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HALF INVERSE PROBLEMS FOR THE IMPULSIVE OPERATOR WITH EIGENVALUE-DEPENDENT BOUNDARY CONDITIONS

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Abstract. In this work we study a Sturm-Liouville operator with a piece-wise continuous coefficient and a spectral parameter in the boundary condition. We show that if the potential function is ascertained in $(\pi/2,\pi)$ then one spectrum suffices to determinate the potential function in the whole of the interval.

1. Introduction

We consider the boundary value problem L for the differential equation

$$-y''+q(x)y=\rho^2 r(x)y,\quad x\in(0,\pi), \tag{1.1}$$

with the boundary conditions

$$\begin{gathered} U(y):=y'(0)-hy(0)=0,\\ V(y):=y'(\pi)+(H_1\rho+H_0)y(\pi)=0. \end{gathered} \tag{1.2}$$

Here we suppose that

$$r(x)=\begin{cases}1, & x<\pi/2,\\ \alpha^2, & x>\pi/2,\end{cases}$$

for $\alpha>1$. Also the coefficients h and H_i, $i=0,1$ are real numbers and $\rho=\sigma+i\tau$ is a spectral parameter. The function $q(x)$ is real-valued in $L_2(0,\pi)$.

Inverse spectral theory is an important research topic in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences. Because of the importance of this subject many researchers have studied in this field in recent years that we can see those in the works [1, 7, 8, 9, 11, 12]. Physical applications of boundary value problems with a spectral parameter have increased in recent years. The Sturm-Liouville problems with discontinuous coefficients are also connected with non-homogeneous material properties. Some aspects of the inverse problem theory for discontinuous Sturm-Liouville operators or spectral boundary conditions have been investigated in [2, 5, 6, 15, 16, 17, 19] and other papers. In [18] inverse spectral problems for the differential pencil with spectral boundary conditions have been studied. Latushkin and Pivovarchik also investigated direct and inverse scattering problems for the forked-shaped graph having one half-infinite and two finite edges and have shown how to recover the potentials from the Jost

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function (see for details [13]). In [3] Chugunova studied inverse spectral problems for the Sturm-Liouville equation with eigenvalue-dependent boundary conditions. The author has reconstructed the potential function $q(x)$ and the parameters applied in the boundary conditions from two spectra (eigenvalues) λ_n and μ_n and has used the Gelfand-Levitan method. To complete our studies in this field, we have investigated inverse problems for the Sturm-Liouville equation with discontinuous coefficients and spectral boundary condition by Hochstadt and Lieberman method. The half inverse problem which is the reconstruction of the operator by eigenvalues and knowing the potential function in the half of the interval were obtained by Hochstadt and Lieberman [10]. Later, Hald gave the boundary value problem by the eigenvalues knowing the potential function in the half of the interval and one boundary condition (see [9]).

In this article we prove that the specification of the spectrum considering the potential function in $\left(\frac{\pi}{2},\pi\right)$ uniquely determines the potential $q(x)$ and the coefficient h of the boundary condition. The survey of this problem by the half inverse method is the novelty of the paper. The similar problem has been studied using the Weyl function in [14].

2. Main results

Let $\varphi(x,\rho)$ be the solution of (1.1) under the initial conditions $\varphi(0,\rho)=1$ and $U(\varphi)=0$. Regarding to reference [1], we have the following asymptotic form for this function

$$\varphi(x,\rho)=\begin{cases}\cos\rho x+O\left(\frac{1}{\rho}\exp(|\tau|x)\right), & x<\pi/2,\\ \frac{1}{2}\left(1+\frac{1}{\alpha}\right)\cos\rho\left(\alpha\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}\right) & \\ +\frac{1}{2}\left(1-\frac{1}{\alpha}\right)\cos\rho\left(\alpha\left(\frac{\pi}{2}-x\right)+\frac{\pi}{2}\right) & \\ +O\left(\frac{1}{\rho}\exp\left(|\tau|\left(\alpha\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}\right)\right)\right), & x>\pi/2.\end{cases} \tag{2.1}$$

The functions $\varphi^{(m)}(x,\rho), m=0,1$ are entire in ρ. Also the zeros of the characteristic function are the eigenvalues of L which the characteristic function is denoted by

$$\Delta(\rho)=\varphi'(\pi,\rho)+(H_1\rho+H_0)\varphi(\pi,\rho) \tag{2.2}$$

(see for details [7]). Since the functions $\varphi^{(m)}(x,\rho)$, $m=0,1$ are entire in ρ, the function $\Delta(\rho)$ is an entire function.

By the well known method [4, 20], we can give eigenvalues of the boundary value problem L of the form

$$\rho_n=\rho_n^0+O\left(1\right), \quad n\to\infty. \tag{2.3}$$

These eigenvalues are zeros of the characteristic function

$$\Delta(\rho)=\Delta_0(\rho)+O\left(\exp\left(|\tau|\left(\alpha+1\right)\frac{\pi}{2}\right)\right), \tag{2.4}$$

in which ρ_n^0 is the roots of the function

$$\begin{aligned}\Delta_0(\rho)=\frac{\rho\sqrt{H_1^2+\alpha^2}}{2}\Big(&\big(1+\frac{1}{\alpha}\big)\cos\Big((\rho(\alpha+1)+\sigma_0)\frac{\pi}{2}\Big)\\ &+\big(1-\frac{1}{\alpha}\big)\cos\Big((\rho(\alpha-1)+\sigma_0)\frac{\pi}{2}\Big)\Big),\end{aligned} \tag{2.5}$$

where

$$\sigma_0=\frac{1}{\pi i}\ln\frac{H_1 i-\alpha}{H_1 i+\alpha}.$$

Let us fix $\delta > 0$ and define $G_\delta := \{\rho \in \mathbb{C}; |\rho - \rho_n| \geq \delta,\ \forall n \geq 0\}$. By using the asymptotic formula for $\Delta(\rho)$, we can find

$$|\Delta(\rho)| \geq C_\delta |\rho| \exp\Big((|\tau|(\alpha+1) + |\sigma_0|)\,\frac{\pi}{2}\Big), \tag{2.6}$$

for some positive constant C_δ.

Now we can show the uniqueness solution of the inverse problem. For this purpose, we will consider a boundary value problem $\widetilde{L} = L(\widetilde{q}(x), \widetilde{h})$ of the form

$$-y'' + \widetilde{q}(x)y = \rho^2 r(x)y, \quad x \in (0,\pi), \tag{2.7}$$

with the boundary conditions

$$\begin{gathered} U(y) := y'(0) - \widetilde{h}y(0) = 0, \\ V(y) := y'(\pi) + (H_1\rho + H_0)y(\pi) = 0. \end{gathered} \tag{2.8}$$

Also if a certain symbol denotes an object related to L, then the corresponding symbol with tilde will denote the analogs object related to $\widetilde{L}$.

Theorem 2.1. *If $\rho_n = \widetilde{\rho}_n$ for all $n \in \mathbb{N}$ and $q(x) = \widetilde{q}(x)$ on $\left(\frac{\pi}{2}, \pi\right)$, then $q(x) = \widetilde{q}(x)$ almost everywhere on $(0,\pi)$ and $h = \widetilde{h}$.*

Proof. The function $\varphi(x,\rho)$ satisfies the integral equation

$$\varphi(x,\rho) = \cos\rho x + \int_0^x K(x,t)\cos\rho t dt, \quad x < \frac{\pi}{2}, \tag{2.9}$$

where $K(x,t)$ is a continuous function which does not depend on ρ. There exists a bounded function $V(x,t)$ such that

$$\varphi(x,\rho)\widetilde{\varphi}(x,\rho) = \frac{1}{2}\Big(1 + \cos 2\rho x + \int_0^x V(x,t)\cos 2\rho t dt\Big), \tag{2.10}$$

assuming the function $\widetilde{\varphi}(x,\rho)$ is a solution of (2.7). Also we can write

$$-\varphi''(x,\rho) + q(x)\varphi(x,\rho) = \rho^2 r(x)\varphi(x,\rho), \tag{2.11}$$

$$-\widetilde{\varphi}''(x,\rho) + \widetilde{q}(x)\widetilde{\varphi}(x,\rho) = \rho^2 r(x)\widetilde{\varphi}(x,\rho). \tag{2.12}$$

First we multiply (2.11) by $\widetilde{\varphi}(x,\rho)$ and then (2.12) by $\varphi(x,\rho)$. After subtracting the obtained equations from each other, we get

$$\widetilde{\varphi}''(x,\rho)\varphi(x,\rho) - \widetilde{\varphi}(x,\rho)\varphi''(x,\rho) + (q(x) - \widetilde{q}(x))\varphi(x,\rho)\widetilde{\varphi}(x,\rho) = 0. \tag{2.13}$$

Taking the assumption of the theorem and then integrating the above relation on $[0,\pi]$, we obtain

$$\int_0^{\pi/2} \big(q(x) - \widetilde{q}(x)\big)\widetilde{\varphi}(x,\rho)\varphi(x,\rho)dx = \big(\varphi'(x)\widetilde{\varphi}(x) - \varphi(x)\widetilde{\varphi}'(x)\big)\big(|_0^{\pi/2} + |_{\frac{\pi}{2}}^{\pi}\big). \tag{2.14}$$

Therefore

$$\int_0^{\pi/2} \big(q(x) - \widetilde{q}(x)\big)\widetilde{\varphi}(x,\rho)\varphi(x,\rho)dx + (h - \widetilde{h}) = \varphi'(\pi)\widetilde{\varphi}(\pi) - \varphi(\pi)\widetilde{\varphi}'(\pi). \tag{2.15}$$

Considering

$$H(\rho) := \int_0^{\pi/2} \big(q(x) - \widetilde{q}(x)\big)\widetilde{\varphi}(x,\rho)\varphi(x,\rho)dx + (h - \widetilde{h}), \tag{2.16}$$

we can write

$$H(\rho) = \varphi'(\pi)\widetilde{\varphi}(\pi) - \varphi(\pi)\widetilde{\varphi}'(\pi). \tag{2.17}$$

Therefore by taking the properties of the functions $\varphi(x,\rho)$ and $\widetilde{\varphi}(x,\rho)$, we result that the function $H(\rho)$ is entire in ρ and $H(\rho_n)=0$.

Next, we show that $H(\rho)=0$ for all ρ. It follows from (2.10) and (2.16) that

$$|H(\rho)| \le M \exp\left(|\tau|\pi\right), \tag{2.18}$$

for some positive constant M. Define

$$\phi(\rho) = \frac{H(\rho)}{\Delta(\rho)}. \tag{2.19}$$

This function is entire and by using (2.6), (2.18) and (2.19), we can give

$$\phi(\rho) = O\left(\frac{1}{\rho}\exp\left(\left(|\tau|(1-\alpha) - |\sigma_0|\right)\frac{\pi}{2}\right)\right), \tag{2.20}$$

for large enough ρ. On the base of the Liouville's theorem, we get $\phi(\rho)=0$ for all ρ, and then $H(\rho)=0$. So from (2.16) we have

$$\int_0^{\pi/2}\big(q(x)-\widetilde{q}(x)\big)\widetilde{\varphi}(x,\rho)\varphi(x,\rho)dx + h - \widetilde{h} = 0. \tag{2.21}$$

Now we can write by applying the Riemann-Lebesgue Lemma that for $|\rho|\to\infty$

$$\int_0^{\pi/2}\big(q(x)-\widetilde{q}(x)\big)\widetilde{\varphi}(x,\rho)\varphi(x,\rho)dx = 0, \tag{2.22}$$

and

$$h - \widetilde{h} = 0. \tag{2.23}$$

Therefore

$$h = \widetilde{h}. \tag{2.24}$$

Substituting (2.10) in (2.22), we get

$$\int_0^{\pi/2} Q(x)\Big(1+\cos 2\rho x + \int_x^{\pi} V(x,t)\cos 2\rho t dt\Big)dx = 0, \tag{2.25}$$

where $Q(x) := q(x)-\widetilde{q}(x)$, and so

$$\int_0^{\pi/2} Q(x)dx + \int_0^{\pi/2} cos2\rho t\Big(Q(t) + \int_0^t V(x,t)Q(x)dx\Big)dt = 0. \tag{2.26}$$

By reusing the Riemann-Lebesgue Lemma, we find that

$$\int_0^{\pi/2} Q(x)dx = 0, \quad \int_0^{\pi/2} cos2\rho t\Big(Q(t) + \int_0^t V(x,t)Q(x)dx\Big)dt = 0. \tag{2.27}$$

From the completeness of the function " cos ", we have

$$Q(t) + \int_0^t V(x,t)Q(x)dx = 0, \quad 0 < t < \frac{\pi}{2}. \tag{2.28}$$

This equation is a homogeneous Volterra integral equation and has only the zero solution. Therefore $Q(x)=0$ on $\left(0,\frac{\pi}{2}\right)$ that is $q(x)=\widetilde{q}(x)$ almost everywhere on $\left(0,\frac{\pi}{2}\right)$. This completes the proof. □

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