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MULTIPLE SOLUTIONS TO FRACTIONAL EQUATIONS WITHOUT THE AMBROSETTI-RABINOWITZ CONDITION

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Abstract. In this article we study a class of fractional Laplace equations which do not satisfy the Ambrosetti-Rabinowitz condition (AR-condition). We establish the existence of three nontrivial solutions and of multiple sign changing solutions by using Morse theory.

1. Introduction

In this article, we consider the non-local fractional equation

$$\begin{gathered} (-\Delta)^s u = f(x,u), \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.1}$$

where $s \in (0,1)$ is a fixed parameter, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N > 2s$ and $(-\Delta)^s$ is the fractional Laplace operator.

In recent years, a great attention has been focused on the study of fractional and non-local operators of elliptic type, both for the pure mathematical research and for real-world applications. Fractional and nonlocal operators appear in many fields such as, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science and water waves. For an elementary introduction to this topic and for a-still not exhaustive-list of related references see, e.g., [6].

In the literature there are many papers devoted to the study of non-local fractional Laplacian with superlinear and subcritical or critical growth (see [1, 3, 14, 16, 22, 21] and the reference therein). We stress that, at least in some of these references, a fractional operator different from the one considered here was taken into account. We refer to [17] for a detailed discussion about similarities and differences between different fractional operators. In particular, Servadei and Valdinoci [16] established the existence of nontrivial solution for (1.1) by the mountain pass theorem due to Ambrosetti and Rabinowitz [12]. Similarly, Servadei and Valdinoci [18] obtained general existence results of nontrivial solutions for (1.1) with the

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Ambrosetti- Rabinowitz condition (A-R condition) by using mountain pass theorem and linking theorem. Zhang and Ferrara [22] established the existence of two nontrivial solutions for (1.1) without the Ambrosetti-Rabinowitz condition by a variant version of the mountain pass theorem. Zhang et al. [2] obtained infinitely many solutions for (1.1) without Ambrosetti-Rabinowitz condition by using the fountain theorem. Secchi [13] studied fractional Schrödinger equations without Ambrosetti-Rabinowitz condition and proved the existence of radially symmetric solutions. Ferrara et al. [7] obtained nontrivial solutions for (1.1) by computing the critical groups and Morse theory. In [8], Iannizzotto et al. studied fractional p-Laplacian equations with p-superlinear and obtained one nontrivial solution by using Morse theory.

There are many interesting problems in the standard framework of the Laplacian (or higher order Laplacian), widely studied in the literature. A natural question is whether or not the existence results of multiple solutions obtained in the classical context can be extended to the non-local framework of the fractional Laplacian operators. Sun [20] showed the existence of three nontrivial solutions and infinitely many sign-changing solutions for a superlinear p-Laplacian equation without AR-condition.

Motivated by the publication above, we study the following non-local problem with homogeneous Dirichlet boundary conditions investigated by Servadei and Valdinoci [19] and the related works [16, 15]:

$$\begin{gathered} -\mathcal{L}_k u = f(x,u), \quad \text{in } \Omega; \\ u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{gathered} \tag{1.2}$$

where $\mathcal{L}_k$ is the integro-differential operator defined by

$$\mathcal{L}_k u(x) = \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^N, \tag{1.3}$$

with the kernel $K : \mathbb{R}^N \backslash \{0\} \to (0, +\infty)$ satisfying

- (A1) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;
- (A2) there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \backslash \{0\}$;
- (A3) $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \backslash \{0\}$.

Throughout this paper, K is the singular kernel $K(x) = |x|^{-(N+2s)}$ which leads to the fractional Laplace operator $-(-\Delta)^s$, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N. \tag{1.4}$$

Obviously, the corresponding fractional equation in model (1.2) changes to problem (1.1).

Let $F(x,t) = \int_0^t f(x,s)ds$, and suppose that the non-linearity f satisfies the following conditions:

- (A4) $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ with $f(x,0) = 0$ and satisfies the improved subcritical polynomial growth condition, i.e.

$$\lim_{t \to \infty} \frac{f(x,t)}{|t|^{2^*-1}} = 0 \quad \text{uniformly for } x \in \bar{\Omega},$$

  where $2^* = 2N/(N-2s)$;

(A5) $\lim_{|t|\to 0} \frac{f(x,t)}{t} = p(x)$, uniformly for $x \in \Omega$, where $p \in L^\infty(\Omega)$ satisfies $p(x) \le \lambda_1$ for all $x \in \Omega$ and $p(x) < \lambda_1$ on some $\Omega_0 \subset \Omega_1$ with $|\Omega_0| > 0$, where $\Omega_1 := \{x \in \Omega : \phi_1(x) \neq 0\}$ and $\lambda_1 > 0$ that has an associated eigenfunction ϕ_1 is the first eigenvalue of $(-\Delta)^s$ with homogeneous Dirichlet boundary data;

(A6) $f(x,t)$ is superlinear at infinity, i.e. $\lim_{|t|\to+\infty} f(x,t)/t = +\infty$ uniformly for all $x \in \Omega$;

(A7) There exist $\theta \ge 1$ and $C_* > 0$ such that $\theta\mathcal{F}(x,t) \ge \mathcal{F}(x,st) - C_*$ for $(x,t) \in \Omega \times \mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t) - 2F(x,t)$.

Theorem 1.1. *Assume conditions* (A4)-(A7) *hold. Then problem* (1.1) *has at least three nontrivial solutions.*

Remark 1.2. Condition (A4) comes from [11] and it is weaker than the usual subcritical growth condition, i.e. there is a constant $q \in (2, 2^*)$ such that

$$\lim_{t\to\infty} \frac{f(x,t)}{|t|^{q-1}} = 0$$

uniformly for all $x \in \Omega$. Comparing with standard Ambrosetti-Rabinowitz condition, that is, there exist $\mu > 2,\ M > 0$ such that

(A7) $0 < \mu F(x,t) \le tf(x,t)$, for all $t \in \mathbb{R}$, $|t| \ge M$ and all $x \in \Omega$.

Conditions (A6) and (A7) are very general. More detailed information for the origin and changing of the generalized superlinear conditions (A5), (A6) can be found in [10]. For conditions (A5)–(A7) and usual subcritical growth condition, two nontrivial solutions can be obtained as in [22] , but the existence of the third solution has some difficulty. However, using the method in [9], we can provide some information for the critical group of the mountain pass solutions and find the third nontrivial solution. Therefore, Theorem 1.1 improves the results in [16, 22, 18].

Our next task is to consider the existence of sign changing solutions of (1.1). We now state the following assumptions:

(A4') $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ with $f(x,0) = 0$ and satisfies the growth condition:

$$|f'(x,t)| \le c(1 + |t|^{q-2}) \quad \forall t \in \mathbb{R},\ x \in \Omega,$$

for some $c > 0$ and $q \in (2, 2^*)$.

Theorem 1.3. *Assume condition* (A4') *holds. Moreover, suppose that the number of positive and negative solutions of* (1.1) *is finite.*

(i) *If* (A5)–(A7) *hold, then* (1.1) *has at least a sign changing solution.*

(ii) *If* (A5)–(A7) *hold and the function* $f(x,t)$ *is odd in* t, *then* (1.1) *has a sequence of pairs of sign changing solutions* $\{u_k, -u_k\}$ *such that*

$$\lim_{k\to\infty} \|u_k\|_\infty = \infty.$$

Here, we have extend [20, Theorem 1.3] to the fractional Laplacian problem (1.1), which is a new result.

This article is organized as follows. In section 2, we present some necessary preliminary knowledge about working space. In section 3, we prove some lemmas in order to prove our main results. In section 4, we give the proofs for our main results.

2. Preliminaries

In this section, we give some preliminary results which will be used in the sequel. We briefly recall the related definition and notes for functional space X_0 introduced in [19].

The functional space X denotes the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function g in X belongs to $L^2((\Omega)$ and the map $(x,y) \mapsto (g(x)-g(y))\sqrt{K(x-y)}$ is in $L^2((\mathbb{R}^N \times \mathbb{R}^N)\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega), dx\,dy)$ (here $\mathcal{C}\Omega = \mathbb{R}^N \backslash \Omega$). Also, we define a linear subspace of X,

$$X_0 := \{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

Note that X and X_0 are non-empty, since $C_0^2(\Omega) \subseteq X_0$ by [19]. Moreover, the space X is endowed with the norm

$$\|g\|_X = \|g\|_{L^2(\Omega)} + \Big(\int_Q |g(x)-g(y)|^2 K(x-y)\,dx\,dy \Big)^{1/2}, \tag{2.1}$$

where $Q = (\mathbb{R}^N \times \mathbb{R}^N)\backslash \mathcal{O}$ and $\mathcal{O} = (\mathcal{C}\Omega) \times (\mathcal{C}\Omega) \subset \mathbb{R}^N \times \mathbb{R}^N$. We equip X_0 with the norm

$$\|g\|_{X_0} = \Big(\int_Q |g(x)-g(y)|^2 K(x-y)\,dx\,dy \Big)^{1/2}, \tag{2.2}$$

which is equivalent to the usual norm defined in (2.1) (see [16]). It is easy to check that $(X_0, \|\cdot\|_{X_0})$ is a Hilbert space with scalar product

$$\langle u, v \rangle_{X_0} = \int_Q (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy. \tag{2.3}$$

Denote by $H^s(\Omega)$ the usual fractional Sobolev space with respect to the Gagliardo norm

$$\|g\|_{H^s(\Omega)} = \|g\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|g(x)-g(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \Big)^{1/2}. \tag{2.4}$$

Now, we give basic facts to be used later.

Lemma 2.1 ([16])**.** *The embedding $j : X_0 \hookrightarrow L^v(\Omega)$ is continuous for any $v \in [1, 2^*]$, while it is compact whenever $v \in [1, 2^*)$.*

3. Some lemmas

First, we observe that problem (1.1) has a variational structure. Indeed it is the Euler-Lagrange equation of the functional $\mathcal{J} : X_0 \to \mathbb{R}$ defined as follows:

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \int_\Omega F(x, u(x))dx.$$

It is well known that the functional $\mathcal{J}$ is Frechét differentiable in X_0 and for any $\varphi \in X_0$,

$$\langle \mathcal{J}'(u), \varphi \rangle = \int_{\mathbb{R}^N \times \mathbb{R}^N} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy - \int_\Omega f(x, u(x))\varphi(x)dx.$$

Thus, critical points of $\mathcal{J}$ are solutions of problem (1.1).

Let

$$f_+(x,t) = \begin{cases} f(x,t), & t > 0, \\ 0, & t \le 0; \end{cases}$$

$$\mathcal{J}_\pm(u) = \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)\,dx\,dy - \int_\Omega F_\pm(x,u(x))dx,$$

where $F_\pm(x,t)=\int_0^t f_\pm(x,s)ds$. Now, we prove the following compactness condition for $\mathcal{J}$ and $\mathcal{J}_\pm$.

Definition 3.1. The functional $\mathcal{J}$ is said to satisfy Cerami condition at level $c\in\mathbb{R}$ ($(C)_c$ condition for short) if every sequence $\{u_n\}\subset E$ with

$$\mathcal{J}(u_n)\to c, (\|u_n\|+1)\mathcal{J}'(u_n)\to 0 \quad \text{as } n\to\infty,$$

possesses a convergent subsequence. $\mathcal{J}$ satisfies the (C) condition if $\mathcal{J}$ satisfies $(C)_c$ condition at every $c\in\mathbb{R}$.

Lemma 3.2. *Under conditions* (A4), (A6), (A7)*, the functionals $\mathcal{J}$ and $\mathcal{J}_\pm$ satisfies the (C) condition.*

Proof. We only give the proof for $\mathcal{J}_+$, the cases of $\mathcal{J}$ and $\mathcal{J}_-$ are similar. Let $\{u_n\}\subset X_0$ be a sequence such that

$$|\mathcal{J}_+'(u_n)|\to c,\ (1+\|u_n\|_{X_0})\|\mathcal{J}_+'(u_n)\|_{X_0^*}\to 0, \quad \text{as } n\to\infty. \tag{3.1}$$

The proof of this lemma, we divide two steps:

Step 1. We first prove that $\{u_n\}$ is bounded in X_0. Let $u_n^+=\max\{u_n,0\}$, $u_n^-=\min\{u_n,0\}$. From (3.1), we obtain

$$|\langle J_+'(u_n),\varphi\rangle|\le \epsilon_n\|\varphi\|_{X_0} \quad \text{for any } \varphi\in X_0, \tag{3.2}$$

where $\epsilon_n\to 0$ as $n\to\infty$, then the boundedness of u_n^- can be directly obtained. For the case of u_n^+, by contradiction, we assume that $\|u_n^+\|_{X_0}\to\infty$ as $n\to\infty$. Let $v_n=\|u_n^+\|_{X_0}^{-1}u_n^+$, then $\|v_n\|_{X_0}=1$. By lemma 2.1, up to a subsequence, we have

$$v_n \rightharpoonup v \quad \text{in } X_0, \tag{3.3}$$
$$v_n\to v \quad \text{in } L^q(\mathbb{R}^N), \tag{3.4}$$
$$v_n\to v \quad \text{a.e. } x\in\mathbb{R}^N. \tag{3.5}$$

Case 1. Suppose that $v\neq 0$, then the Lebesgue measure of $\Omega_0=\{x\in\Omega: v(x)\neq 0\}$ is positive. Using (3.1), we obtain

$$\langle \mathcal{J}_+'(u_n),u_n^+\rangle = o(1),$$

which implies that

$$\int_\Omega \frac{f_+(x,u_n^+)u_n^+}{\|u_n^+\|_{X_0}^2}dx = \int_\Omega \frac{f_+(x,u_n^+)u_n^+}{|u_n^+|^2}|v_n|^2dx = 1+o(1). \tag{3.6}$$

By (A6), there is a constant $M>0$ such that

$$f_+(x,u_n^+)u_n^+>0, \quad \text{as } |u_n|>M,$$

then we have

$$\int_{\Omega\setminus\Omega_0} \frac{f_+(x,u_n^+)u_n^+}{(u_n^+)^2}|v_n|^2dx \ge -C. \tag{3.7}$$

On the other hand, for $x\in\Omega_0$, $u_n^+\to\infty$ as $n\to\infty$. Then by the Fatou's lemma and (A6) we have

$$\int_{\Omega_0} \frac{f_+(x,u_n^+)u_n^+}{(u_n^+)^2}|v_n|^2dx\to\infty, \quad \text{as } n\to\infty.$$

Combining this with (3.7) gives

$$\int_\Omega \frac{f_+(x,u_n^+)u_n^+}{(u_n^+)^2}|v_n|^2dx \to \infty, \quad \text{as } n\to\infty. \tag{3.8}$$

This contradicts (3.6). Then this case is impossible.

Case 2. Assume that $v=0$, let $\{t_n\}\subset\mathbb{R}$ such that

$$\mathcal{J}_+(t_nu_n^+) = \max_{t\in[0,1]} \mathcal{J}_+(tu_n^+).$$

For any $m>0$, we assume that $w_n = 2\sqrt{m}v_n$. Then $w_n\to 0$ in $L^q(\mathbb{R}^N)$. So from conditions (A4) and (A5), for every $\epsilon>0$, we can find a constant $C(\epsilon)>0$ such that

$$F(x,w_n) \le C(\epsilon)(w_n)^2 + \epsilon(w_n)^{2^*}, \tag{3.9}$$

which implies

$$\lim_{n\to\infty}\int_\Omega F_+(x,w_n)dx = 0. \tag{3.10}$$

Since $2\sqrt{m}\|u_n^+\|_{X_0}^{-1}\in(0,1)$ for n large enough, by (3.10) we obtain

$$\mathcal{J}_+(t_nu_n^+) \ge \mathcal{J}_+(w_n) = 2m - \int_\Omega F_+(x,w_n)dx \ge m,$$

which implies

$$\mathcal{J}_+(t_nu_n^+)\to\infty, \quad \text{as } n\to\infty. \tag{3.11}$$

From$\mathcal{J}_+(0)=0$ and $\mathcal{J}_+(u_n^+)\to c$ we have $t_n\in(0,1)$, then

$$\langle \mathcal{J}_+'(t_nu_n^+), t_nu_n^+\rangle = t_n\frac{d}{dt}\Big|_{t=t_n}\mathcal{J}_+(tu_n) = 0.$$

Then, from (A7) it follows that

$$\begin{aligned}
\frac{1}{\theta}\mathcal{J}_+(t_nu_n^+) &= \frac{1}{\theta}\Big(\mathcal{J}_+(t_nu_n^+) - \frac12\langle \mathcal{J}_+'(t_nu_n^+), t_nu_n^+\rangle\Big)\\
&= \frac{1}{2\theta}\int_\Omega \mathcal{F}(x,t_nu_n^+)dx\\
&\le \frac12\int_\Omega \mathcal{F}(x,u_n^+)dx + \frac{1}{2\theta}|\Omega|C_*\\
&= \mathcal{J}_+(u_n^+) - \frac12\langle \mathcal{J}_+'(u_n^+), u_n^+\rangle + c \to C.
\end{aligned}$$

This contradicts that $\mathcal{J}_+(t_nu_n^+)\to\infty$. Hence $\{u_n\}$ is bounded; that is, there exists a positive constant M such that

$$\|u_n\|_{X_0}\le M, \quad \text{for all } n\in N.$$

Step 2. We prove $\{u_n\}$ has a convergent subsequence. In fact, we can suppose that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } X_0,\\
u_n \to u \quad \text{in } L^q(\Omega),\ \forall 1\le q<2^*,\\
u_n(x)\to u(x) \quad \text{a.e. } x\in\Omega.
\end{gathered}$$

Now, since Ω is a bounded set, for every $\epsilon > 0$, we can find a constant $C(\epsilon) > 0$ such that

$$f_+(x,s) \le C(\epsilon) + \epsilon |s|^{2^*-1}, \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

then

$$\begin{aligned}
&\Big|\int_\Omega f_+(x,u_n)(u_n-u)dx\Big| \\
&\le C(\epsilon)\int_\Omega |u_n-u|dx + \epsilon\int_\Omega |u_n-u||u_n|^{2^*-1}dx \\
&\le C(\epsilon)\int_\Omega |u_n-u|dx + \epsilon\Big(\int_\Omega \big(|u_n|^{2^*-1}\big)^{\frac{2^*}{2^*-1}}dx\Big)^{\frac{2^*-1}{2^*}}\Big(\int_\Omega |u_n-u|^{2^*}\Big)^{1/2^*} \\
&\le C(\epsilon)\int_\Omega |u_n-u|dx + \epsilon C(\Omega).
\end{aligned}$$

Similarly, since $u_n \rightharpoonup u$ in X_0, it follows that $\int_\Omega |u_n-u|dx \to 0$. Since $\epsilon > 0$ is arbitrary, we can conclude that

$$\int_\Omega (f_+(x,u_n) - f_+(x,u))(u_n-u)dx \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

By (3.12), we have

$$\langle \mathcal{J}_+'(u_n) - \mathcal{J}_+'(u), (u_n-u)\rangle \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

From (3.12) and (3.13), we obtain $\|u_n\|_{X_0} \to \|u\|_{X_0}$, as $n \to \infty$. Thus we have

$$\|u_n - u\|_{X_0} \to 0, \text{ as } n \to \infty,$$

which means that $\mathcal{J}_+$ satisfies condition (C). $\square$

Before stating our next lemma, we recall some concepts and results of Morse theory. For the details, we refer to [4]. Let X be a real Banach space and $\mathcal{J} \in C^1(X,R)$. $K = \{u \in X | \mathcal{J}'(u) = 0\}$ is the critical set of $\mathcal{J}$. Let $u \in K$ be an isolated critical point of $\mathcal{J}$ with $\mathcal{J}(u) = c \in \mathbb{R}$, and U be an isolated neighborhood of u, i.e. $K \cap U = \{u\}$. The group

$$C_*(\mathcal{J},u) = H_*(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U\backslash\{u\}), \quad * = 0,1,2,\dots,$$

is called the $*$-th critical group of $\mathcal{J}$ at u, where $\mathcal{J}^c = \{u \in X | \mathcal{J}(u) \le c\}$.

$H_*(\cdot,\cdot)$ is the singular relative homology group of $\mathcal{J}$ at infinity is defined by

$$C_*(\mathcal{J},\infty) = H_*(X, \mathcal{J}^a), \quad * = 0,1,2,\dots.$$

We denote

$$P(u,t) = \sum_i \operatorname{rank} C_i(\mathcal{J},u)t^i, \quad P(\infty,t) = \sum_i \operatorname{rank} C_i(\mathcal{J},\infty)t^i.$$

Let $\alpha < \beta$ be the regular values of $\mathcal{J}$ and set

$$P(\alpha,\beta,t) = \sum_i \operatorname{rank} C_i(\mathcal{J},\infty)t^i.$$

If $K = \{u_1,u_2,\dots,u_k\}$, then there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that

$$\sum_j P(u_j,t) = P(\infty,t) + (1+t)Q(t), \tag{3.14}$$

$$\sum_{\alpha<\mathcal{J}(u_j)<\beta} P(u_j,t)=P(\alpha,\beta,t)+(1+t)Q(t). \tag{3.15}$$

Lemma 3.3. *Assume that conditions* (A4), (A6), (A7) *hold. Then we have*

$$C_*(\mathcal{J},\infty)=C_*(\mathcal{J}_\pm,\infty)=\{0\},\quad *=0,1,2,\dots.$$

Proof. We only give the proof of J_+; the others are similar. Let $S=\{u\in X_0:\|u\|_{X_0}=1,\ u^+\neq 0\}$ and $B^\infty=\{u\in X_0:\|u\|_{X_0}\le 1\}$. By (A6), for any $M>0$ there exists $c>0$, such that $F(x,t)\ge Mt^2-c$, for $(x,t)\in\Omega\times\mathbb{R}$, which implies $\mathcal{J}_+(tu)\to-\infty$, as $t\to+\infty$, for any $u\in S$. Using (A7), we have

$$f_+(x,t)t-2F_+(x,t)\ge-\frac{C_*}{\theta},\quad \text{for } (x,t)\in\Omega\times\mathbb{R}. \tag{3.16}$$

Choose

$$a<\min\big\{\inf_{u\in B^\infty}\mathcal{J}_+(u),\ -\frac{C_*}{p\theta}|\Omega|\big\}.$$

Then for any $u\in S$, there exists $t>1$ such that $\mathcal{J}_+(tu)\le a$, that is

$$\mathcal{J}_+(tu)=\frac{t^2}{2}-\int_\Omega F_+(x,tu)dx\le a,$$

which (3.16) implies

$$\frac{d}{dt}\mathcal{J}_+(tu)=t-\int_\Omega f_+(x,tu)u\le\frac{1}{t}\big(2a+\frac{C_*}{\theta}|\Omega|\big)<0\,.$$

Therefore, by the implicit function theorem, there exists a unique $T\in C(S,\mathbb{R})$ such that

$$\mathcal{J}_+(T(u)u)=a,\quad \text{for } u\in S.$$

Let $S_1=\{u\in E:\|u\|_{X_0}\ge 1,\ u^+\neq 0\}$. We construct a strong deformation retract $\tau:[0,1]\times S_1\to S_1$ which satisfies $\tau(s,u)=(1-s)u+sT\big(\frac{u}{\|u\|}\big)\frac{u}{\|u\|}$ if $\mathcal{J}_+(u)\ge a$ and $\tau(s,u)=u$ if $\mathcal{J}_+(u)<a$. Hence, It follows from the construction of τ that $\mathcal{J}_+^a$ is a strong deformation retract of S_1, which is homotopy equivalent to the set S. By the homotopy invariance of homology group, we have

$$C_*(\mathcal{J}_+,\infty)=H_*(X_0,\mathcal{J}_+^a)\cong H_*(X_0,S)\cong H_*(X_0,X_0\setminus\{0\})=0.$$

□

4. Proofs of main results

Proof of Theorem 1.1. By Lemma 3.2, we know that $\mathcal{J}$ and $\mathcal{J}_\pm$ satisfy the (C) condition. By conditions (A4) and (A5), we can easily prove that 0 is a local minimum of $\mathcal{J}$ and $\mathcal{J}_\pm$. So, we have

$$C_*(J,0)=C_*(J_\pm,0)=\delta_{*,0}G. \tag{4.1}$$

Using the mountain pass theorem in [12] and maximum principle in [8], we obtain $\mathcal{J}_+$ ($\mathcal{J}_-$) has a critical point $u_+>0$ ($u_-<0$), and $u_\pm$ are also the nontrivial critical points of the functional $\mathcal{J}$. Without loss of generality, we assume that $u_\pm$ are isolated and the only nontrivial critical points of the functional $\mathcal{J}$. Now we claim that

$$C_*(\mathcal{J}_\pm,u_\pm)=\delta_{*,1}G. \tag{4.2}$$

Indeed, using the methods of [9], we let $\mathcal{J}_+(u_+) = c > 0$. It follows from the homology exact sequence of the triple $\mathcal{J}_+^A \subset \mathcal{J}_+^{\frac{c}{2}} \subset X_0$, we have

$$\dots \to H_*(X_0, \mathcal{J}_+^A) \to H_*(X_0, \mathcal{J}_+^{\frac{c}{2}}) \to H_{*-1}(\mathcal{J}_+^{\frac{c}{2}}, \mathcal{J}_+^A) \to H_{*-1}(X_0, \mathcal{J}_+^A) \to \dots, \tag{4.3}$$

where $A < 0$ is a constant. Since 0 is the only critical point of $\mathcal{J}_+$ in the set $\mathcal{J}_+^{\frac{c}{2}}$, by (4.1), we obtain

$$H_*(\mathcal{J}_+^{\frac{c}{2}}, \mathcal{J}_+^A) = C_*(\mathcal{J}_+, 0) = \delta_{*,0}G. \tag{4.4}$$

Similarly, since u_+ is the only critical point of $\mathcal{J}_+$ in the set $\{u \in X_0 | \mathcal{J}_+(u) \geq \frac{c}{2}\}$, we have

$$H_*(X_0, \mathcal{J}_+^{\frac{c}{2}}) = C_*(\mathcal{J}_+, u_1), \quad * = 0, 1, 2, \dots. \tag{4.5}$$

From Lemma 3.3, we have

$$H_*(X_0, \mathcal{J}_+^A) = C_*(\mathcal{J}_+, \infty) = 0, \quad * = 0, 1, 2, \dots. \tag{4.6}$$

From (4.3) to (4.6), we deduce that

$$C_*(\mathcal{J}_+, u_1) = C_{*-1}(\mathcal{J}_+, 0) = \delta_{*,1}G.$$

The case for u_- is similar.

By the claim and [9, Lemma 2.4], we have

$$C_*(J, u_\pm) = \delta_{*,1}G.$$

The Morse equality (3.14) with $t = -1$ implies that

$$(-1)^0 + (-1)^1 + (-1)^1 = 0,$$

which is a contradiction. Then (1.1) has at least three nontrivial solutions. □

Proof of Theorem 1.3. Our proof is similar to proof in [5], which studies equations with condition (A7).

(i) By contradiction, we assume that there is no sign changing solution of problem (1.1). Let $\{u_i^+\}_1^s$ and $\{u_j^-\}_1^m$ be the sets of positive and negative solutions, respectively. Let

$$\chi_\pm(u^\pm) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} C_k(\mathcal{J}_\pm, u^\pm),$$
$$\chi(u^\pm) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank} C_k(\mathcal{J}, u^\pm).$$

Using the results in [5], we know that $\chi_\pm(u^\pm)$ and $\chi(u^\pm)$ are well defined, and by the results of in [9], we obtain

$$\chi_\pm(u^\pm) = \chi(u^\pm). \tag{4.7}$$

Lemma 3.3 implies that

$$C_*(\mathcal{J}, \infty) = C_*(J_\pm, \infty) = 0, \quad * = 0, 1, 2, \dots.$$

This together with the Morse equality (3.14) for $\mathcal{J}_+$, $\mathcal{J}_-$, $\mathcal{J}$ gives

$$\chi_+(0) + \sum_1^s \chi_+(u_i^+) = 0, \tag{4.8}$$

$$\chi_-(0) + \sum_1^m \chi_-(u_j^-) = 0, \tag{4.9}$$

$$\chi(0) + \sum_1^s \chi(u_i^+) + \sum_1^m \chi(u_j^-) = 0. \tag{4.10}$$

Similar to the proof of [5, Theorem 5.1], we also have

$$\chi_+(0) = \chi_-(0) = 1. \tag{4.11}$$

From (4.7) to (4.11), we obtain

$$1 = \chi(0) = \chi_+(0) + \chi_-(0) = 2\chi_+(0) = 2. \tag{4.12}$$

This is a contradiction. Then problem (1.1) has at least a sign changing solution.

(ii) By [8, Corollary 3.2], condition (A4') and simple integration, we know that the $L^\infty(\Omega)$ boundedness of solutions of problem (1.1) is equivalent to the X_0 boundedness. Then by contradiction we can assume that there exists a positive constant R such that all solutions of (1.1) are located in the ball $B_R = \{u \in X_0 : \|u\|_{X_0} < R\}$. Therefore, there are constants $\beta < \inf \mathcal{J}(K) < \alpha$ such that all critical points of $\mathcal{J}$ are in the set $\mathcal{J}^\alpha$ and

$$C_*(\mathcal{J}, \infty) = H_*(X_0, \mathcal{J}^\beta) = H_*(\mathcal{J}^\alpha, \mathcal{J}^\beta) = 0, \quad * = 0, 1, 2, \dots. \tag{4.13}$$

From (3.15) and (4.13), we have the Moser equality

$$0 = \chi(0) + \sum_{u \neq 0, u \in K} \chi(u). \tag{4.14}$$

Since the nonzero solutions of (1.1) appear in pairs $\{u, -u\}$, $\chi(u) = \chi(-u)$, the right hand side of (4.14) is odd. This is a contradiction. Therefore, there exists an unbounded sequence of pairs of sign changing solutions $\{u_k, -u_k\}$ of (1.1). □

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