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# SOLUTIONS TO POLYTROPIC FILTRATION EQUATIONS WITH A CONVECTION TERM

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Abstract. We introduce a new type of the weak solution of the polytropic filtration equations with a convection term,

$$u_t = \operatorname{div}(a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u) + \frac{\partial b^i(u^m)}{\partial x_i}.$$

Here, $\Omega \subset \mathbb{R}^N$ is a domain with a $C^2$ smooth boundary $\partial\Omega$, $a(x) \in C^1(\overline{\Omega})$, $p > 1$, $m = 1 + \frac{\alpha}{p-1}$, $\alpha > 0$, $a(x) > 0$ when $x \in \Omega$ and $a(x) = 0$ when $x \in \partial\Omega$. Since the equation is degenerate on the boundary, its weak solutions may lack the needed regularity to have a trace on the boundary. The main aim of the paper is to establish the stability of the weak solution without any boundary value condition.

## 1. Introduction

Consider the polytropic filtration equation with a convection term

$$u_t = \operatorname{div}(a(x)|u|^\alpha |\nabla u|^{p-2}\nabla u) + \frac{\partial b^i(u^m)}{\partial x_i}, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where $p > 1$, $m = 1 + \frac{\alpha}{p-1}$, $\alpha > 0$, $\Omega \subset \mathbb{R}^N$ is with a $C^2$ smooth boundary $\partial\Omega$, $a(x) \in C^1(\overline{\Omega})$, $a(x) \geqslant 0$. The equations like (1.1) arise from a variety of diffusion phenomena, such as soil physics, fluid dynamics, combustion theory, reaction chemistry, one can see [1, 10] and the references therein.

In particular, when $\alpha > 0$, $a(x) \equiv 1$, the well-posedness of equation (1.1) with the usual initial-boundary value conditions

$$u|_{t=0} = u_0(x), \ x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \Gamma_T = \partial\Omega \times (0,T), \tag{1.3}$$

has been studied thoroughly, one can refer to [2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16]. In this article, we assume that

$$a(x) > 0, x \in \Omega,$$
$$a(x) = 0, x \in \partial\Omega.$$

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Consequently, equation (1.1) is always degenerate on the boundary. Not only the degeneracy comes from the physics quantity $u$ itself, but also comes from the diffusion coefficient $a(x)$.

Now, let us introduce some basic definitions and the main results. For every fixed $t \in [0,T]$, the Banach space

$$V_t(\Omega) = \big\{u(x,t) : u(x,t) \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u(x,t)|^p \in L^1(\Omega)\big\},$$

is with the norm

$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p,\Omega},$$

and we denote its dual space as $V_t'(\Omega)$. By $W(Q_T)$ we denote the Banach space

$$W(Q_T) = \{u : [0,T] \to V_t(\Omega) | u \in L^2(Q_T), |\nabla u|^p \in L^1(Q_T), u = 0 \quad \text{on } \partial\Omega\},$$

$$\|u\|_{W(Q_T)} = \|\nabla u\|_{p,Q_T} + \|u\|_{2,Q_T}.$$

Here $W'(Q_T)$ is the dual of $W(Q_T)$ (the space of linear functionals over $W(Q_T)$, $w \in W'(Q_T)$ if

$$w = w_0 + \sum_{i=1}^{n} D_i w_i, \quad w_0 \in L^2(Q_T), w_i \in L^{p'}(Q_T),$$

$$\forall \phi \in W(Q_T),\ \langle w, \phi\rangle = \iint_{Q_T} \Big(w_0\phi + \sum_i w_i D_i\phi\Big)\,dx\,dt.$$

The norm in $W'(Q_T)$ is defined by

$$\|v\|_{W'(Q_T)} = \sup\{\langle v, \phi\rangle : \phi \in \mathbf{W(Q_T)}, \|\phi\|_{W(Q_T)} \leqslant 1\}.$$

**Definition 1.1.** A nonnegative function $u(x,t)$ is said to be a weak solution of (1.1) with the initial value (1.2), if $u$ satisfies

$$u \in L^\infty(Q_T),\ \frac{\partial u}{\partial t} \in W'(Q_T),\ a(x)|u|^\alpha|\nabla u|^p \in L^1(Q_T), \tag{1.4}$$

and for any function $\varphi_1 \in L^1(0,T; C_0^1(\Omega))$, $\varphi_2 \in L^\infty(Q_T)$ such that for any given $t \in [0,T)$, $\varphi_2(x,\cdot) \in W_{\rm loc}^{1,p}(\Omega)$, we have

$$\begin{aligned}\iint_{Q_T} \big[&\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + a(x)|u|^\alpha|\nabla u|^{p-2}\nabla u \cdot \nabla(\varphi_1\varphi_2)\\ &+ b^i(u^m)(\varphi_1\varphi_2)_{x_i}\big]\,dx\,dt = 0.\end{aligned} \tag{1.5}$$

The initial value (1.2) is satisfied in the sense that

$$\lim_{t\to 0}\int_\Omega u(x,t)\phi(x)\,dx = \int_\Omega u_0(x)\phi(x)\,dx, \forall \phi(x) \in C_0^\infty(\Omega). \tag{1.6}$$

If $u \in L^\infty(0,T; W^{1,\gamma}(\Omega))$ for some constant $\gamma > 1$, the boundary value condition (1.3) is satisfied in the sense of the trace, then we say $u$ is a weak solution of the initial-boundary problem of equation (1.1).

Clearly, if noticing $m = 1 + \frac{\alpha}{p-1}$, by (1.4), then

$$a(x)|\nabla u^m|^p \in L^1(Q_T),$$

and (1.5) is equivalent to

$$\begin{aligned}\iint_{Q_T} \big[&\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + \frac{1}{m^{p-1}}a(x)|\nabla u^m|^{p-2}\nabla u^m \cdot \nabla(\varphi_1\varphi_2)\\ &+ b^i(u^m)(\varphi_1\varphi_2)_{x_i}\big]\,dx\,dt = 0.\end{aligned} \tag{1.7}$$

In general, since (1.1) is always degenerate on the boundary, instead of $u(x,t) \in L^\infty(0,T;W_0^{1,p}(\Omega))$, we only have $u(x,t) \in L^\infty(0,T;W_{\rm loc}^{1,p}(\Omega))$. Thus, we can not define the trace of the weak solution $u$ on the boundary. If $u, v$ are two weak solutions of equation (1.1), to prove the stability (or uniqueness) of the weak solutions, one generally must choose a test function with the form $f(x,t,u-v)$ which involves the boundary value condition

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Gamma_T = \partial\Omega \times (0,T). \tag{1.8}$$

However, the weak solution defined in this paper can not guarantee this condition. This is the main reason that we need to choose the test function $\varphi_1\varphi_2$ in Definition 1.1.

If $\alpha = 0$, $m = 1$, $b^i \equiv 0$, the existence of the weak solutions had been proved in our previous paper [14]. In this paper, we mainly concern with the stability of the weak solutions of equation (1.1).

**Theorem 1.2.** *Let $u, v$ be two nonnegative solutions of* (1.1) *with the same homogeneous boundary value condition* (1.3) *and with the different initial values $u_0$, $v_0$ respectively. Then*

$$\int_\Omega |u(x,t) - v(x,t)|\,dx \leqslant \int_\Omega |u_0 - v_0|\,dx. \tag{1.9}$$

**Theorem 1.3.** *Let $u, v$ be two nonnegative solutions of equation* (1.1) *with the initial values $u_0$, $v_0$ respectively. If $1 < p \leqslant 2$, and*

$$\int_\Omega a^{-\frac{1}{p-1}}(x)dx < \infty, \tag{1.10}$$

*then the stability of the weak solutions is true in the sense of* (1.9).

**Theorem 1.4.** *Let $u, v$ be two nonnegative solutions of* (1.1) *with the initial values $u_0$, $v_0$ respectively. If $p > 1$ and for small enough $\lambda > 0$, $u(x)$ and $v(x)$ satisfy*

$$\frac{1}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\Big)^{\frac{p-1}{p}} \leqslant c, \quad \frac{1}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^p dx\Big)^{\frac{p-1}{p}} \leqslant c, \tag{1.11}$$

*then* (1.9) *is true. Here $\Omega_\lambda = \{x \in \Omega : a(x) > \lambda\}$*

**Theorem 1.5.** *Let $u, v$ be two weak solutions of problem* (1.1) *with the initial values $u_0(x), v_0(x)$ respectively. If $p > 1$, $m > 0$,*

$$\int_\Omega \frac{|\nabla a|}{a}|u^m|dx \leqslant c, \quad \int_\Omega \frac{|\nabla a|}{a}|v^m|dx \leqslant c, \tag{1.12}$$

*then* (1.9) *is true.*

At the end, we suggest that not any boundary value condition is required in Theorems 1.3–1.5. However, from my own perspective, the condition (1.12) in Theorem 1.5 makes a substitute of the boundary value condition. Moreover, if $b^i \equiv 0$, i.e. equation (1.1) has no convection term, Theorem 1.5 is true without the condition (1.12).

## 2. Proof of Theorem 1.2

Let $u, v$ are two nonnegative solutions of equation (1.1) with the same homogeneous boundary value and with the different initial values $u_0$, $v_0$ respectively. From the definition of the weak solution, we let $\varphi_1 = \varphi \in L^1(0,T;C_0^1(\Omega))$, $\varphi_2 \equiv 1$. Then

$$\begin{aligned}
&\int_\Omega \varphi \frac{\partial(u-v)}{\partial t}\,dx + \int_\Omega a(x)(u^\alpha|\nabla u|^{p-2}\nabla u - v^\alpha|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi\,dx \\
&+ \int_\Omega [b^i(u^m) - b^i(v^m)]\varphi_{x_i}\,dx = 0,
\end{aligned} \tag{2.1}$$

or equivalently

$$\begin{aligned}
&\int_\Omega \varphi \frac{\partial(u-v)}{\partial t}\,dx + \frac{1}{m^{p-1}}\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m - |\nabla v^m|^{p-2}\nabla v^m)\cdot\nabla\varphi\,dx \\
&+ \int_\Omega [b^i(u^m) - b^i(v^m)]\varphi_{x_i}dx = 0.
\end{aligned} \tag{2.2}$$

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau)d\tau, \quad h_\eta(s) = \frac{2}{\eta}\big(1 - \frac{|s|}{\eta}\big)_+ . \tag{2.3}$$

Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$\begin{gathered}
h_\eta(s) \geqslant 0, \quad |sh_\eta(s)| \leqslant 1, \quad |S_\eta(s)| \leqslant 1, \\
\lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn} s, \quad \lim_{\eta\to 0} sS'_\eta(s) = 0.
\end{gathered} \tag{2.4}$$

We can choose $\varphi = S_\eta(u^m - v^m)$ as the test function, then

$$\begin{aligned}
&\int_\Omega S_\eta(u^m - v^m)\frac{\partial(u-v)}{\partial t}dx + \frac{1}{m^{p-1}}\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m - |\nabla v^m|^{p-2}\nabla v^m) \\
&\quad \cdot \nabla(u^m - v^m)S'_\eta(u^m - v^m)dx \\
&= -\int_\Omega [b^i(u^m) - b^i(v^m)](u^m - v^m)_{x_i}S'_\eta(u^m - v^m)dx.
\end{aligned} \tag{2.5}$$

Clearly,

$$\begin{aligned}
\lim_{\eta\to 0}\int_\Omega S_\eta(u^m - v^m)\frac{\partial(u-v)}{\partial t}\,dx &= \int_\Omega \operatorname{sgn}(u^m - v^m)\frac{\partial(u-v)}{\partial t}\,dx \\
&= \int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}\,dx \\
&= \frac{d}{dt}\|u-v\|_{L^1(\Omega)},
\end{aligned} \tag{2.6}$$

and

$$\int_\Omega a(x)|(|\nabla u^m|^{p-2}\nabla u - |\nabla v^m|^{p-2}\nabla v^m)\cdot\nabla(u^m - v^m)S'_\eta(u^m - v^m)\mathrm{d}x \geqslant 0. \tag{2.7}$$

At the same time,

$$\int_\Omega a^{\frac{-1}{p-1}}(x)dx < \infty,$$

using Lebesgue dominated convergence theorem, by (2.4), we have

$$\begin{aligned} &\lim_{\eta\to 0}\Big|\int_\Omega [b^i(u^m)-b^i(v^m)](u^m-v^m)_{x_i}S'_\eta(u^m-v^m)dx\Big| \\ &\leqslant \lim_{\eta\to 0}\Big(\int_\Omega |[b^i(u^m)-b^i(v^m)]S'_\eta(u^m-v^m)a^{-\frac{1}{p}}|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}} \\ &\quad\times\Big(\int_\Omega a(x)(|\nabla u^m|^p+|\nabla v^m|^p)dx\Big)^{1/p}=0. \end{aligned} \tag{2.8}$$

Let $\eta\to 0$ in (2.2). Then

$$\frac{d}{dt}\|u-v\|_{L^1(\Omega)}\leqslant 0. \tag{2.9}$$

This implies

$$\int_\Omega |u(x,t)-v(x,t)|\,dx\leqslant \int_\Omega |u_0-v_0|\,dx,\quad \forall t\in[0,T).$$

## 3. Proofs of Theorem 1.3 and 1.4

*Proof of Theorem 1.3.* By Definition 1.1, for any function $\varphi_1\in L^1(0,T;C_0^1(\Omega))$, $\varphi_2\in L^\infty(Q_T)$ such that for any given $t\in[0,T)$, $\varphi_2(x,\cdot)\in W^{1,p}_{\rm loc}(\Omega)$, we have

$$\begin{aligned} &\iint_{Q_T}\big[\frac{\partial(u-v)}{\partial t}(\varphi_1\varphi_2)+\frac{1}{m^{p-1}}a(x)(|\nabla u^m|^{p-2}\nabla u^m \\ &-|\nabla v^m|^{p-2}\nabla v^m)\cdot\nabla(\varphi_1\varphi_2)+(b^i(u^m)-b^i(v^m))(\varphi_1\varphi_2)_{x_i}\big]\,dx\,dt=0. \end{aligned} \tag{3.1}$$

For a small positive constant $\lambda>0$, let

$$\begin{gathered} \Omega_\lambda=\{x\in\Omega: a(x)>\lambda\}, \\ \phi_\lambda(x)=\begin{cases} 1, & \text{if } x\in\Omega_\lambda, \\ \frac{1}{\lambda}a(x), & \text{if } x\in\Omega\setminus\Omega_\lambda. \end{cases} \end{gathered} \tag{3.2}$$

Now, we choose $\varphi_1=\phi_\lambda(x)\chi_{[\tau,s]}$, $\varphi_2=S_\eta(u^m-v^m)$, and then integrate it over $\Omega$, to have

$$\begin{aligned} &\int_\tau^s\int_\Omega \phi_\lambda(x)S_\eta(u^m-v^m)\frac{\partial(u-v)}{\partial t}\,dx\,dt+\frac{1}{m^{p-1}}\int_\tau^s\int_\Omega \phi_\lambda(x)a(x) \\ &\times\big(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m\big)\cdot\nabla(u^m-v^m)S'_\eta(u^m-v^m)\,dx\,dt \\ &+\frac{1}{m^{p-1}}\int_\tau^s\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m) \\ &\cdot\nabla\phi_\lambda(x)S_\eta(u^m-v^m)\,dx\,dt \\ &+\int_\tau^s\int_\Omega [b^i(u^m)-b^i(v^m)][\phi_\lambda(x)S'_\eta(u^m-v^m)(u^m-v^m)_{x_i} \\ &+S_\eta(u^m-v^m)\phi_{\lambda x_i}(x)]\,dx\,dt=0. \end{aligned} \tag{3.3}$$

Clearly,

$$\begin{aligned} &\int_\Omega \phi_\lambda(x)a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m) \\ &\cdot\nabla(u^m-v^m)S'_\eta(u^m-v^m)\,dx\geqslant 0. \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&\Big|\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m)\cdot\nabla\phi_\lambda(x)S_\eta(u^m-v^m)\,dx\Big|\\
&\leqslant \int_{\Omega\setminus\Omega_\lambda} a(x)|(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m)\cdot\nabla\phi_\lambda(x)S_\eta(u^m-v^m)|dx\\
&\leqslant \int_{\Omega\setminus\Omega_\lambda} a(x)|(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m)||\nabla\phi_\lambda(x)|dx\\
&\leqslant \frac{c}{\lambda}\Big[\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^{p-1}|\nabla a|\,dx+\int_\tau^s\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^{p-1}|\nabla a|dx\Big].
\end{aligned}\tag{3.5}
$$

Since $1<p\leqslant 2$, $|\nabla a|\leqslant c$ and

$$
\int_{\Omega\setminus\Omega_\lambda}|\nabla a|^p dx\leqslant c\lambda\leqslant c\lambda^{p-1},
$$

it follows that

$$
\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p}\leqslant\frac{c}{\lambda}\Big(\lambda\int_{\Omega\setminus\Omega_\lambda}|\nabla a|^p dx\Big)^{1/p}\leqslant c. \tag{3.6}
$$

By (3.5)-(3.6), using the Hölder inequality,

$$
\begin{aligned}
&\Big|\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m)\cdot\nabla\phi_\lambda(x)S_\eta(u^m-v^m)\,dx\Big|\\
&\leqslant \frac{c}{\lambda}\Big[\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^{p-1}|\nabla a|\,dx+\int_\tau^s\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^{p-1}|\nabla a|dx\Big]\\
&\leqslant \frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\Big)^{\frac{p-1}{p}}\\
&\quad+\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^p dx\Big)^{\frac{p-1}{p}}\\
&\leqslant c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\Big)^{\frac{p-1}{p}}+c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^p dx\Big)^{\frac{p-1}{p}}.
\end{aligned}\tag{3.7}
$$

Then, we have

$$
\begin{aligned}
&\lim_{\lambda\to0}\Big|\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\\
&\cdot\nabla\phi_\lambda(x)S_\eta(u^m-v^m)\,dx\Big|=0.
\end{aligned}\tag{3.8}
$$

At the same time, by that $\int_\Omega a^{-\frac{1}{p-1}}(x)dx<c$, using (2.4) and the Lebesgue dominated convergence theorem, we also have

$$
\lim_{\eta\to0}\int_\Omega \phi_\lambda[b^i(u^m)-b^i(v^m)]S'_\eta(u^m-v^m)(u^m-v^m)_{x_i}dx=0, \tag{3.9}
$$

and

$$
\begin{aligned}
&\lim_{\lambda\to0}\Big|\int_\Omega \phi_{\lambda x_i}[b_i(u^m)-b_i(v^m)]S_\eta(u^m-v^m)dx\Big|\\
&\leqslant \lim_{\lambda\to0}\frac{c}{\lambda}\int_{\Omega\setminus\Omega_\lambda}|\nabla a|dx\\
&\leqslant \lim_{\lambda\to0}\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a^{-\frac{1}{p-1}}(x)dx\Big)^{\frac{p-1}{p}}=0,
\end{aligned}\tag{3.10}
$$

by (3.6) and $\int_\Omega a^{-\frac{1}{p-1}}(x)dx < c$. At last,

$$\begin{aligned}
&\lim_{\eta\to 0}\lim_{\lambda\to 0}\int_\tau^s\int_\Omega \phi_\lambda(x)S_\eta(u^m-v^m)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&=\lim_{\eta\to 0}\int_\tau^s\int_\Omega S_\eta(u^m-v^m)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&=\int_\tau^s\int_\Omega \operatorname{sgn}(u^m-v^m)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&=\int_\tau^s\int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&=\int_\tau^s \frac{d}{dt}\|u-v\|_{L^1(\Omega)}dt.
\end{aligned}\tag{3.11}$$

Now, after letting $\lambda\to 0$, let $\eta\to 0$ in (3.2). Then by (3.4), (3.8)-(3.11),

$$\int_\Omega |u(x,t)-v(x,t)|dx \leqslant \int_\Omega |u_0-v_0|dx.$$

□

*Proof of Theorem 1.4.* As in the proof of Theorem 1.3, we have (3.3)- (3.5). Since $u(x)$ and $v(x)$ satisfy (1.11) by (3.6)-(3.7), using the Hölder inequality, we have

$$\begin{aligned}
&\Big|\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^2\nabla v^m)\cdot\nabla\phi_\lambda(x)S_\eta(u^m-v^m)\,dx\Big| \\
&\leqslant \frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla u^m|^p dx\Big)^{\frac{p-1}{p}} \\
&\quad+\frac{c}{\lambda}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p}\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla v^m|^p dx\Big)^{\frac{p-1}{p}} \\
&\leqslant c\Big(\int_{\Omega\setminus\Omega_\lambda} a|\nabla a|^p dx\Big)^{1/p}+c\Big(\int_{\Omega\setminus\Omega_\lambda} a(x)|\nabla a|^p dx\Big)^{1/p},
\end{aligned}\tag{3.12}$$

which approaches zero as $\lambda\to 0$ since that $a(x)\in C^1(\overline{\Omega})$, we have (3.8). At last, since $\int_\Omega a^{-\frac{1}{p-1}}(x)dx<\infty$, similar as the proof of Theorem 1.3, we have (3.9)-(3.10). So, as the proof of Theorem 1.3, we know that the stability (1.9) is true. □

## 4. Proof Theorem 1.5

It is not difficult to show that the following definition is equivalent to Definition 1.1.

**Definition 4.1.** A function $u(x,t)$ is said to be a weak solution of (1.1) with initial value (1.2), if

$$u\in L^\infty(Q_T),\quad u_t\in L^2(Q_T),\quad a(x)|\nabla u|^p\in L^1(Q_T),\tag{4.1}$$

and for any function $g(s)\in C^1(\mathbb{R})$, $g(0)=0$, $\varphi_1\in C_0^1(\Omega)$, $\varphi_2\in L^\infty(0,T;W^{1,p}_{\rm loc}(\Omega))$,

$$\begin{aligned}
&\iint_{Q_T}[u_t g(\varphi_1\varphi_2)+a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla g(\varphi_1\varphi_2)+u\big(b_{ix_i}(x)g(\varphi_1\varphi_2) \\
&\quad+b_i(x)g_{x_i}(\varphi_1\varphi_2)\big)-c(x,t)ug(\varphi_1\varphi_2)+f(x,t)g(\varphi_1\varphi_2)]\,dx\,dt=0.
\end{aligned}\tag{4.2}$$

The initial value is satisfied in the sense that

$$\lim_{t\to 0}\int_\Omega u(x,t)\phi(x)\,dx=\int_\Omega u_0(x)\phi(x)\,dx, \forall\phi(x)\in C_0^\infty(\Omega). \tag{4.3}$$

*Proof of Theorem 1.5.* Let $u$, $v$ be two solutions of equation (1.1) with the initial values $u_0(x), v_0(x)$. We can choose $S_\eta(a^\beta(u^m-v^m))$ as the test function. Then

$$\begin{aligned}
&\int_\Omega S_\eta(a^\beta(u^m-v^m))\frac{\partial(u-v)}{\partial t}dx+\frac{1}{m^{p-1}}\int_\Omega a^{\beta+1}(x)\big(|\nabla u^m|^{p-2}\nabla u^m\\
&-|\nabla v^m|^{p-2}\nabla v^m\big)\cdot\nabla(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\,dx\\
&+\int_\Omega a(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\\
&\cdot\nabla a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\,dx\\
&+\int_\Omega[b_i(u^m)-b_i(v^m)][S_\eta'(a^\beta(u^m-v^m))\\
&(a^\beta_{x_i}(u^m-v^m)+a^\beta(u^m-v^m)_{x_i}dx=0.
\end{aligned}\tag{4.4}$$

Thus

$$\lim_{\eta\to 0}\int_\Omega S_\eta(a^\beta(u^m-v^m))\frac{\partial(u-v)}{\partial t}dx=\frac{d}{dt}\|u-v\|_1, \tag{4.5}$$

$$\begin{aligned}
&\int_\Omega a^{\beta+1}(x)(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\\
&\cdot\nabla(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\,dx\geqslant 0.
\end{aligned}\tag{4.6}$$

From $|\nabla a(x)|\leqslant c$ in $\Omega$, we have

$$\begin{aligned}
&\Big|\int_\Omega a(x)(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\\
&\cdot\nabla a^\beta\,dx\Big|\\
&\leqslant c\Big|\int_\Omega a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\\
&\quad\times(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\,dx\Big|,
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
&\Big|\int_\Omega a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\\
&\times(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\,dx\Big|\\
&=\Big|\int_{\Omega:a^\beta|u^m-v^m|<\eta}a^{-\frac{p-1}{p}}a^\beta(u^m-v^m)S_\eta'\\
&\quad\cdot(a^\beta(u^m-v^m))a^{\frac{p-1}{p}}(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)\,dx\Big|\\
&\leqslant\Big(\int_{\Omega:a^\beta|u-v|<\eta}|a^{-\frac{p-1}{p}}a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))|^pdx\Big)^{1/p}\\
&\quad\times\Big(\int_{\Omega:a^\beta|u-v|<\eta}a(x)(|\nabla u^m|^p+|\nabla v^m|^p)dx\Big)^{\frac{p-1}{p}}.
\end{aligned}\tag{4.8}$$

If $\{x\in\Omega:u^m-v^m=0\}$ has 0 measure, since $\int_\Omega a^{p-1}(x)dx<\infty$, we have

$$\Big|\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}|a^{-\frac{p-1}{p}}a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))|^pdx\Big|$$

and

$$\begin{aligned}&\lim_{\eta\to 0}\Big(\int_{\{\Omega:a^{\beta}|u^m-v^m|<\beta\}}a(x)(|\nabla u^m|^p+|\nabla v^m|^p)dx\Big)^{\frac{p-1}{p}}\\&=\Big(\int_{\{\Omega:|u^m-v^m|=0\}}a(x)(|\nabla u^m|^p+|\nabla v^m|^p)dx\Big)^{\frac{p-1}{p}}=0.\end{aligned}\tag{4.9}$$

If $\{x\in\Omega: u^m-v^m=0\}$ has a positive measure, obviously

$$\begin{aligned}&\lim_{\eta\to 0}\Big(\int_{\{\Omega:a^{\beta}|u^m-v^m|<\eta\}}|a^{-\frac{p-1}{p}}a^{\beta}(u^m-v^m)S'_\eta(a^{\beta}(u^m-v^m))|^pdx\Big)^{1/p}\\&=\Big(\int_{\{\Omega:|u^m-v^m|=0\}}|a^{-\frac{p-1}{p}}a^{\beta}(u^m-v^m)S'_\eta(a^{\beta}(u^m-v^m))|^pdx\Big)^{1/p}=0.\end{aligned}\tag{4.10}$$

By (2.2) and (2.4), using the Lebesgue controlled convergence theorem, in both cases, we have

$$\lim_{\eta\to 0}|\int_\Omega a^{\beta}(u^m-v^m)S'_\eta(a^{\beta}(u^m-v^m))(|\nabla u^m|^{p-2}\nabla u^m-|\nabla v^m|^{p-2}\nabla v^m)dx|=0.$$

In addition,

$$\begin{aligned}&|\int_\Omega[b_i(u^m)-b_i(v^m)]a^{\beta}_{x_i}(u^m-v^m)S'_\eta(a^{\beta}(u^m-v^m))dx|\\&\leqslant c\int_\Omega(|u^m|+|v^m|)\frac{|\nabla a|}{a}a^{\beta}(u^m-v^m)S'_\eta(a^{\beta}(u^m-v^m))dx\to 0,\end{aligned}\tag{4.11}$$

as $\eta\to 0$ by (1.12),

$$\begin{aligned}&|\int_\Omega[b_i(u^m)-b_i(v^m)]a^{\beta}(u^m-v^m)_{x_i}S'_\eta(a^{\beta}(u^m-v^m))dx|\\&=|\int_\Omega a^{\beta-\frac1p}[b_i(u^m)-b_i(v^m)]S'_\eta(a^{\beta}(u^m-v^m))a^{-\frac1p}(u^m-v^m)_{x_i}dx|\\&\leqslant c\Big(|a^{-\frac1p}a^{\beta}(u^m-v^m)S'_\eta(a^{\beta}(u^m-v^m))|^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\\&\quad\times\Big(\int_\Omega a(x)(|\nabla u^m|^p+|\nabla v^m|)\Big)^{1/p}\to 0,\end{aligned}\tag{4.12}$$

as $\eta\to 0$ by (2.4).

Now, let $\eta\to 0$ in (4.4). Then

$$\int_\Omega|u(x,t)-v(x,t)|\,dx\leqslant\int_\Omega|u_0-v_0|\,dx,\quad\forall t\in[0,T).\tag{4.13}$$

Theorem 1.5 is proved. □

**Corollary 4.2.** *Let $u,v$ be two weak solutions of equation* (1.1) *with the initial values $u_0(x),v_0(x)$ respectively. If $b_i\equiv 0$, then* (4.13) *is true without any boundary value condition.*

*Proof.* We notice that, in the proof of Theorem 1.5, condition (1.12) is only used to deal with the convection term to obtain (4.11) and (4.12). Consequently, when $b_i\equiv 0$, the stability is (4.13) is true. □

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