha)]\\
&\quad + \frac{1}{K_2}[d_0 y(\beta) - c_0 y^{\Delta}(\beta)][c_1 \overline{z}^{\Delta}(\beta) - d_1 \overline{z}(\beta)]\\
&\quad - \frac{1}{K_2}[c_1 y^{\Delta}(\beta) - d_1 y(\beta)][d_0 \overline{z}(\beta) - c_0 \overline{z}^{\Delta}(\beta)] = 0.
\end{aligned}$$

□

Corollary 2.3. *All eigenvalues of problem* (2.1)-(2.3) *are real numbers and two eigenfunctions $y(t)$, $z(t)$ corresponding to different eigenvalues λ_1, λ_2 are orthogonal, i.e.,*

$$\begin{aligned}
&\int_{\alpha}^{\beta} y(t)\overline{z(t)}\Delta t + \frac{1}{K_1}[a_1 y^{\Delta}(\alpha) - b_1 y(\alpha)][a_1 z^{\Delta}(\alpha) - b_1 z(\alpha)]\\
&+ \frac{1}{K_2}[c_1 y^{\Delta}(\beta) - d_1 y(\beta)][c_1 z^{\Delta}(\beta) - d_1 z(\beta)] = 0
\end{aligned}$$

Let $s(t,\lambda)$, $c(t,\lambda)$, $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ be the solutions of (2.1) under the initial conditions

$$s(\alpha,\lambda) = 0, \quad s^{\Delta}(\alpha,\lambda) = 1, \tag{2.4}$$

$$c(\alpha,\lambda) = 1, \quad c^{\Delta}(\alpha,\lambda) = 0, \tag{2.5}$$

$$\varphi(\alpha,\lambda) = a_1\lambda + a_0, \quad \varphi^{\Delta}(\alpha,\lambda) = b_1\lambda + b_0, \tag{2.6}$$

$$\psi(\beta,\lambda) = c_1\lambda + c_0, \quad \psi^{\Delta}(\beta,\lambda) = d_1\lambda + d_0, \tag{2.7}$$

respectively. It is clear that the Wronskian

$$W[c(t,\lambda), s(t,\lambda)] = c(t,\lambda)s^{\Delta}(t,\lambda) - c^{\Delta}(t,\lambda)s(t,\lambda)$$

does not depend on t. From (2.4) and (2.5) we have $W[c(t,\lambda), s(t,\lambda)] = 1$ and so $s(t,\lambda)$ and $c(t,\lambda)$ are linearly independent. Moreover, it can be seen that the relation $\varphi(t,\lambda_n) = \chi_n\psi(t,\lambda_n)$ is valid for each eigenvalue λ_n, where

$$\chi_n = \frac{[d_1\varphi(\beta,\lambda_n) - c_1\varphi^{\Delta}(\beta,\lambda_n)]}{K_2}.$$

Theorem 2.4. *$s(t,\lambda)$, $c(t,\lambda)$, $\varphi(t,\lambda)$, $\psi(t,\lambda)$ and their Δ-derivatives are entire functions of λ for each fixed t.*

Proof. Let us prove that $\varphi(t,\lambda)$ is analytic for arbitrary $\lambda_0 \in \mathbb{C}$ and fixed $t \in \mathbb{T}^{k^2}$. Analyticity of the other functions can be proven similarly. Put $\phi(t,\lambda,\lambda_0) =$

$\varphi(t,\lambda)-\varphi(t,\lambda_0)$. From (2.1) and (2.6), the function $\phi(t,\lambda,\lambda_0)$ is the solution of the initial-value problem

$$\begin{gathered}
-\phi^{\Delta\Delta}(t)+q(t)\phi^{\sigma}(t)=\lambda_0\phi^{\sigma}(t)+(\lambda-\lambda_0)\varphi^{\sigma}(t,\lambda)\\
\phi(\alpha)=a_1(\lambda-\lambda_0)\\
\phi^{\Delta}(\alpha)=b_1(\lambda-\lambda_0).
\end{gathered}\tag{2.8}$$

From the variation of parameters formula, we have

$$\begin{aligned}
\phi(t,\lambda,\lambda_0)&=Ac(t,\lambda_0)+Bs(t,\lambda_0)\\
&\quad+\int_{\alpha}^{t}[c^{\sigma}(\xi,\lambda_0)s(t,\lambda_0)-s^{\sigma}(\xi,\lambda_0)c(t,\lambda_0)](\lambda-\lambda_0)\varphi^{\sigma}(\xi,\lambda)\Delta t
\end{aligned}$$

where $A=a_1(\lambda-\lambda_0)$ and $B=(\lambda-\lambda_0)[b_1+\mu(\alpha)\varphi^{\sigma}(\alpha,\lambda)]$. Therefore,

$$\begin{aligned}
\frac{\varphi(t,\lambda)-\varphi(t,\lambda_0)}{\lambda-\lambda_0}&=a_1c(t,\lambda_0)+[b_1+\mu(\alpha)\varphi^{\sigma}(\alpha,\lambda)]s(t,\lambda_0)+\\
&\quad+\int_{\alpha}^{t}[c^{\sigma}(\xi,\lambda_0)s(t,\lambda_0)-s^{\sigma}(\xi,\lambda_0)c(t,\lambda_0)]\varphi^{\sigma}(\xi,\lambda)\Delta t.
\end{aligned}$$

Since the solution $\varphi(t,\lambda)$ is continuous, according to λ, it follows that $\varphi(t,\lambda)$ is also analytic on λ. □

Put

$$\Delta(\lambda)=W[\psi,\varphi]=(c_1\lambda+c_0)\varphi^{\Delta}(\beta,\lambda)-(d_1\lambda+d_0)\varphi(\beta,\lambda).\tag{2.9}$$

From Theorem 2.4, $\Delta(\lambda)$ is an entire function.

Theorem 2.5. *The zeros of the function $\Delta(\lambda)$ coincide with the eigenvalues of the problem* (2.1)-(2.3).

Proof. It is clear that, if $\Delta(\lambda_0)=0$ for a number λ_0, then λ_0 is also an eigenvalue of (2.1)-(2.3).

On the other hand, if λ_0 is an eigenvalue and $y(t,\lambda_0)=C_1s(t,\lambda_0)+C_2c(t,\lambda_0)\neq 0$ is the corresponding eigenfunction, then $y(t,\lambda_0)$ satisfies (2.2) and (2.3). Therefore,

$$\begin{gathered}
(a_1\lambda_0+a_0)C_1-(b_1\lambda_0+b_0)C_2=0,\\
[(c_1\lambda_0+c_0)s^{\Delta}(\beta)-(d_1\lambda_0+d_0)s(\beta)]C_1\\
+[(c_1\lambda_0+c_0)c^{\Delta}(\beta)-(d_1\lambda_0+d_0)c(\beta)]C_2=0
\end{gathered}$$

and so

$$\det\begin{pmatrix}(a_1\lambda_0+a_0) & -(b_1\lambda_0+b_0)\\ (c_1\lambda_0+c_0)s^{\Delta}(\beta)-(d_1\lambda_0+d_0)s(\beta) & (c_1\lambda_0+c_0)c^{\Delta}(\beta)-(d_1\lambda_0+d_0)\end{pmatrix}=0.$$

Since $\varphi(t,\lambda_0)=(a_1\lambda_0+a_0)c(t,\lambda_0)+(b_1\lambda_0+b_0)s(t,\lambda_0)$, we obtain $\Delta(\lambda_0)=0$. □

Since $\Delta(\lambda)$ is entire function, problem (2.1)-(2.3) has a discrete spectrum.

Theorem 2.6. *The eigenvalues of problem* (2.1)-(2.3) *are algebraically simple.*

Proof. Let λ_n be an eigenvalue of (2.1)-(2.3). Obviously,

$$\frac{d\Delta(\lambda)}{d\lambda}=(c_1\lambda+c_0)\varphi_{\lambda}^{\Delta}(\beta,\lambda)-(d_1\lambda+d_0)\varphi_{\lambda}(\beta,\lambda)+c_1\varphi^{\Delta}(\beta,\lambda)-d_1\varphi(\beta,\lambda),$$

where $\varphi_{\lambda}=\frac{\partial\varphi}{\partial\lambda}$ and $\varphi_{\lambda}^{\Delta}=\frac{\partial\varphi^{\Delta}}{\partial\lambda}$. It will be sufficient to see $\frac{d\Delta(\lambda)}{d\lambda}|_{\lambda=\lambda_n}\neq 0$.

We write equation (2.1) for $\varphi(t,\lambda)$ as

$$-\varphi^{\Delta\Delta}(t,\lambda)+q(t)\varphi^{\sigma}(t,\lambda)=\lambda\varphi^{\sigma}(t,\lambda). \tag{2.10}$$

By derivation according to λ, we obtain

$$-\varphi_{\lambda}^{\Delta\Delta}(t,\lambda)+q(t)\varphi_{\lambda}^{\sigma}(t,\lambda)=\lambda\varphi_{\lambda}^{\sigma}(t,\lambda)+\varphi^{\sigma}(t,\lambda). \tag{2.11}$$

Now by the standard procedure of multiplying (2.10) by $\varphi_{\lambda}^{\sigma}$, (2.11) by φ^{σ}, subtracting and integrating, we obtain

$$[\varphi(t,\lambda)\varphi_{\lambda}^{\Delta}(t,\lambda)-\varphi^{\Delta}(t,\lambda)\varphi_{\lambda}(t,\lambda)]_{\alpha}^{\beta}=-\int_{\alpha}^{\beta}[\varphi^{\sigma}(t,\lambda)]^{2}\Delta t. \tag{2.12}$$

Taking into account the initial conditions (2.6) we obtain

$$\varphi^{\Delta}(\alpha,\lambda)\varphi_{\lambda}(\alpha,\lambda)-\varphi(\alpha,\lambda)\varphi_{\lambda}^{\Delta}(\alpha,\lambda)=K_1.$$

Therefore,

$$-\int_{\alpha}^{\beta}[\varphi^{\sigma}(t,\lambda)]^{2}\Delta t=K_1+\varphi(\beta,\lambda)\varphi_{\lambda}^{\Delta}(\beta,\lambda)-\varphi^{\Delta}(\beta,\lambda)\varphi_{\lambda}(\beta,\lambda). \tag{2.13}$$

Putting λ_n instead of λ in (2.13) and doing the necessary calculations we obtain

$$\begin{aligned}
&-\int_{\alpha}^{\beta}[\varphi^{\sigma}(t,\lambda_n)]^{2}\Delta t=K_1+\varphi(\beta,\lambda_n)\varphi_{\lambda}^{\Delta}(\beta,\lambda_n)-\varphi^{\Delta}(\beta,\lambda_n)\varphi_{\lambda}(\beta,\lambda_n)\\
&=K_1+\chi_n[(c_1\lambda_n+c_0)\varphi_{\lambda}^{\Delta}(\beta,\lambda_n)-(d_1\lambda_n+d_0)\varphi_{\lambda}(\beta,\lambda_n)]\\
&=K_1+\chi_n[(c_1\lambda_n+c_0)\varphi_{\lambda}^{\Delta}(\beta,\lambda_n)-(d_1\lambda_n+d_0)\varphi_{\lambda}(\beta,\lambda_n)]+\\
&\quad+\chi_n[c_1\varphi^{\Delta}(\beta,\lambda_n)-d_1\varphi(\beta,\lambda_n)]-\chi_n[c_1\varphi^{\Delta}(\beta,\lambda_n)-d_1\varphi(\beta,\lambda_n)]\\
&=K_1+\chi_n^2K_2+\chi_n\frac{d\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_n}
\end{aligned}$$

Hence, we obtain

$$-\frac{d\Delta(\lambda)}{d\lambda}\Big|_{\lambda=\lambda_n}=\frac{1}{\chi_n}\Big\{\int_{\alpha}^{\beta}[\varphi^{\sigma}(t,\lambda_n)]^{2}\Delta t+K_1+\chi_n^2K_2\Big\}\neq 0\,.$$

This completes the proof. □

The next theorem gives the number of eigenvalues of problem (2.1)-(2.3) on a finite time scale.

Theorem 2.7. *If $\mathbb{T}$ is finite, then the number of eigenvalues of* (2.1)-(2.3) *is*

$$n-\operatorname{def}(c_1)-\operatorname{def}(a_1+\mu(\alpha)b_1) \tag{2.14}$$

where n is number of elements of $\mathbb{T}$ and $\operatorname{def}(u)=\begin{cases}0, & u\neq 0\\ 1, & u=0\end{cases}$.

Proof. Since $\mathbb{T}$ consists of finitely many elements, all points of $\mathbb{T}$ are isolated. Let $\mathbb{T}=\{\alpha,\sigma(\alpha),\sigma^2(\alpha),\dots,\sigma^{n-2}(\alpha),\sigma^{n-1}(\alpha)\}$, $\sigma^{n-2}(\alpha)=\beta$, where $\sigma^j=\sigma^{j-1}\circ\sigma$, for $j\geq 2$. It can be calculated from (2.1) and (2.6) that

$$\varphi^{\sigma}(\alpha)=(\mu(\alpha)b_1+a_1)\lambda+\mu(\alpha)b_0+a_0 \tag{2.15}$$

$$\varphi^{\sigma^2}(t)=[f(t)\lambda+g(t)]\varphi^{\sigma}(t)+h(t)\varphi(t),\quad t\in\mathbb{T}^{k^2} \tag{2.16}$$

where $f(t) = -\mu(t)\mu^\sigma(t)$, $g(t) = \mu(t)\mu^\sigma(t)q(t) + \frac{\mu^\sigma(t)}{\mu(t)} + 1$ and $h(t) = -\frac{\mu^\sigma(t)}{\mu(t)}$. It can be obtained that the following equality is valid for $r \geq 3$,

$$\frac{\varphi^{\sigma^r}(\alpha)}{\prod_{j=0}^{r-2} f(\sigma^j(\alpha))} = \begin{cases} A_1\lambda^r + p_{r-1}(\lambda), & A_1 \neq 0 \\ A_2\lambda^{r-1} + p_{r-2}(\lambda), & A_1 = 0, \end{cases} \tag{2.17}$$

where $A_1 = \mu(\alpha)b_1 + a_1$, $A_2 = \mu(\alpha)b_0 + a_0 + \frac{1}{(\mu(\alpha))^2}a_1$, $p_{r-1}(\lambda)$ and $p_{r-2}(\lambda)$ are polynomials with degrees $r-1$ and $r-2$, respectively. We must note that, if $A_1 = 0$, then $A_2 \neq 0$. Indeed, $A_1 = 0$ yields $\mu(\alpha)b_1 = -a_1$. Since $K_1 > 0$, we obtain

$$0 < K_1\mu(\alpha) = a_1(b_0\mu(\alpha) + a_0).$$

Thus, the signs of a_1 and $b_0\mu(\alpha) + a_0$ are the same and so $A_2 \neq 0$. It is obvious from (2.9) that

$$\Delta(\lambda) = \frac{1}{\mu(\beta)}\{(c_1\lambda + c_0)\varphi^\sigma(\beta,\lambda) - [(c_1 + d_1\mu(\beta))\lambda + (c_0 + d_0\mu(\beta))]\varphi(\beta,\lambda)\}. \tag{2.18}$$

Taking into account (2.17) and (2.18), it is obtained that

$$\deg\Delta(\lambda) = \begin{cases} n, & c_1 \neq 0 \text{ and } \mu(\alpha)b_1 + a_1 \neq 0 \\ n-1, & c_1 \neq 0 \text{ and } \mu(\alpha)b_1 + a_1 = 0 \\ n-1, & c_1 = 0 \text{ and } \mu(\alpha)b_1 + a_1 \neq 0 \\ n-2, & c_1 = 0 \text{ and } \mu(\alpha)b_1 + a_1 = 0\,. \end{cases} \tag{2.19}$$

Hence, we conclude that equality (2.14) holds. □

Corollary 2.8. *If the number of points in* $\mathbb{T}$ *is not less than three, then the boundary-value problem* (2.1)-(2.3) *has at least one eigenvalue.*

Example 2.9. Let us consider the following boundary-value problems on $\mathbb{T} = \{0, 1, 2, \dots, n-1\}$:

$$\begin{gathered} -y^{\Delta\Delta}(t) = \lambda y^\sigma(t) \\ \lambda y^\Delta(0) + (\lambda - 1)y(0) = 0 \\ y^\Delta(n-2) - \lambda y(n-2) = 0, \end{gathered}$$

$$\begin{gathered} -y^{\Delta\Delta}(t) = \lambda y^\sigma(t) \\ \lambda y^\Delta(0) - y(0) = 0 \\ y^\Delta(n-2) - \lambda y(n-2) = 0 \end{gathered}$$

and

$$\begin{gathered} -y^{\Delta\Delta}(t) = \lambda y^\sigma(t) \\ \lambda y^\Delta(0) - y(0) = 0 \\ \lambda y^\Delta(n-2) + y(n-2) = 0\,. \end{gathered}$$

According to Theorem 2.7, the numbers of the eigenvalues of these problems are $n-2$, $n-1$ and n, respectively.

Theorem 2.10. *If λ is not an eigenvalue, then the operator $(L-\lambda I^\sigma): D(L)\to L_2(\mathbb{T}^k)\oplus\{0\}^2$ is bijective and its inverse operator is*
$$(L-\lambda I^\sigma)^{-1}\begin{pmatrix} f(t)\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} y(t)\\ Y_\alpha\\ Y_\beta\end{pmatrix}$$
such that $y(t)=\int_\alpha^\beta G(t,\xi,\lambda)f(\xi)\Delta\xi$,
$$\begin{gathered}Y_\alpha=\int_\alpha^\beta [a_1G^\Delta(\alpha,\xi,\lambda)-b_1G(\alpha,\xi,\lambda)]f(\xi)\Delta\xi,\\ Y_\beta=\int_\alpha^\beta [c_1G^\Delta(\beta,\xi,\lambda)-d_1G(\beta,\xi,\lambda)]f(\xi)\Delta\xi,\end{gathered}$$
where
$$G(t,\xi,\lambda)=\frac{1}{\Delta(\lambda)}\begin{cases}\varphi^\sigma(\xi,\lambda)\psi(t,\lambda), & \alpha\le\xi<t\\ \varphi(t,\lambda)\psi^\sigma(\xi,\lambda), & t<\xi\le\beta\,.\end{cases}$$

Proof. If λ is not an eigenvalue, then $\Delta(\lambda)\neq 0$ so $L-\lambda I^\sigma$ is injective on $D(L)$. On the other hand, from the method of variation of parameters, the boundary-value problem
$$\begin{gathered}-y^{\Delta\Delta}(t)+\{q(t)-\lambda\}y^\sigma(t)=f(t),\quad t\in\mathbb{T}^{k^2},\\ (a_1\lambda+a_0)y^\Delta(\alpha)-(b_1\lambda+b_0)y(\alpha)=0,\\ (c_1\lambda+c_0)y^\Delta(\beta)-(d_1\lambda+d_0)y(\beta)=0\end{gathered}$$
has a solution for each $f(t)$ in $L_2(\mathbb{T}^k)$ such that $y(t)=\int_\alpha^\beta G(t,\xi,\lambda)f(\xi)\Delta\xi$ and
$$G(t,\xi,\lambda)=\frac{1}{\Delta(\lambda)}\begin{cases}\varphi^\sigma(\xi,\lambda)\psi(t,\lambda), & \alpha\le\xi<t\\ \varphi(t,\lambda)\psi^\sigma(\xi,\lambda), & t<\xi\le\beta\,.\end{cases}$$
Therefore, $L-\lambda I^\sigma$ is surjective operator and its inverse operator is given as
$$(L-\lambda I^\sigma)^{-1}\begin{pmatrix} f(t)\\ 0\\ 0\end{pmatrix}=\begin{pmatrix} y(t)\\ Y_\alpha\\ Y_\beta\end{pmatrix}.$$
Finally, since $Y=\begin{pmatrix} y(t)\\ Y_\alpha\\ Y_\beta\end{pmatrix}\in D(L)$, it follows that
$$\begin{gathered}Y_\alpha=\int_\alpha^\beta [a_1G^\Delta(\alpha,\xi,\lambda)-b_1G(\alpha,\xi,\lambda)]f(\xi)\Delta\xi,\\ Y_\beta=\int_\alpha^\beta [c_1G^\Delta(\beta,\xi,\lambda)-d_1G(\beta,\xi,\lambda)]f(\xi)\Delta\xi.\end{gathered}$$
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