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PERIODIC SOLUTIONS AND ASYMPTOTIC BEHAVIOR FOR CONTINUOUS ALGEBRAIC DIFFERENCE EQUATIONS

EL HADI AIT DADS, LAHCEN LHACHIMI

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Abstract. Many phenomena in mathematical physics and in the theory of dynamical populations are described by difference equations. The aim of this work is to study existence of periodic solutions and the asymptotic behavior for some algebraic difference equations. The technique used is based on convergence of series associated with the forcing term and the characterization by Fourier coefficients. Our results generalize the main results of our previous results in [3]. For illustration, we provide some examples.

1. Introduction

The existence problem of bounded solutions has been one of the most attractive topics in the qualitative theory of ordinary or functional differential equations for its significance in the physical sciences. In many cases, it is of interest to model the evolution of some system over time. There are two distinct cases. One can think of time as a continuous variable, or one can think of time as a discrete variable. The first case often leads to differential equations, the second case leads to difference equations. We will not discuss differential equations in this note.

Difference equations have many applications in population dynamics, they can be used to describe the evolution of many phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$th generation $x(n+1)$ is a function of the nth generation $x(n)$. This relation express in the following difference equation (see [6])

$$x(n+1) = f(x(n)). \tag{1.1}$$

The main goal of this article is to study the problem of existence of periodic solutions and their asymptotic behaviour of the equation

$$x(t+1) - x(t) = f(t). \tag{1.2}$$

Before, in its first part we consider the qualitative properties and behavior at infinity of real continuous solutions of algebraic difference equations of the form

$$P(x(t+m), \dots, x(t), t) = 0. \tag{1.3}$$

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Where P is a polynomial with real coefficients in its arguments $x(t+m),\dots,x(t)$ and t. The problem was first treated by Lancaster [12], who obtained an upper bound for the rate of increase of the solutions of algebraic difference equations of a given order and pointed out the surprising dissimilarity with the known rates of increase for solutions of differential equations of the same order. The main goal of the second part of this work is to investigate the problem of existence of periodic solutions of the special difference equation (1.2). The work is motivated by some quantitative and qualitative results of the difference equation considered in Ait Dads et al [3].

In the first section, we are concerned with the algebraic equation: Let $d \in \mathbb{N}^*$, $(a_0, a_1, .., a_{d-1}) \in \mathbb{C}^d$, with $a_0 \neq 0$. Let us consider the following difference equation:

$$f(t+d) = \sum_{i=0}^{d-1} a_i f(t+i), \quad \forall t \in \mathbb{R}, \tag{1.4}$$

where $f \in C(\mathbb{R}, \mathbb{C})$, for which we associate the polynomial

$$P(X) = X^d - \sum_{i=0}^{d-1} a_i X^i = \prod_{k=1}^{r} (X - \lambda_k)^{m_k}, \quad r \leq d$$

with the $\lambda_k' s$ are non null complexes which are two by two distinct. And we have the first result concerning this equation.

The organization of the paper is as follows. Section 2 concerns the study of algebraic properties of some difference equation. In section 3, we consider the asymptotic behavior for solutions of special difference equation (1.2). Section 4 deals with the characterization of bounded solutions of the equation (1.2), this last section is divided into three subsections, in the first one, we study the existence of periodic solutions in connection with the forcing term of equation (1.2), the second one is concerned with the charaterization of periodic solution with respect to the Fourier coefficients. The third subsection considers the case where the period is an irrational number. In the last section, to illustrate the work, some examples are given in the paper.

2. Difference Equations

2.1. General solution of (1.4).

Definition 2.1. The function f is called general solution of equation (1.4) if f is continuous from $\mathbb{R}$ to $\mathbb{C}$ and satisfies (1.4).

Proposition 2.2. *If we put $\lambda_k = e^{c_k}$. Then the solutions of* (1.4) *are of the form:*

$$f(t) = \sum_{k=1}^{r} e^{c_k t} \sum_{i=0}^{m_k - 1} p_i(t) t^i$$

where the $p_i' s$ are continuous and 1*-periodic functions.*

Proof. Let $E = C(\mathbb{R}, \mathbb{C})$, and $\tau : E \to E$, defined by

$$\tau(f)(t) = f(t+1), \forall f \in E.$$

Then the set of solutions of (1.4) is $\ker P(\tau)$, and by the Kernels decomposition theorem we have

$$\ker P(\tau) = \bigoplus_{k=1}^{r} \ker(\tau - \lambda_k \,\mathrm{id})^{m_k}.$$

Let $M_k : E \to E$ be defined for all $f \in E$ by $M_k(f)(t) = e^{c_k t} f(t)$. One has

$$\tau \circ M_k(f)(t) = e^{c_k(t+1)} f(t+1) = \lambda_k e^{c_k t} f(t+1) = \lambda_k M_k \circ \tau(f)(t).$$

Then $\tau \circ M_k = \lambda_k M_k \circ \tau$, M_k is reversible and $M_k^{-1}(f)(t) = e^{-c_k t} f(t)$. Moreover

$$M_k^{-1} \circ \tau \circ M_k = \lambda_k \tau,$$

henceforth

$$M_k^{-1} \circ (\tau - \lambda_k \operatorname{id}) \circ M_k = \lambda_k(\tau - \operatorname{id}),$$

and consequently

$$M_k^{-1} \circ (\tau - \lambda_k \operatorname{id})^{m_k} \circ M_k = \lambda_k^{m_k}(\tau - \operatorname{id})^{m_k}.$$

Then

$$\ker(\tau - \lambda_k \operatorname{id})^{m_k} = \ker(\tau - \operatorname{id})^{m_k} \circ M_k^{-1} = M_k(\ker(\tau - \operatorname{id})^{m_k}).$$

So, to complete the proof, it suffices to show that for all $n \geq 1$,

$$\ker(\tau - \operatorname{id})^n = \{t \mapsto \sum_{i=0}^{n-1} p_i(t) t^i, \text{ with the } p_i \text{ continuous and 1-periodic}\}.$$

Denote by E_n the set of polynomial applications from $\mathbb{R}$ to $\mathbb{C}$ with degree $\leq n$. Let us verify that $\forall n \geq 1$, $(\tau - \operatorname{id})(E_n) = E_{n-1}$. Indeed, one has $\forall k \leq n$, $t \mapsto (t+1)^k - t^k$ is a polynomial of degree $\leq k-1$, then

$$(\tau - \operatorname{id})(E_n) \subset E_{n-1.}$$

Since all polynomial which is periodic is a constant function. Then $\dim \ker(\tau - \operatorname{id}) \cap E_n = 1$. Consequently, by Rank formula one has

$$\dim(\tau - \operatorname{id})(E_n) = (\dim E_n) - 1 = \dim E_{n-1}.$$

So, $(\tau - \operatorname{id})(E_n) = E_{n-1}$, it results that

$$(\tau - \operatorname{id})^n(E_i) = \{0\} \quad \forall i \leq n-1.$$

If p is continuous and 1-periodic, one has for $f \in E$,

$$(\tau - \operatorname{id})(pf)(t) = p(t+1)f(t+1) - p(t)f(t) = p(t)(f(t+1) - f(t)).$$

Then

$$(\tau - \operatorname{id})(pf) = p(\tau - \operatorname{id})(f),$$

it follows that

$$(\tau - \operatorname{id})^n(pf) = p(\tau - \operatorname{id})^n(f), \quad \forall i \leq n-1,\ \forall f_i \in E_i,\ (\tau - \operatorname{id})^n(pf_i) = 0,$$

from where, we have that

$$\{t \mapsto \sum_{i=0}^{n-1} p_i(t) t^i, \text{ with the } p_i \text{ continuous and 1-periodic}\} \subset \ker(\tau - \operatorname{id})^n.$$

Let us prove the other inclusion by recurrence on n. For $n = 1$, one has all element of $\ker(\tau - \operatorname{id})$ is continuous and 1-periodic. Let $n \geq 2$, assume that the result is hold for $n-1$, and let us prove that the result remains true for n. Let $f \in \ker(\tau - \operatorname{id})^n$, then $(\tau - \operatorname{id})(f) \in \ker(\tau - \operatorname{id})^{n-1}$ and by recurrent hypothesis one has $(\tau - \operatorname{id})(f)(t) = \sum_{i=0}^{n-2} p_i(t) t^i$, but the map $t \mapsto t^i$ is in $E_i = (\tau - \operatorname{id})(E_{i+1})$, then there exists $q_{i+1} \in E_{i+1}$ such that $t^i = (\tau - \operatorname{id})(q_{i+1})(t)$, from which one has $f - \sum_{i=0}^{n-2} p_i q_{i+1} \in \ker(\tau - \operatorname{id})$. Then there exists a continuous and 1-periodic function p such that $f - \sum_{i=0}^{n-2} p_i q_{i+1} = p$, thus $f = p + \sum_{i=0}^{n-2} p_i q_{i+1}$ with $q_{i+1} \in E_{n-1}$, consequently f is in the required form. $\square$

3. Asymptotic behavior for solutions of the special difference equation

Proposition 3.1. *Let $f \in C(\mathbb{R}, \mathbb{C})$, then the following three properties are equivalent:*

1. *Equation (1.2) admits a solution $x \in C(\mathbb{R}, \mathbb{C})$ such that $\lim_{t\to+\infty} x(t) = 0$.*
2. *for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t+n)$ converges uniformly on $[a, +\infty[$.*
3. *The series $\sum_{n\geq 0} f(t+n)$ converges pointwise on $\mathbb{R}$ and $\lim_{t\to+\infty} \sum_{n=0}^{+\infty} f(t+n) = 0$.*

   Under these conditions, x is unique and is given by $x(t) = -\sum_{n=0}^{+\infty} f(t+n)$.

Proof. (1) $\Rightarrow$ (2) One has $\sum_{k=0}^{n} f(t+k) = x(t+n+1) - x(t)$ with $\lim_{t\to+\infty} x(t) = 0$, then the series $\sum_{n\geq 0} f(t+n)$ converges pointwise on $\mathbb{R}$ and $x(t) = -\sum_{k=0}^{+\infty} f(t+k)$ which ensures the uniqueness of x. On the other hand, for $t \in [a, +\infty[$ one has $-\sum_{k=n+1}^{+\infty} f(t+k) = x(t+n+1)$. For $\varepsilon > 0$, $\exists\ A > 0$, $\forall t \geq A$, $|x(t)| \leq \varepsilon$. Let $n_0 \in \mathbb{N}$ be such that $a + n_0 + 1 \geq A$, then for all $n \geq n_0$, for all $t \in [a, +\infty[$, $|\sum_{k=n+1}^{+\infty} f(t+k)| \leq \varepsilon$, which leads to the uniformly convergence of the series $\sum_{n\geq 0} f(t+n)$ on $[a, +\infty[$.

(2) $\Rightarrow$ (1) One has that for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t+n)$ converges uniformly on $[a, +\infty[$, and $t \mapsto f(t+n)$ is continuous on $\mathbb{R}$, then by the continuity theorem, the map $x : t \mapsto -\sum_{n=0}^{+\infty} f(t+n)$ is continuous on $\mathbb{R}$, and one has $\lim_{t\to+\infty} x(t) = 0$,

$$x(t+1) - x(t) = -\sum_{n=1}^{+\infty} f(t+n) + \sum_{n=0}^{+\infty} f(t+n) = f(t).$$

(2) $\Rightarrow$ (3) for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t+n)$ converges uniformly on $[a, +\infty[$. In particular, the series $\sum_{n\geq 0} f(t+n)$ converges pointwise on $\mathbb{R}$, in the other part, the uniform convergence on $\mathbb{R}^+$ gives that $\sup_{t\in\mathbb{R}^+} |f(t+n)| \to 0$ as $n \to +\infty$. Then for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, and all $t \in \mathbb{R}^+$, $|f(t+n)| \leq \varepsilon$ implies $|f(t)| \leq \varepsilon$ for all $t \geq n_0$, hence $\lim_{t\to+\infty} f(t) = 0$; on the other hand the series $\sum_{n\geq 0} f(t+n)$ converges uniformly on $\mathbb{R}^+$, hence by the inversion limit theorem one has

$$\lim_{t\to+\infty} \sum_{n=0}^{+\infty} f(t+n) = \sum_{n=0}^{+\infty} \lim_{t\to+\infty} f(t+n) = 0.$$

(3) $\Rightarrow$ (2) Let $a \in \mathbb{R}$ and $\varepsilon > 0$, since $\lim_{t\to+\infty} \sum_{n=0}^{+\infty} f(t+n) = 0$, there exists $A > 0$, $|\sum_{k=0}^{+\infty} f(t+k)| \leq \varepsilon$ for all $t \geq A$. Let $n_0 \in \mathbb{N}$ such that $a + n_0 + 1 \geq A$ and $t \in [a, +\infty[$, for $n \geq n_0$ one has

$$|\sum_{k=n+1}^{+\infty} f(t+k)| = |\sum_{k=0}^{+\infty} f(t+1+n+k)| \leq \varepsilon,$$

because $t + 1 + n \geq a + 1 + n_0 \geq A$, from what the series $\sum_{n\geq 0} f(t+n)$ converges uniformly on $[a, +\infty[$. $\square$

Proposition 3.2. *Let $f \in C(\mathbb{R}, \mathbb{C})$. Then the following three properties are equivalent:*

(1) *Equation (1.2) has a solution $x \in C(\mathbb{R}, \mathbb{C})$ such that $\lim_{t \to -\infty} x(t) = 0$.*
(2) *for all $a \in \mathbb{R}$, the series $\sum_{n \geq 1} f(t-n)$ converges uniformly on $]-\infty, a]$.*
(3) *The series $\sum_{n \geq 1} f(t-n)$ converges pointwise on $\mathbb{R}$ and $\lim_{t \to -\infty} \sum_{n=1}^{+\infty} f(t-n) = 0$. Under these conditions x is unique and is defined by $x(t) = \sum_{n=1}^{+\infty} f(t-n)$.*

Proof. Putting $G(t) = x(-t)$ and $F(t) = -f(-t-1)$, the equation can be rewritten as

$$G(t+1) - G(t) = F(t), \quad \forall t \in \mathbb{R}$$

and the proposition 3.1 yields the conclussion. $\square$

Theorem 3.3 (Tauberian Theorem of Hardy [11]). *Let $(u_n)_{n \geq 0}$ be a complex sequence such that $u_n = O(1/n)$ as $n \to +\infty$, $s_n = \sum_{k=0}^{n} u_k$, $\sigma_n = \frac{1}{n} \sum_{k=0}^{n-1} s_k$. If $(\sigma_n)_{n \geq 0}$ is convergent, then $(s_n)_{n \geq 0}$ converges to the same limit.*

Proposition 3.4 (Poisson summation Formula). *Let $f \in C(\mathbb{R}, \mathbb{C})$ such that for all $a \in \mathbb{R}$, the series $\sum_{n \geq 0} f(t+n)$ converges uniformly on $[a, +\infty[$ and the series $\sum_{n \geq 0} f(t-n)$ is also uniformly convergent on $]-\infty, a]$. Then:*

(1) *The function $p(t) = \sum_{n \in \mathbb{Z}} f(t+n)$ is defined and continuous 1-periodic on $\mathbb{R}$, for all $n \in \mathbb{Z}$, $\int_{\mathbb{R}} f(t) e^{i2n\pi t} dt$ converges and*
$$c_n(p) = \hat{f}(2\pi n) = \int_{\mathbb{R}} f(t) e^{-i2\pi n t} dt.$$
(2) *If moreover $\hat{f}(2\pi n) = O(1/|n|)$ as $|n| \to +\infty$, then the Fourier series of p converges pointwise on $\mathbb{R}$ and one has. The Poisson summation formula is*
$$\sum_{n \in \mathbb{Z}} f(t+n) = \sum_{n \in \mathbb{Z}} \sum \hat{f}(2\pi n) e^{i2n\pi t} \quad \forall t \in \mathbb{R}.$$

Proof. By uniform convergence of the series $\sum_{n \geq 0} f(t+n)$ and $\sum_{n \geq 0} f(t-n)$, one has p is well defined and continuous on $\mathbb{R}$, and it is clear that p is 1-periodic,

$$c_k(p) = \int_0^1 p(t) e^{-2ik\pi t} dt = \int_0^1 \sum_{n \in \mathbb{Z}} f(n+t) e^{-2ik\pi t} dt.$$

One has $\sum_{n \geq 0} f(t+n)$ converges uniformly on $[0,1]$. Since $e^{-2ik\pi t}$ is independent from n and $|e^{-2ik\pi t}| = 1$, it follows that $\sum_{n \geq 0} f(t+n) e^{-2ik\pi t}$ is also uniformly convergent on $[0,1]$. Then

$$\begin{aligned}
\int_0^N f(t) e^{-2ik\pi t} dt &= \sum_{n=0}^{N-1} \int_n^{n+1} f(t) e^{-2ik\pi t} dt \\
&= \sum_{n=0}^{N-1} \int_0^1 f(t+n) e^{-2ik\pi t} dt \\
&= \int_0^1 \sum_{n=0}^{N-1} f(t+n) e^{-2ik\pi t} dt
\end{aligned}$$

$$\to \int_0^1 \sum_{n=0}^{+\infty} f(t+n)e^{-2ik\pi t}dt$$

as $N \to +\infty$. If $A > 0$, let $N = E(A)$ (where E denotes the greatest integer function) one has

$$\Big|\int_N^A f(t)e^{-2ik\pi t}dt\Big| \leq \int_N^A |f(t)|dt \leq \int_N^{N+1} |f(t)|dt = \int_0^1 |f(t+N)|dt$$

approaches 0 because $f(t+N)$ converges uniformly to 0. Then we conclude that $\int_{\mathbb{R}^+} f(t)e^{-2ik\pi t}dt$ converges and

$$\int_{\mathbb{R}^+} f(t)e^{-2ik\pi t}dt = \int_0^1 \sum_{n=0}^{+\infty} f(t+n)e^{-2ik\pi t}dt.$$

In the same manner the uniform convergence on $[0,1]$ of the series $\sum_{n\geq 1} f(t-n)$ implies that $\int_{\mathbb{R}^-} f(t)e^{-2ik\pi t}dt$ converges and

$$\int_{\mathbb{R}^-} f(t)e^{-2ik\pi t}dt = \int_0^1 \sum_{n=1}^{+\infty} f(t-n)e^{-2ik\pi t}dt.$$

So $\int_{\mathbb{R}} f(t)e^{-2ik\pi t}dt$ converges and

$$\int_{\mathbb{R}} f(t)e^{-2ik\pi t}dt = \int_0^1 \sum_{n\in\mathbb{Z}} f(t+n)e^{-2ik\pi t}dt = c_k(p).$$

Putting $u_0 = c_0$ and for $n \geq 1$, $u_n = c_n e^{i2\pi nt} + c_{-n}e^{-i2\pi nt}$, $s_n = \sum_{k=0}^n u_k$, s_n is the Fourier series of p, moreover, since p is continuous, then from Fejer, we know that σ_n (The Césaro mean of s_n) converges to $p(t)$ and since $c_n(p) \underset{|n|\to+\infty}{=} O(\frac{1}{|n|})$, $u_n = O(\frac{1}{n})$ from Hardy one has s_n converges to $p(t)$ which yields the Poisson summation formula : $p(t) = \sum_{n\in\mathbb{Z}} c_n e^{i2n\pi t}$. $\square$

Remark 3.5. In the literature, one finds often the main conditions as for example: there exists $\alpha > 1$ such that $f(t) = O(1/|t|^\alpha)$ as $|t| \to +\infty$, and $\sum_{n\in\mathbb{Z}} |\hat{f}(2\pi n)| < +\infty$. Here we have conditions which are weaker namely: $\sum_{n\geq 0} f(t+n)$ (resp $\sum_{n\geq 0} f(t-n)$) converges uniformly on $[a,+\infty[$ (resp. $]-\infty,a]$) and $\hat{f}(2\pi n) = O(1/n)$.

Proposition 3.6. *Let $f \in C(\mathbb{R},\mathbb{C})$, then the following properties are equivalent:*

(1) *Equation (1.2) has a solution $x \in C(\mathbb{R},\mathbb{C})$ such that $\lim_{|t|\to+\infty} x(t) = 0$;*
(2) *for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t+n)$ (resp. $\sum_{n\geq 1} f(t-n)$) converges uniformly on $[a,+\infty[$ (resp. $]-\infty,a]$) and for all $t \in \mathbb{R}$, $\sum_{n\in\mathbb{Z}} f(t+n) = 0$;*
(3) *for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t+n)$ (resp. $\sum_{n\geq 1} f(t-n)$) converges uniformly on $[a,+\infty[$ (resp. on $]-\infty,a]$) and for all $n \in \mathbb{Z}$, $\int_{\mathbb{R}} f(t)e^{-2in\pi t}dt = 0$.*

Under these conditions x is unique and is given by

$$x(t) = \sum_{n=1}^{+\infty} f(t-n) = -\sum_{n=0}^{+\infty} f(t+n).$$

The above proposition is an immediate consequence of the propositions 3.1, 3.2 and 3.4.

Proposition 3.7. *Let $f \in C(\mathbb{R}, \mathbb{C})$ such that f is C^1 piecewise.*

(1) If f' is integrable on $\mathbb{R}^+$ and $\int_{\mathbb{R}^+} f\ (t)dt$ converges, then for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t+n)$ converges uniformly on $[a, +\infty[$.

(2) If f' is integrable on $\mathbb{R}^-$ and $\int_{\mathbb{R}^-} f(t)dt$ converges, then for all $a \in \mathbb{R}$, the series $\sum_{n\geq 0} f(t-n)$ converges uniformly on $]-\infty, a]$.

Proof. (1) Let $s \in \mathbb{R}$. One has

$$\begin{aligned}\int_n^{n+1} f(s+t)dt &= [(t-n-1)f(s+t)]_n^{n+1} - \int_n^{n+1}(t-n-1)f'(s+t)dt \\ &= f(s+n) - \int_n^{n+1}(t-n-1)f'(s+t)dt,\end{aligned}$$

then

$$|f(s+n) - \int_n^{n+1} f(s+t)dt| \leq \int_n^{n+1} |f'(s+t)|dt.$$

Since f' is integrable on $\mathbb{R}^+$ and $\int_{\mathbb{R}^+} f(t)dt$ converges, then the series

$$\sum_{n\geq 0}\int_n^{n+1} |f'(s+t)|dt$$

and

$$\sum_{n\geq 0}\int_n^{n+1} f(s+t)dt$$

are convergent. Hence the series $\sum_{n\geq 0} f(s+n)$ is also convergent and

$$\Big|\sum_{k=n+1}^{+\infty} f(s+k) - \int_{n+1+s}^{+\infty} f(t)dt\Big| \leq \int_{n+1+s}^{+\infty} |f'(t)|dt.$$

For $a \in \mathbb{R}$ and $s \in [a, +\infty[$, we have

$$|\sum_{k=n+1}^{+\infty} f(s+k)\big| \leq \big|\int_{n+1+s}^{+\infty} f(t)dt| + \int_{n+1+a}^{+\infty} \big|f'(t)|dt.$$

Since $\int_{\mathbb{R}^+} f(t)dt$ converges, for $\varepsilon > 0$ there exists $A > 0$ such that

$$|\int_s^{+\infty} f(t)dt| \leq \varepsilon \quad \forall s \geq A, .$$

Let $n_0 \in \mathbb{N}$ such that $n_0 + 1 + a \geq A$, then for all $n \geq n_0$ and all $s \in [a, +\infty[$,

$$|\int_{n+1+s}^{+\infty} f(t)dt| \leq \varepsilon.$$

It results that the series $\sum_{n\geq 0} f(s+n)$ converges uniformly on $[a, +\infty[$.

(2) follows from (1) by putting $\varphi(t) = f(-t)$. $\square$

Proposition 3.8. *Let $f \in C(\mathbb{R}, \mathbb{C})$ such that f is C^1 piecemeal, f' is integrable on $\mathbb{R}$ and $\int_{\mathbb{R}} f(t)dt$ converges. Then $p(t) = \sum_{n\in\mathbb{Z}} f(n+t)$ is well defined continuous and 1-periodic on $\mathbb{R}$, for all $n \in \mathbb{Z}$, $\int_{\mathbb{R}} f(t)e^{i2n\pi t}dt$ converges and*

$$c_n(p) = \hat{f}(2\pi n) = \int_{\mathbb{R}} f(t)e^{-i2\pi nt}dt.$$

The Fourier series of p converges pointwise on $\mathbb{R}$ and one has the Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} f(n+t) = \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n)e^{i2n\pi t} \quad \forall t \in \mathbb{R}.$$

Proof. From Propositions 3.4 and 3.7, it suffices to verify that $c_n(p) = O(1/|n|)$ as $|n| \to +\infty$. Since f' is integrable on $\mathbb{R}$ and $\int_{\mathbb{R}} f(t)dt$ converges, it follows that $\lim_{|t|\to+\infty} f(t) = 0$. An integration by parts leads to

$$|c_n(p)| = |\frac{1}{2ni\pi}\int_{\mathbb{R}} f'(t)e^{-i2n\pi t}dt| \le \frac{1}{2|n|\pi}\int_{\mathbb{R}} |f'(t)|dt,$$

from which, we have that $c_n(p) = O(1/|n|)$ as $|n| \to +\infty$. □

3.0.1. *Characterization of bounded solutions.* Let $\mathcal{B} = \mathbf{C}_B(\mathbb{R},\mathbb{C})$ be the space of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{C}$. We have seen for $f \in \mathbf{C}(\mathbb{R},\mathbb{C})$, that equation (1.2) has a solution in $\mathcal{B}$ if and only if there exists $c > 0$ such that for all $t \in \mathbb{R}$, and all$n \in \mathbb{N}$, $|\sum_{k=0}^{n} f(t+k)| \le c$, see [3]. In the sequel , we consider the case where the period of f is in $\mathbb{R}\setminus\mathbb{Q}$. Let C_T be the space of continuous and T-periodic functions from $\mathbb{R}$ to $\mathbb{C}$.

Proposition 3.9. *Let $f \in C_T$ with $T \notin \mathbb{Q}$, then equation* (1.2) *has a solution in $\mathcal{B}$ if and only if the sequence $\sum_{k=0}^{n} f(k)$ is bounded.*

Proof. If (1.2) admits a solution in $\mathcal{B}$, then there exists $c > 0$ such that for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, $|\sum_{k=0}^{n} f(t+k)| \le c$ in particular for $t = 0$ one has the sequence $\sum_{k=0}^{n} f(k)$ is bounded. Reciprocally suppose that there exists $c > 0$ such that for all $n \in \mathbb{N}$, $|\sum_{k=0}^{n} f(k)| \le c$, then

$$\big|\sum_{k=0}^{n} f(k+m)\big| = \big|\sum_{k=m}^{n+m} f(k)\big| = \big|\sum_{k=0}^{n+m} f(k) - \sum_{k=0}^{m-1} f(k)\big| \le 2c \quad \forall m \in \mathbb{N},$$

and since $f \in C_T$, we have

$$|\sum_{k=0}^{n} f(k+m-Tq)| \le 2c \quad \forall q \in \mathbb{N},$$

but $T \notin \mathbb{Q}$ then $\mathbb{N} - T\mathbb{N}$ is dense in $\mathbb{R}$, from which

$$\big|\sum_{k=0}^{n} f(k+t)\big| \le 2c, \quad \forall t \in \mathbb{R},\ \forall n \in \mathbb{N}.$$

It follows that equation (1.2) has a solution in $\mathcal{B}$. □

3.0.2. *Existence of periodic solutions.*

Lemma 3.10. *Let g be a continuous function which has a period $m \in \mathbb{N}^*$. Then*

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(t+k) = \frac{1}{m}\sum_{k=0}^{m-1} g(t+k) \quad \forall t \in \mathbb{R}.$$

Proof. Since $u \mapsto g(u+t)$ has a same period as g, it suffices to consider the case $t = 0$. Let $n \in \mathbb{N}^*$, making the Euclidean division of n by $m : n = q_n m + r_n$, $0 \le r_n \le m-1$.

$$\sum_{k=0}^{n-1} g(k) = \sum_{k=0}^{q_n-1}\sum_{i=0}^{m-1} g(km+i) + \sum_{i=0}^{r_n-1} g(q_n m+i) = q_n\sum_{i=0}^{m-1} g(i) + \sum_{i=0}^{r_n-1} g(i),$$

then

$$\frac{1}{n}\sum_{k=0}^{n-1} g(k) = \frac{q_n}{n}\sum_{i=0}^{m-1} g(i) + \frac{1}{n}\sum_{i=0}^{r_n-1} g(i) \underset{n\to+\infty}{\to} \frac{1}{m}\sum_{i=0}^{m-1} g(i).$$

□

Proposition 3.11. *Let f be a continuous function on $\mathbb{R}$, which has a period $m \in \mathbb{N}^*$. If equation* (1.2) *has a continuous solution x defined on $\mathbb{R}$ and bounded on $\mathbb{R}^+$, then*

$$\sum_{k=0}^{m-1} f(t+k) = 0 \quad \forall t \in \mathbb{R}.$$

Proof. Assume that (1.2) has a solution x which is bounded on $\mathbb{R}^+$. Then

$$\sum_{k=0}^{n-1} f(t+k) = \sum_{k=0}^{n-1} x(t+k+1) - x(t+k) = x(t+n) - x(t),$$

then $\lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(t+k) = 0$ and from Lemma 3.10 one has

$$\sum_{k=0}^{m-1} f(t+k) = 0 \quad \forall t \in \mathbb{R}.$$

□

Corollary 3.12. *Let f be a continuous function defined on $\mathbb{R}$, which is not zero with period* 1*. Then* (1.2) *has not a continuous solution x on $\mathbb{R}$ and bounded on $\mathbb{R}^+$.*

Proposition 3.13. *Let f be a continuous function on $\mathbb{R}$, with a period $m \in \mathbb{N}^*$ and $m \geq 2$. Then* (1.2) *has a solution x which is continuous on $\mathbb{R}$ and bounded on $\mathbb{R}^+$ if and only if f is of the form*

$$f(t) = \sum_{k=1}^{m-1} f_k(t)\exp(\frac{2ik\pi}{m}t),$$

where the f_k are continuous and 1*-periodic. In this case all solutions of* (1.2) *are m-periodic of the form*

$$f(t) = f_0(t) + \sum_{k=1}^{m-1} f_k(t)\frac{\exp(\frac{2ik\pi}{m}t)}{\exp(\frac{2ik\pi}{m}) - 1}$$

where f_0 is continuous and 1*-periodic.*

Proof. If $f(t) = \sum_{k=1}^{m-1} f_k(t)\exp(\frac{2ik\pi}{m}t)$, then it is clear that

$$f_0(t) = \sum_{k=1}^{m-1} f_k(t)\frac{\exp(\frac{2ik\pi}{m}t)}{\exp(\frac{2ik\pi}{m}) - 1}$$

is a particular solution of equation (1.2), since $\ker(\tau - \mathrm{id})$ is formed by continuous and 1-periodic functions, then x is of the request form and is m-periodic. Conversely, if equation (1.2) has a bounded solution defined on $\mathbb{R}^+$, then from proposition 3.11, for all $t \in \mathbb{R}$, $\sum_{k=0}^{m-1} f(t+k) = 0$. On the other hand,

$$1 + X + X^2 + .. + X^{m-1} = \prod_{k=1}^{m-1}(X - e^{\frac{2ik\pi}{m}})$$

and from proposition 2.2,

$$f(t)=\sum_{k=1}^{m-1} f_k(t)\exp(\frac{2ik\pi}{m}t)$$

where the f_k are any continuous and 1-periodic. □

Proposition 3.14. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function which is T-periodic with $T=\frac{n}{p}\in\mathbb{Q}^{+*}$, $n\wedge p=1$. Then the following statements are equivalent:*

1. *Equation (1.2) has a solution x continuous on $\mathbb{R}$ and bounded on $\mathbb{R}^+$;*
2. *for all $t\in\mathbb{R}$, $\sum_{k=0}^{p-1} f(t+k)=0$;*
3. *for all $k\in\mathbb{Z}$, $c_{2k\pi}(f)=0$.*

Under these conditions equation (1.2) has a unique T-periodic solution x such that for all $k\in\mathbb{Z}$, $c_{2k\pi}(x)=0$. Moreover x is given by

$$x(t)=\frac{1}{p}\sum_{k=1}^{p} kf(t+k-1).$$

Proof. Let x_1,x_2 be two solutions of equation (1.2) satisfying $c_{2k\pi}(x_1)=c_{2k\pi}(x_2)=0$ for all $k\in\mathbb{Z}$. Then $x=x_1-x_2$ is 1-periodic and for all $k\in\mathbb{Z}$, $c_{2k\pi}(x)=0$. Then $x=0$ and $x_1=x_2$.

(1) $\Longrightarrow$ (2) If f is $T=\frac{n}{p}$ periodic, then it is p periodic and the end of the proof results from proposition 3.11.

(2) $\Leftrightarrow$ (3) The application $x:t\mapsto\sum_{k=0}^{p-1} f(t+k)$ is 1 periodic, then $x=0\Leftrightarrow$ $\forall m\in\mathbb{Z}$, $c_{2m\pi}(x)=0$. However

$$c_{2m\pi}(x)=\sum_{k=0}^{p-1} e^{2mk\pi i}c_{2m\pi}(f)=pc_{2m\pi}(f)$$

then (2) $\Leftrightarrow$ (3).

(2) $\Longrightarrow$ (1) One has

$$\begin{aligned}\sum_{k=0}^{p-1}(X^k-1)&=(X-1)\sum_{k=1}^{p-1}\sum_{j=0}^{k-1}X^j=(X-1)\sum_{j=0}^{p-2}\sum_{k=j+1}^{p-1}X^j\\&=(X-1)\sum_{j=0}^{p-2}(p-1-j)X^j=(X-1)\sum_{k=1}^{p-1}(p-k)X^{k-1}\\&=(X-1)\sum_{k=1}^{p}(p-k)X^{k-1},\end{aligned}$$

it follows that

$$\sum_{k=0}^{p-1}\tau^k=p\,\mathrm{id}+(\tau-\mathrm{id})\sum_{k=1}^{p}(p-k)\tau^{k-1}.$$

Then

$$0=\sum_{k=0}^{p-1} f(t+k)=pf(t)+(\tau-\mathrm{id})\sum_{k=1}^{p}(p-k)f(t+k-1).$$

But

$$\sum_{k=1}^{p} pf(t+k-1)=p\sum_{k=0}^{p-1} f(t+k)=0,$$

hence

$$f(t) = \frac{1}{p}(\tau - \mathrm{id}) \sum_{k=1}^{p} k f(t+k-1).$$

Consequently

$$x(t) = \frac{1}{p} \sum_{k=1}^{p} k f(t+k-1),$$

is a solution of (1.2) which is continuous on $\mathbb{R}$ and T-periodic, in particular it is bounded on $\mathbb{R}^+$. Moreover one has that for all $m \in \mathbb{Z}$,

$$c_{2m\pi}(x) = \frac{1}{p} \sum_{k=1}^{p} k e^{2m(k-1)i\pi} c_{2m\pi}(f) = 0,$$

because $c_{2m\pi}(f) = 0$. $\square$

Proposition 3.15. *Suppose that f is continuous T-periodic with mean value zero for $T \notin \mathbb{Q}$. Then* (1.2) *has a T-periodic solution if and only if $\frac{1}{n}\sum_{k=0}^{n-1}(n-k)\tau^k f$ converges uniformly on $\mathbb{R}$. Under these conditions,*

$$x_0(t) = -\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} (n-k) f(t+k),$$

is a unique T-periodic solution of (1.2) *with mean value zero.*

Proof. One has f is T periodic with $T \notin \mathbb{Q}$ then for all $\lambda \in 2\pi\mathbb{Z}^*$, $c_\lambda(f) = 0$. In fact if there exists $k \in \mathbb{Z}$ such that $\lambda = \frac{2k\pi}{T}$, we will have $T \in \mathbb{Q}$ which is absurd. Moreover one has f is mean value zero then for all $\lambda \in 2\pi\mathbb{Z}$, $c_\lambda(f) = 0$ namely $f \in F$. From the almost periodic case [4] equation (1.2) has a solution in $AP(\mathbb{R}, \mathbb{C})$ if and only if the sequence $\frac{1}{N+1}\sum\limits_{n=0}^{N} s_n$ converges uniformly on $\mathbb{R}$, where $s_n = \sum_{k=0}^{n} f(t+k)$ and under these conditions $x_0(t) = -\lim_{N\to+\infty} \frac{1}{N+1}\sum\limits_{n=0}^{N} s_n(t)$ is the unique solution in F of equation (1.2) and as f is T-periodic, then x is also, on the other hand the set of functions which are T-periodic with mean value zero is include in F, (since $T \notin \mathbb{Q}$) then x is the unique T-periodic solution with mean value zero. As

$$\frac{1}{N+1} \sum_{n=0}^{N} s_n(t) = \frac{1}{N+1} \sum_{k=0}^{N} (N+1-k) f(t+k),$$

then x_0 is written as

$$x_0(t) = -\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} (n-k) f(t+k).$$

$\square$

Remark 3.16. The set of T-periodic solutions of equation (1.2) is of the form $x_0 + c$ where $c \in \mathbb{C}$.

3.0.3. *Characterization of sequences which are Fourier coefficients of periodic and continuous functions.* Let $f \in C_T$, $\omega_n = \frac{2n\pi}{T}$, $(c_n(f))_{n\in\mathbb{Z}}$ the family of Fourier coefficients of f:

$$c_n(f) = \frac{1}{T}\int_0^T f(t)e^{-i\omega_n t}dt.$$

From Féjer results, we know that the Césaro mean of the sequence $S_n(f)(t) = \sum_{k=-n}^{n} c_k(f)e^{i\omega_k t}$ converges uniformly to f on $\mathbb{R}$. Now, we give the proof of the reciprocal result.

Proposition 3.17. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a family of complex numbers such that the Césaro mean of the sequence $\sum\limits_{k=-n}^{n} \lambda_k e^{i\omega_k t}$ is uniformly convergent on $\mathbb{R}$, then its limit x is the unique function in C_T, such that $\forall n \in \mathbb{Z}$, $c_n(x) = \lambda_n$.*

Proof. Denote $S_n(t) = \sum_{k=-n}^{n} \lambda_k e^{i\omega_k t}$ and $x(t) = \lim_{N\to+\infty} \frac{1}{N}\sum_{n=0}^{N-1} S_n(t)$. Since $\frac{1}{N}\sum_{n=0}^{N-1} S_n \in C_T$ and the uniform convergence of the series, then $f \in C_T$. Now let us verify that for all $p \in \mathbb{Z}$, $c_p(x) = \lambda_p$. In fact, for $n \geq |p|$, $c_p(S_n) = \lambda_p$, then

$$c_p(\frac{1}{N}\sum_{n=|p|}^{N-1} S_n) = \frac{N-|p|}{N}\lambda_p \quad \forall N > |p|,$$

and as $\frac{1}{N}\sum_{n=|p|}^{N-1} S_n$ converges uniformly to f, then

$$\lim_{N\to+\infty} c_p(\frac{1}{N}\sum_{n=|p|}^{N-1} S_n) = c_p(x)$$

from what, for all $p \in \mathbb{Z}$, $c_p(x) = \lambda_p$. The uniqueness of x results from that the elements of C_T which have the same Fourier coefficients are equal . □

Let $T \in \mathbb{R} \setminus \mathbb{Q}$ which is non negative, $f \in C_T$: We will give a characterization of the solutions $x \in C_T$ of (1.2) by the Fourier coefficients of f.

Proposition 3.18. *Equation (1.2) has a solution $f \in C_T$ if and only if $c_0(f) = 0$ and the Césaro mean of the sequence $\sum_{1\leq|k|\leq n} \frac{c_k(f)}{e^{i\omega_k}-1}e^{i\omega_k t}$ is uniformly convergent, its limit x_0 is the unique solution of (1.2) satisfying $c_0(x_0) = 0$. The solutions in C_T of (1.2) are of the form $x = c + x_0$, where c is a constant.*

Proof. Note that for $k \neq 0$, $\frac{c_k(f)}{e^{i\omega_k}-1}$ is well definite because $T \notin \mathbb{Q}$.

(⇒) Let $x \in C_T$ be a solution of (1.2), then

$$c_0(x) = c_0(t \to x(t+1)) - c_0(x) = c_0(x) - c_0(x) = 0,$$

for $k \neq 0$ one has $e^{i\omega_k}c_k(x) - c_k(x) = c_k(f)$, from where $c_k(x) = \frac{c_k(f)}{e^{i\omega_k}-1}$ and then

$$\sum_{1\leq|k|\leq n} \frac{c_k(f)}{e^{i\omega_k}-1}e^{i\omega_k t} = \sum_{1\leq|k|\leq n} c_k(x)e^{i\omega_k t}.$$

From Fejer, we conclude that the Césaro mean of the following sequence $\sum_{1\leq|k|\leq n} \frac{c_k(f)}{e^{i\omega_k}-1}e^{i\omega_k t}$ converges uniformly to $x_0 = x - c_0(x)$. If moreover $c_0(x) = 0$, then the Fourier coefficients of x are unique; which gives the uniqueness of the solution $x \in C_T$ such that $c_0(x) = 0$, and the solutions in C_T of (1.2) are of the form $x = c + x_0$ where c is a constant.

($\Leftarrow$) We have the Césaro mean of the sequence

$$S_n = \sum_{1\le|k|\le n} \frac{c_k(f)}{e^{i\omega_k}-1} e^{i\omega_k t}$$

is uniformly convergent, then thinks to proposition 3.17 its limit $x_0 \in C_T$ is such that $c_0(x_0) = 0$ and for all $k \in \mathbb{Z}^*$, $c_k(x_0) = \frac{c_k(f)}{e^{i\omega_k}-1}$, furthermore one has $c_0(f) = 0$, then the application $t \mapsto x_0(t+1) - x_0(t)$ has the same Fourier coefficients as f, from which $x_0(t+1) - x_0(t) = f(t)$. $\square$

3.0.4. *Characterization of irrational periodic solutions.* In this section, we discuss the characterization of irrational periodic solution. Let T be a nonnegative irrational number, then $\mathbb{R}/T\mathbb{Z}$ is a metric space with respect to the distance defined by

$$d(\bar{x},\bar{y}) = |\exp(\frac{2\pi i}{T}x) - \exp(\frac{2\pi i}{T}y)|.$$

We denote

$$\mathbb{U} = \{z \in \mathbb{C} : \ |z| = 1\}.$$

Proposition 3.19. *(1) The map $\phi : \mathbb{R} \to \mathbb{R}/T\mathbb{Z}$ defined by $\phi(x) = \bar{x}$ is continuous on $\mathbb{R}$.*

(2) The map $r : \mathbb{R} \to [0, T[$, $x \mapsto x - T[\frac{x}{T}]$ ($[\cdot]$ denotes the greatest integer function) is T-periodic and $\overline{r(x)} = \bar{x}$.

(3) The metric space $(\mathbb{R}/T\mathbb{Z}, d)$ is compact.

(4) The map $h : \mathbb{R}/T\mathbb{Z} \to \mathbb{U}$, $\bar{x} \mapsto \exp(\frac{2\pi i}{T}x)$ is an homeomorphism.

(5) Let $\bar{x}_0 \in \mathbb{R}/T\mathbb{Z}$, then $\{\bar{x}_0 + \bar{n},\ n \in \mathbb{N}\}$ is dense in $\mathbb{R}/T\mathbb{Z}$.

Proof. (1) results from $x \mapsto \bar{x}$ begin $\frac{2\pi}{T}$ Lipschitzian, thus continuous.

(2) is a consequence of the facts that $E(\frac{x}{T}) \le \frac{x}{T} < 1 + E(\frac{x}{T})$ and $x \mapsto E(x)$ is 1-periodic.

(3) Let $(\bar{x}_n)_n$ be a sequence of $\mathbb{R}/T\mathbb{Z}$, $r(x_n)$ is bounded, which has a convergent subsequence $r(x_{\varphi(n)})$, let ℓ be its limit, and by continuity of $x \mapsto \bar{x}$ one has $\bar{x}_{\varphi(n)}$ tends to $\bar{\ell}$.

(4) It is clear that h is bijective and one has $d(\bar{x}_n, \bar{x}) = |h(\bar{x}_n) - h(\bar{x})|$, then h is bicontinuous.

(5) As T is a nonnegative irrational number, then $\mathbb{N} - T\mathbb{N}$ is dense in $\mathbb{R}$, it follows that $x_0 + \mathbb{N} - T\mathbb{N}$ is also dense, and by continuity of $x \mapsto \bar{x}$, we have that $\{\bar{x}_0 + \bar{n}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}/T\mathbb{Z}$. $\square$

Notation We denote by $\mathcal{P}_T$ the space of continuous functions defined from $\mathbb{R}$ to $\mathbb{C}$ which are T-periodic.

Proposition 3.20. *The map $\psi : \mathcal{P}_T \to C(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$, defined by $f \longmapsto \tilde{f}$, where $\tilde{f}(\bar{x}) = f(x)$, is a bijection.*

Proof. $\tilde{f}$ is well defined since f is T-periodic. Let us prove that it is continuous: In deed if $\bar{x}_n$ tends to $\bar{x}$. Then using (4) of proposition 3.19 one has that for all $k \in \mathbb{Z}$, $[\exp(\frac{2\pi i}{T}x_n)]^k$ tends to $[\exp(\frac{2\pi i}{T}x)]^k$, hence for all trigonometric polynomial $P \in \mathcal{P}_T$, one has $P(x_n)$ goes to $P(x)$ and since f is a uniform limit of trigonometric polynomials, then $f(x_n)$ tends to $f(x)$, from where $\tilde{f}$ is continuous, it is clear that

ψ is one to one. It remains to prove that it is surjective, let $g \in C(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$, putting $f = g \circ p$ where $p : x \mapsto \bar{x}$, then

$$f(x+T) = g(\overline{x+T}) = g(\bar{x}) = f(x)$$

and f is continuous as composition of continuous functions, then $g = \tilde{f}$, so ψ is surjective. $\square$

Corollary 3.21. *Let $u \in \mathcal{P}_T$. Then f is a solution in $\mathcal{P}_T$ of the equation $f(x+1) - f(x) = u(x)$ if and only if $\tilde{f}$ is a solution in $C(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$ of the following equation:*

$$g(\bar{x} + \bar{1}) - g(\bar{x}) = \tilde{u}(\bar{x}). \tag{3.1}$$

Proposition 3.22. *Let $u \in \mathcal{P}_T$ and g a solution in $C(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$ of equation (3.1). Let $K = \{(\bar{x}, f(\bar{x}))$, for $\bar{x} \in \mathbb{R}/T\mathbb{Z}\}$. Then K is a non empty compact subset which is stable by the map*

$$s : (\bar{x}, y) \mapsto (\bar{x} + \bar{1}, y + u(x))$$

and K is minimal for the inclusion, namely for all non empty compact K' which is invariant under s, one has $K' \subset K \Rightarrow K' = K$.

Proof. One has K is a range of the compact $\mathbb{R}/T\mathbb{Z}$ by a continuous map $\bar{x} \mapsto (\bar{x}, g(\bar{x}))$, then K is a non empty compact set. Let $(\bar{x}, g(\bar{x})) \in K$. Then

$$s(\bar{x}, g(\bar{x})) = (\bar{x} + \bar{1}, g(\bar{x}) + u(x)) = (\bar{x} + \bar{1}, g(\bar{x} + \bar{1})),$$

from where K is invariant under s. Let K' a non empty compact stable by s such that $K' \subset K$, $(\bar{x}_0, f(\bar{x}_0)) \in K'$, hence for all $n \geq 1$, $s^n((\bar{x}_0, f(\bar{x}_0))) \in K'$, namely $(\bar{x}_0 + \bar{n}, f(\bar{x}_0 + \bar{n})) \in K'$. We have $\{\bar{x}_0 + \bar{n},\ n \in \mathbb{N}\}$ is dense in $\mathbb{R}/T\mathbb{Z}$, but as g is continuous and K' is closed, then for all $\bar{x} \in \mathbb{R}/T\mathbb{Z}$, $(\bar{x}, f(\bar{x})) \in K'$, namely $K \subset K'$ so $K' = K$. $\square$

Proposition 3.23. *Let $u \in \mathcal{P}_T$ and K a non empty compact invariant under the application $s : (\bar{x}, y) \mapsto (\bar{x} + \bar{1}, y + u(x))$ and K is minimal for the inclusion. Then* (3.1) *has a solution f in $C(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$ such that $K = \{(\bar{x}, f(\bar{x}))$, for all $\bar{x} \in \mathbb{R}/T\mathbb{Z}\}$.*

Proof. One has $K \neq \emptyset$, let $(\bar{x}_0, y_0) \in K$, stable by s, then for all $n \geq 1$, $s^n(\bar{x}_0, y_0) \in K$, namely

$$(\bar{x}_0 + \bar{n}, y_0 + \sum_{k=0}^{n-1} u(x+k)) \in K,$$

one has $p : (\bar{x}, y) \mapsto \bar{x}$ is continuous and $\bar{x}_0 + \bar{n} \in p(K)$ which is compact, but $\{\bar{x}_0 + \bar{n}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}/T\mathbb{Z}$. Then $\mathbb{R}/T\mathbb{Z} \subset p(K)$ from where $p(K) = \mathbb{R}/T\mathbb{Z}$. Let $(\bar{x}, y) \in K$ and $d \in \mathbb{C}^*$. Consider $f : \mathbb{R}/T\mathbb{Z} \times \mathbb{C} \to \mathbb{R}/T\mathbb{Z} \times \mathbb{C}$ defined by

$$f(\bar{w}, \lambda) = (\bar{w}, \lambda + d).$$

One has f is continuous and commutes with s. Then $f(K) \cap K$ is also a compact stable by s. As K is minimal then

$$f(K) \cap K = K \quad \text{or} \quad f(K) \cap K = \emptyset,$$

similarly $f^{-1}(K) \cap K$ is a compact stable by s, leads to

$$f^{-1}(K) \cap K = K \quad \text{or} \quad f^{-1}(K) \cap K = \emptyset$$

which is equivalent to $K \cap f(K) = \emptyset$. Then if we assume that $f(K) \cap K \neq \emptyset$, we will have at once

$$f(K) \cap K = K \quad \text{and} \quad f^{-1}(K) \cap K = K$$

which gives

$$K \subset f(K) \quad \text{and} \quad K \subset f^{-1}(K),$$

then $f(K) = K$. It follows that for all $n \geq 1$, $f^n(K) = K$ and for all $n \geq 1$, $(\bar{x}, y+nd) \in K$ which contradicts the boundedness of K, then $K \cap f(K) = \emptyset$, hence for all $d \in \mathbb{C}^*$, $(\bar{x}, y+d) \notin K$. So $(\bar{x}, y) \in K$ and $(\bar{x}, z) \in K$ implies that $y = z$. Moreover $p(K) = \mathbb{R}/T\mathbb{Z}$, then for all $\bar{x} \in \mathbb{R}/T\mathbb{Z}$, there exists a unique $y \in \mathbb{C}$ such that $(\bar{x}, y) \in K$; so we will define a map $g : \mathbb{R}/T\mathbb{Z} \to \mathbb{C}$ such that

$$K = \{(\bar{x}, g(\bar{x})) : \text{ such that } \bar{x} \in \mathbb{R}/T\mathbb{Z}\}.$$

Let us prove that g is continuous, indeed if $\bar{x}_n$ goes to $\bar{a}$, as K is compact then its projections are compact, then $g(\bar{x}_n)$ is a sequence in the compact. To prove that it converges to $g(\bar{a})$ it suffices to prove that $g(\bar{a})$ is the alone adhesion value of $g(\bar{x}_n)$. Let ℓ an adhesion value of $g(\bar{x}_n)$, $\ell = \lim_{n\to+\infty} g(\bar{x}_{\varphi(n)})$, one has

$$(\bar{x}_{\varphi(n)}, g(\bar{x}_{\varphi(n)})) \underset{n\to+\infty}{\longrightarrow} (\bar{a}, \ell) \in K,$$

hence $\ell = g(\bar{a})$. As K is invariant under s, then $(\bar{x} + \bar{1}, g(\bar{x}) + u(x)) \in K$; from where

$$g(\bar{x} + \bar{1}) = g(\bar{x}) + u(x).$$

□

Proposition 3.24. *Let $T \in \mathbb{R}^{+*}\backslash\mathbb{Q}$, $u \in \mathcal{P}_T$, such that the sequence $\sum_{k=0}^{n} u(k)$ is bounded, then for all $(x_0, y_0) \in \mathbb{R} \times \mathbb{C}$, the equation $f(x+1) - f(x) = u(x)$ has a unique solution in $\mathcal{P}_T$ such that $f(x_0) = y_0$.*

Proof. Uniqueness: Let f and g be two solutions, then $f - g \in \mathcal{P}_T \cap \mathcal{P}_1$, but the set of periods of $f - g$ is a group G, and as it contains $T\mathbb{Z} + \mathbb{Z}$ which is dense in $\mathbb{R}$ and that G is closed then $G = \mathbb{R}$, hence $f - g$ is constant, but $(f-g)(x_0) = 0$ consequently $f - g = 0$ which leads to the uniqueness.

Existence: Let $s : (\bar{x}, y) \mapsto (\bar{x} + \bar{1}, y + u(x))$ and $A = \{s^n(\bar{0}, 0), n \in \mathbb{N}\}$, one has that for all $n \geq 1$, $s^n(\bar{0}, 0) = (\bar{n}, \sum_{k=0}^{n-1} u(k))$, then there exists $c > 0$ such that $A \subset (\mathbb{R}/T\mathbb{Z}) \times \{z \in \mathbb{C}, |z| \leq c\}$ which is compact and non empty then $L = \bar{A}$ is a non empty compact and as A is invariant under s and s is continuous, then L is compact and invariant under s. Let

$$E = \{K \text{ non empty compact subset of } (\mathbb{R}/T\mathbb{Z}) \times \mathbb{C} \text{ stable by } s\},$$

one has E is non empty because $L \in E$. Let $(K_i)_{i\in I}$ a family totally ordinate of E, then $\cap_{i\in I} K_i$ is also compact invariant under s, if it was empty, it could exist a finite subset J of I, such that $\cap_{i\in J} K_i = \emptyset$ and as the family $(K_i)_{i\in I}$ is a family totally ordinate of E, then there exists $j \in J$ such that $\cap_{i\in J} K_i = K_j$; hence $K_j = \emptyset$ absurd. Then $\cap_{i\in I} K_i \neq \emptyset$ and by the Zorn lemma one has E has a minimal element then thanks to proposition 3.23, the equation $g(\bar{x} + \bar{1}) - g(\bar{x}) = \tilde{u}(\bar{x})$ has a solution in $C(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$ and from the corollary 3.21 the equation $f(x+1) - f(x) = u(x)$ has a solution f_0 in $\mathcal{P}_T$. Then if we put $f(x) = f_0(x) + y_0 - f_0(x_0)$ then f is an answer to the question. □

4. Examples

In this section, we give some examples to demonstrate the results obtained in previous sections.

Example 4.1 (Solutions which go to 0)**.** Let $\lambda \in \mathbb{R}^*$, I an interval in $\mathbb{R}$ and $\varphi : I \mapsto \mathbb{C}$ such that $u \mapsto \frac{\varphi(u)}{e^{i\lambda u}-1}$ is integrable on I. Then for all $n \in \mathbb{Z}$,

$$\int_{\mathbb{R}} \Big(\int_I \varphi(u) e^{it(\lambda u + 2n\pi)} du \Big) dt = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} \int_I \varphi(u) e^{i\lambda(t+n)u} du = 0.$$

In fact, we consider the equation (1.2), where

$$f(t) = \int_I \varphi(u) e^{i\lambda t u} du.$$

Since $u \mapsto \frac{\varphi(u)}{e^{i\lambda u}-1}$ is integrable on I, then φ is also. Thus f is defined and continuous on $\mathbb{R}$, on the other hand, thanks to Riemann Lebesgue lemma, we have $\lim_{|t|\to+\infty} f(t) = 0$, and

$$x(t) = \int_I \frac{\varphi(u)}{e^{i\lambda u} - 1} e^{i\lambda t u} du$$

is the unique solution of equation (1.2) such that $\lim_{|t|\to+\infty} x(t) = 0$. Hence, from proposition 3.6, it follows that for all $n \in \mathbb{Z}$,

$$\int_{\mathbb{R}} f(t) e^{2in\pi t} dt = 0 \quad \text{and} \quad \sum_{n\in\mathbb{Z}} f(t+n) = 0.$$

Example 4.2 (Examples of periodic solutions)**.** (1) Thanks to proposition (3.17), if the Césaro mean of

$$s_n(t) = \sum_{1\le|k|\le n} \frac{c_k(f)}{e^{i\omega_k} - 1} e^{i\omega_k t}$$

converges uniformly, in particular if s_n is uniformly convergent, then its limit x_0 is a solution of equation (1.2).

(2) Let $(\alpha_n)_{n\in\mathbb{N}^*}$ be a nonnegative non increasing sequence such that $\alpha_n = o(1/n)$ (for example $\alpha_n = \frac{1}{n\ln(n+1)}$),

$$f(t) = \sum_{n=1}^{+\infty} \alpha_n \sin(n)\cos(2t+1)n,$$

then equation (1.2) has as a π-periodic solution

$$x_0(t) = \sum_{n=1}^{+\infty} \frac{\alpha_n \sin(2nt)}{2}$$

which has mean value zero. In fact we have

$$\alpha_n \sin(n)\cos(2t+1)n = \frac{1}{2}\alpha_n \sin(n)(e^{(2t+1)in} + e^{-(2t+1)in})$$

Hence, if we assume in first time that the series which defines f is uniformly convergent, we will have $c_0(f) = 0$, and for $n \ge 1$,

$$c_n(f) = \frac{1}{2}\alpha_n \sin(n) e^{in}, \ c_{-n}(f) = \overline{c_n(f)}.$$

So

$$c_n(x_0) = \frac{c_n(f)}{e^{2in} - 1} = \frac{\alpha_n \sin(n)}{2} \frac{e^{in}}{e^{2in} - 1} = -\frac{i}{4}\alpha_n$$

Thus we obtain

$$c_n(x_0)e^{2int} + c_{-n}(x_0)e^{-2int} = 2\operatorname{Re}(c_n(x_0)e^{2int})$$

$$= 2\frac{\alpha_n}{4}\,\mathrm{Re}(-ie^{2int}) = \frac{\alpha_n \sin(2nt)}{2}$$

it follows that

$$x_0(t) = \sum_{n=1}^{+\infty} \frac{\alpha_n \sin(2nt)}{2}$$

and we have

$$\begin{aligned}
x_0(t+1) - x_0(t) &= \sum_{n=1}^{+\infty} \alpha_n \frac{\sin(2nt+2n) - \sin(2nt)}{2} \\
&= \sum_{n=1}^{+\infty} \frac{2\alpha_n \sin(\frac{2nt+2n-2nt}{2})\cos(\frac{2nt+2n+2nt}{2})}{2} \\
&= \sum_{n=1}^{+\infty} \alpha_n \sin(n)\cos(2t+1)n = f(t).
\end{aligned}$$

To complete the proof, it suffices to verify the uniform convergence on $\mathbb{R}$ of the series

$$\sum_{n\geq 1} \alpha_n \sin(2nt).$$

For this, it suffices to prove the uniform convergence of $\sum_{n\geq 1} \alpha_n \sin(nt)$ on $[0, \pi]$. For $m > n \geq 1$; and $t \in [0; \pi]$ one has

$$\begin{aligned}
&\sum_{k=n}^{n} 2\alpha_k \sin(kt)\sin(\frac{t}{2}) \\
&= \sum_{k=n}^{m} \alpha_k(\cos(k-\frac{1}{2})t - \cos(k+\frac{1}{2})t) \\
&= \sum_{k=n-1}^{m-1} \alpha_{k+1}\cos(k+\frac{1}{2})t - \sum_{k=n}^{m} \alpha_k \cos(k+\frac{1}{2})t \\
&= \alpha_n \cos(n-\frac{1}{2})t - \alpha_m \cos(m+\frac{1}{2})t + \sum_{k=n}^{m-1}(\alpha_{k+1}-\alpha_k)\cos(k+\frac{1}{2})t.
\end{aligned}$$

Hence

$$|\sum_{k=n}^{m} 2\alpha_k \sin(kt)\sin\frac{t}{2}| \leq \alpha_n + \alpha_m + \sum_{k=n}^{m-1}(\alpha_k - \alpha_{k+1})$$

which implies

$$|\sum_{k=n}^{m} \alpha_k \sin(kt)\big|\sin\frac{t}{2} \leq \alpha_n$$

which allows us to see that the series

$$\sum_{n\geq 1} \alpha_n \sin(nt)$$

is pointwise convergent on $[0, \pi]$; and if m goes to $+\infty$ and $t \in]0, \pi]$; one has

$$|\sum_{k=n}^{+\infty} \alpha_k \sin(kt)| \leq \frac{\alpha_n}{\sin\frac{t}{2}}$$

Let $t \in]0,\pi]$, and put $E = [\frac{\pi}{t}]$, the greatest integer part of $\frac{\pi}{t}$, and $u_n = \sup_{k\geq n} k\alpha_k$. Then we

$$\begin{aligned} \Big|\sum_{k=n+E}^{+\infty}\alpha_k\sin(kt)\Big| &\leq \frac{\alpha_{n+E}}{\sin\frac{t}{2}} \leq \frac{\alpha_{n+E}}{\frac{2}{\pi}\frac{t}{2}} \\ &\leq \frac{\pi}{t}\alpha_{n+E} \leq (1+E)\alpha_{n+E} \\ &\leq (n+E)\alpha_{n+E} \leq u_n \end{aligned}$$

which leads to

$$|\sum_{k=n}^{+\infty}\alpha_k\sin(kt)| \leq (1+\pi)u_n$$

and the inequality remains valid for $t=0$. Tt follows that the series $\sum_{n\geq 1}\alpha_n\sin(nt)$ converges uniformly on $[0,\pi]$.

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El Hadi Ait Dads
Cadi Ayyad University, Faculty of Sciences, Department of Mathematics B.P. 2390, Marrakesh, Morocco
E-mail address: aitdads@uca.ac.ma

Lahcen Lhachimi
Unité associée au CNRST (Morocco) URAC 02.
UMI- UMMISCO (IRD- UPMC) France
E-mail address: lllahcen@gmail.com