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ITERATIVE OSCILLATION RESULTS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

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Abstract. This article concerns the oscillation of solutions to a linear second-order differential equation with advanced argument. Sufficient oscillation conditions involving limit inferior are given which essentially improve known results. We base our technique on the iterative construction of solution estimates and some of the recent ideas developed for first-order advanced differential equations. We demonstrate the advantage of our results on Euler-type advanced equation. Using MATLAB software, a comparison of the effectiveness of newly obtained criteria as well as the necessary iteration length in particular cases are discussed.

1. Introduction

We consider the linear second-order advanced differential equation

$$y''(t) + q(t)y(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $q \in \mathcal{C}([t_0, \infty))$ and $\sigma \in \mathcal{C}^1([t_0, \infty))$ are such that $q(t) > 0$, $\sigma(t) \geq t$ and $\sigma'(t) \geq 0$.

By a solution of (1.1), we understand a nontrivial function $y \in \mathcal{C}^2([t_0, \infty))$, which satisfies (1.1) on $[t_0, \infty)$. We restrict our attention to those solutions y of (1.1) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$, for all $T \geq t_0$. We recall that a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. Equation (1.1) is called oscillatory if all of its solutions are oscillatory as well.

Differential equations with deviating argument are deemed to be adequate in modeling of the countless processes in all areas of science. As is well known, a distinguishing feature of *delay differential equations* under consideration is the dependence of the evolution rate of the processes described by such equations on the past history. This consequently results in predicting the future in a more reliable and efficient way, explaining at the same time many qualitative phenomena such as periodicity, oscillation or instability. The concept of the delay incorporation into systems plays an essential role in modeling to represent time taken to complete some hidden processes, see [8, 11].

Contrariwise, *advanced differential equations* can find use in many applied problems whose evolution rate depends not only on the present, but also on the future.

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Therefore, an advance could be introduced into the equation to highlight the influence of potential future actions, which are available at the presence and should be beneficial in the process of decision making. For instance, population dynamics, economical problems or mechanical control engineering are typical fields where such phenomena is believed to occur (see [8] for details).

The first oscillation results for differential equations with deviating argument were obtained in the classical paper by Fite [10] in 1921. Since then, a great deal of the effort has been made by many researchers in order to advance the knowledge further (for the summary of most essential contributions on the subject, see, e.g., monographs [1, 2, 11, 9, 18] and the references cited therein).

Most of the literature, however, has been devoted to the investigation of differential equations with delay argument, and very little is known up to now about those with advanced arguments. In particular, two main approaches for the investigation of (1.1) have appeared (see [2, Chapter 2], [5, 15, 16]). Taking Kusano's and Naito's comparison theorem [16, Theorem 1] into account, the oscillatory behavior of (1.1) can be treated as that of the ordinary differential equation

$$y''(t) + q(t)y(t) = 0. \tag{1.2}$$

It seems obvious that in such a case, all impact of the advanced argument is completely neglected. On the other hand, an another approach has been based on the comparison with the first-order advanced differential equation

$$y'(t) - \Big(\int_t^\infty q(s)\mathrm{d}s\Big)y(\sigma(t)) = 0, \tag{1.3}$$

in the sense that oscillation of (1.1) is inherited from that of (1.3) (see [2, Theorem 2.1.12]). Here, the advance may generate oscillations. In particular, by applying the famous Hille's result [13] and the well-known oscillation criterion due to Ladas [17] to (1.2) and (1.3), respectively, one can immediately get the following couple of oscillation criteria for (1.1):

$$\liminf_{t\to\infty} t\int_t^\infty q(s)\mathrm{d}s > \frac{1}{4}, \tag{1.4}$$

$$\liminf_{t\to\infty} \int_t^{\sigma(t)}\int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u > \frac{1}{\mathrm{e}}. \tag{1.5}$$

The question naturally arises:

> Is it possible to establish an effective oscillation result of Hille type which simultaneously takes into account the presence of the advance and the second order nature of the equation studied as well?

The purpose of this article is to give an affirmative answer to this quastion, i.e., to propose an approach for investigation the (1.1) when both above-mentioned conditions (1.4) and (1.5) fail. The use is made of some of the recent results developed for first-order delay/advanced differential equations which have been based on the iterative application of the Grönwall's inequality (see [4, 7]). This technique enables one to obtain sufficient conditions for oscillation of (1.1) involving $\liminf$, which essentially use value of the advanced argument. Our method of the proof that is quite different from the very recent study [3] is essentially new.

Finally, we demonstrate the advantage of our results on Euler-type advanced equations. Using MATLAB software, a comparison of the effectiveness of newly obtained criteria is provided as well as the necessary iteration length in particular cases.

2. Main results

In this section, we establish a number of new oscillation criteria for (1.1).

In the sequel, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough.

Remark 2.1. As $-y(t)$ is also a solution of (1.1), we may restrict ourselves only to the case where $y(t)$ is eventually positive.

Remark 2.2. In view of the well-known Leighton's criterion [19] and the comparison theorem [16, Theorem 1], equation (1.1) is oscillatory if $\int^\infty q(s)\mathrm{d}s = \infty$. Therefore, we assume throughout the paper that $\int^\infty q(s)\mathrm{d}s < \infty$.

We define

$$\tilde{q}(t) = q(t)\Big(1 + \int_t^{\sigma(t)} \int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u\Big).$$

Theorem 2.3. *Assume that the second-order differential equation*

$$y''(t) + \tilde{q}(t)y(t) = 0 \tag{2.1}$$

is oscillatory. Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that y is a positive solution of (1.1) on $[t_0,\infty)$. Obviously, there exists $t_1 \ge t_0$ such that

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) \le 0, \quad \text{for} \quad t \ge t_1. \tag{2.2}$$

An integration of (1.1) from t to ∞ in view of (2.2) leads to

$$y'(t) \ge \int_t^\infty q(s)y(\sigma(s))\mathrm{d}s \tag{2.3}$$

$$\ge y(\sigma(t))\Big(\int_t^\infty q(s)\mathrm{d}s\Big). \tag{2.4}$$

Integrating (2.4) from t to $\sigma(t)$, we have

$$y(\sigma(t)) \ge y(t) + \int_t^{\sigma(t)} y(\sigma(u)) \int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u. \tag{2.5}$$

Using that $y(\sigma(t)) \ge y(t)$ in (2.5), one obtains

$$\begin{aligned} y(\sigma(t)) &\ge y(t) + y(\sigma(t)) \int_t^{\sigma(t)} \int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u \\ &\ge y(t)\Big(1 + \int_t^{\sigma(t)} \int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u\Big). \end{aligned}$$

Combining the last inequality and (1.1) yields

$$y''(t) + \tilde{q}(t)y(t) \le 0. \tag{2.6}$$

Define $w(t) = y'(t)/y(t)$ to see that $w(t)$ satisfies the first-order Riccati inequality

$$w'(t) - \tilde{q}(t) - w^2(t) \le 0,$$

which in turn implies (see [1, Lemma 2.2.1]) that the equation (2.1) has a positive solution; a contradiction. The proof is complete. $\square$

Corollary 2.4. *If*

$$\liminf_{t\to\infty} t\int_t^\infty \tilde{q}(s)\mathrm{d}s > \frac{1}{4}, \tag{2.7}$$

then (1.1) *is oscillatory.*

Remark 2.5. The criterion (2.7) of Hille type takes the presence of the advanced argument into account and thus can be applied even if the corresponding known one (1.4) fails.

The lemma below is a slight modification of [14, Lemma 1] originally given for the first-order equation with delayed argument. For the sake of clarity, we also include its complete proof.

Lemma 2.6. *Let $y(t)$ be an eventually positive solution of* (1.1). *Then*

$$\rho := \liminf_{t\to\infty} \int_t^{\sigma(t)}\int_u^\infty q(s)\mathrm{d}s\mathrm{d}u \le \frac{1}{\mathrm{e}}, \tag{2.8}$$

$$\liminf_{t\to\infty} \frac{y(\sigma(t))}{y(t)} \ge \lambda, \tag{2.9}$$

where λ is the smaller root of the transcendental equation $\lambda = \mathrm{e}^{\rho\lambda}$.

Proof. Let

$$\alpha = \liminf_{t\to\infty} \frac{y(\sigma(t))}{y(t)}.$$

Dividing (2.4) by $y(t)$ and integrating from t to $\sigma(t)$, we have

$$\ln\Big(\frac{y(\sigma(t))}{y(t)}\Big) \ge \int_t^{\sigma(t)} \frac{y(\sigma(u))}{y(u)}\int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u,$$

or

$$\frac{y(\sigma(t))}{y(t)} \ge \exp\Big(\int_t^{\sigma(t)} \frac{y(\sigma(u))}{y(u)}\int_u^\infty q(s)\mathrm{d}s\,\mathrm{d}u\Big),$$

which clearly implies

$$\alpha \ge \mathrm{e}^{\rho\alpha}. \tag{2.10}$$

Note that (2.10) is impossible when $\rho > 1/\mathrm{e}$, since $\lambda < \exp\rho\lambda$ for all $\lambda > 0$ and so (1.1) has no positive solutions. If $\rho \le 1/\mathrm{e}$, then the equation $\lambda = \exp\rho\lambda$ has roots $\lambda \le \tilde{\lambda}$, with $\lambda = \tilde{\lambda} = \mathrm{e}$ if and only if $\rho = 1/\mathrm{e}$ and (2.10) holds if and only if $\lambda \le \alpha \le \tilde{\lambda}$. □

As an immediate consequence of Lemma 2.9, we have the following result, which applies when (1.5) fails.

Theorem 2.7. *Let* (2.8) *hold and λ be as in Lemma 2.6. Assume that the second-order differential equation*

$$y''(t) + k\lambda q(t)y(t) = 0 \tag{2.11}$$

is oscillatory for some $k \in (0,1)$. Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that y is a positive solution of (1.1) on $[t_0,\infty)$. Then it follows from Lemma 2.6 that there exists $t_1 \in [t_0,\infty)$ such that, for every $k \in (0,1)$,

$$\frac{y(\sigma(t))}{y(t)} \ge k\lambda \quad \text{on } [t_1,\infty). \tag{2.12}$$

Using (2.12) in (1.1), it is easy to see that y is a positive solution of the inequality

$$y''(t) + k\lambda y(t) \le 0.$$

The same as in the proof of Theorem 2.3, we can conclude that the corresponding equation (2.11) also has a positive solution, a contradiction. The proof is complete. $\square$

Corollary 2.8. *Let* (2.8) *hold and* λ *be as in Lemma 2.6. If*

$$\liminf_{t\to\infty} t\int_t^\infty q(s)\mathrm{d}s > \frac{1}{4\lambda}, \tag{2.13}$$

then (1.1) *is oscillatory.*

In the next lemma, we derive some useful estimates which are based on the iterative application of the Grönwall inequality and permit us to improve all the previous results.

Lemma 2.9. *Let* $y(t)$ *be an eventually positive solution of* (1.1)*. Define*

$$a_1(s,t) = \exp\Big(\int_t^s \int_u^\infty q(x)\mathrm{d}x\,\mathrm{d}u\Big),$$
$$a_{n+1}(s,t) = \exp\Big(\int_t^s \int_u^\infty q(x)a_n(\sigma(x),u)\mathrm{d}x\,\mathrm{d}u\Big), \quad n \in \mathbb{N}.$$

Then

$$y(s) \ge y(t)a_n(s,t), \quad s \ge t, \tag{2.14}$$

for t large enough.

Proof. We will prove Lemma 2.9 by mathematical induction. Since y is an eventually positive solution of (1.1), there exists $t_1 \ge t_0$ such that y satisfies (2.2) on $[t_1,\infty)$. Thus $y(\sigma(t)) \ge y(t)$ and by virtue of (2.4), we have

$$y'(t) \ge y(t)\int_t^\infty q(s)\mathrm{d}s.$$

Applying the Grönwall inequality, we obtain

$$y(s) \ge y(t)\exp\Big(\int_t^s \int_u^\infty q(x)\mathrm{d}x\,\mathrm{d}u\Big), \quad s \ge t \ge t_1, \tag{2.15}$$

that is, the estimate (2.14) is valid for $n = 1$.

Next, we assume that (2.14) holds for some $n > 1$. Then

$$y(\sigma(s)) \ge y(t)a_n(\sigma(s),t), \quad \sigma(s) \ge t. \tag{2.16}$$

Substituting (2.16) into (2.3) yields

$$y'(t) \ge \int_t^\infty q(s)y(\sigma(s))\mathrm{d}s \ge y(t)\int_t^\infty q(s)a_n(\sigma(s),t)\mathrm{d}s.$$

Again, applying the Grönwall inequality, we have

$$y(s) \ge y(t)\exp\Big(\int_t^s \int_u^\infty q(x)a_n(\sigma(x),u)\mathrm{d}x\,\mathrm{d}u\Big), \tag{2.17}$$

i.e.,

$$y(s) \ge y(t)a_{n+1}(s,t).$$

This established the induction step and completes the proof. $\square$

Theorem 2.10. *Let $a_n(t,s)$ be as in Lemma 2.9. Assume that the first-order advanced differential equation*
$$y'(t) - \Big(\int_t^\infty q(s)a_n(\sigma(s),\sigma(t))\mathrm{d}s\Big)y(\sigma(t)) = 0 \tag{2.18}$$
is oscillatory for some $n \in \mathbb{N}$. Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that y is a positive solution of (1.1) on $[t_0,\infty)$. Then there exists $t_1 \geq t_0$ such that y satisfies (2.2) on $[t_1,\infty)$. It follows from Lemma 2.9 that
$$y(\sigma(s)) \geq y(\sigma(t))a_n(\sigma(s),\sigma(t)), \quad s \geq t, \tag{2.19}$$
for some $n \in \mathbb{N}$ and t large enough. Integrating (1.1) from t to ∞ and using (2.19), we are led to
$$y'(t) \geq \int_t^\infty q(s)y(\sigma(s))\mathrm{d}s \geq y(\sigma(t))\int_t^\infty q(s)a_n(\sigma(s),\sigma(t))\mathrm{d}s, \tag{2.20}$$
which means that y is a positive solution of the first-order advanced differential inequality
$$y'(t) - \Big(\int_t^\infty q(s)a_n(\sigma(s),\sigma(t))\mathrm{d}s\Big)y(\sigma(t)) \geq 0.$$
In view of [20, Theorem 1], the equation (2.18) also has a positive solution, a contradiction. The proof is complete. □

Corollary 2.11. *Let $a_n(t,s)$ be as in Lemma 2.9. If*
$$\liminf_{t\to\infty}\int_t^{\sigma(t)}\int_u^\infty q(s)a_n(\sigma(s),\sigma(u))\mathrm{d}s\,\mathrm{d}u > \frac{1}{\mathrm{e}}, \tag{2.21}$$
for some $n \in \mathbb{N}$, then (1.1) *is oscillatory.*

Remark 2.12. The above theorem permits us to deduce oscillation of (1.1) from that of the first-order advanced differential equation (2.18). One can see that, even for $n = 1$, the criterion (2.21) is sharper than (1.5) and thus provides a better result.

Theorem 2.13. *Assume that the second-order differential equation*
$$y''(t) + q(t)a_n(\sigma(t),t)y(t) = 0 \tag{2.22}$$
is oscillatory for some $n \in N$. Then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that y is a positive solution of (1.1) on $[t_0,\infty)$. Then there exists $t_1 \geq t_0$ such that y satisfies (2.2) on $[t_1,\infty)$. It follows from Lemma 2.9 that
$$y(\sigma(t)) \geq y(t)a_n(\sigma(t),t) \tag{2.23}$$
for some $n \in \mathbb{N}$ and t large enough. Using (2.23) in (1.1), we see that y is a positive solution of
$$y''(t) + q(t)a_n(\sigma(t),t)y(t) \leq 0.$$
As in the proof of Theorem 2.3, we can see that the corresponding equation (2.22) also has a positive solution, a contradiction. The proof is complete. □

Corollary 2.14. *If*

$$\liminf_{t\to\infty} t\int_t^\infty q(s)a_n(\sigma(s),s)\mathrm{d}s > \frac{1}{4} \tag{2.24}$$

for some $n\in\mathbb{N}$, then (1.1) is oscillatory.

We define

$$\tilde{q}_n(t) = q(t)\Big(1+\int_t^{\sigma(t)}\int_u^\infty q(s)a_n(\sigma(s),t)\mathrm{d}s\,\mathrm{d}u\Big), \quad n\in\mathbb{N},$$

where $a_n(s,t)$ is as in Lemma 2.9.

Theorem 2.15. *Assume that the second-order differential equation*

$$y''(t)+\tilde{q}_n(t)y(t)=0 \tag{2.25}$$

is oscillatory for some $n\in\mathbb{N}$. Then (1.1) is oscillatory.

Proof. Suppose to the contrary that y is a positive solution of (1.1) on $[t_0,\infty)$. Then there exists $t_1\ge t_0$ such that y satisfies (2.2) on $[t_1,\infty)$. As in the proof of Theorem 2.10, we obtain (2.20), that is,

$$y'(t)\ge y(\sigma(t))\int_t^\infty q(s)a_n(\sigma(s),\sigma(t))\mathrm{d}s. \tag{2.26}$$

Integrating (2.26) from t to $\sigma(t)$ and using (2.14), i.e.,

$$y(\sigma(u))\ge y(t)a_n(\sigma(u),t), \quad \sigma(u)\ge t,$$

we obtain

$$\begin{aligned} y(\sigma(t)) &\ge y(t)+\int_t^{\sigma(t)} y(\sigma(u))\int_u^\infty q(s)a_n(\sigma(s),\sigma(u))\mathrm{d}s\,\mathrm{d}u \\ &\ge y(t)\Big(1+\int_t^{\sigma(t)} a_n(\sigma(u),t)\int_u^\infty q(s)a_n(\sigma(s),\sigma(u))\mathrm{d}s\,\mathrm{d}u\Big). \end{aligned}$$

The rest of the proof is similar to that of Theorem 2.3 and so we omit it. □

Corollary 2.16. *If*

$$\liminf_{t\to\infty} t\int_t^\infty \tilde{q}_n(s)\mathrm{d}s > \frac{1}{4} \tag{2.27}$$

for some $n\in\mathbb{N}$, then (1.1) is oscillatory.

Lemma 2.17. *Let $y(t)$ be an eventually positive solution of (1.1). Then*

$$\rho_n := \liminf_{t\to\infty}\int_t^{\sigma(t)}\int_u^\infty q(s)a_n(\sigma(s),\sigma(u))\mathrm{d}s\,\mathrm{d}u \le \frac{1}{\mathrm{e}}, \tag{2.28}$$

and

$$\liminf_{t\to\infty}\frac{y(\sigma(t))}{y(t)}\ge\lambda_n,$$

where $a_n(t,s)$ is as in Lemma 2.9 and λ_n is the smaller root of the equation

$$\lambda_n = \mathrm{e}^{\rho_n\lambda_n}.$$

Proof. We proceed as in the proof of Theorem 2.10 to obtain that y satisfies (2.20). The next arguments are the same as in the proof of Lemma 2.6 so we can omit them. □

Theorem 2.18. *Let* (2.28) *hold and* λ_n *be as in Lemma 2.17. Assume that the second-order differential equation*
$$y''(t) + k\lambda_n q(t)y(t) = 0 \tag{2.29}$$
is oscillatory for some $n \in \mathbb{N}$ *and* $k \in (0,1)$*. Then* (1.1) *is oscillatory.*

Corollary 2.19. *Let* (2.28) *hold and* λ_n *be as in Lemma 2.17. If*
$$\liminf_{t\to\infty} t\int_t^\infty q(s)\mathrm{d}s > \frac{1}{4\lambda_n}, \tag{2.30}$$
for some $n \in \mathbb{N}$*, then* (1.1) *is oscillatory.*

Finally, we discuss the efficiency of newly obtained criteria on Euler-type differential equations.

Example 2.20. Consider the second-order advanced Euler differential equation
$$y''(t) + \frac{a}{t^2}y(ct) = 0, \quad c \ge 1, \quad a > 0, \quad t \ge 1. \tag{2.31}$$
Known oscillation criteria (1.4) and (1.5) give
$$a > \frac{1}{4} \tag{2.32}$$
and
$$a\ln c > \frac{1}{\mathrm{e}}, \tag{2.33}$$
respectively.

The recent result [3, Corollary 1] gives
$$a\Big(\frac{c^\beta - 1}{\beta} + \frac{1}{1-a} + \frac{c^\beta}{1-\beta}\Big) > 1, \tag{2.34}$$
where $\beta = \frac{1-\sqrt{1-4a}}{2}$ and $a \le 1/4$. From Corollary 2.4, we have that (2.31) is oscillatory if
$$a(1 + a\ln c) > \frac{1}{4}. \tag{2.35}$$
To apply Corollary 2.8, we set $\rho := a\ln c \le 1/\mathrm{e}$. Then the smaller root of the equation $\lambda = \mathrm{e}^{\rho\lambda}$ is
$$\lambda = -\frac{W(-\ln \mathrm{e}^\rho)}{\ln \mathrm{e}^\rho} = -\frac{W(-\rho)}{\rho},$$
where $W(\cdot)$ denotes the principal branch of the Lambert function, see [6] for details. Consequently, the oscillation criterion (2.13) becomes
$$-a\frac{W(-\rho)}{\rho} > \frac{1}{4},$$
that is,
$$-\frac{W(-a\ln c)}{\ln c} > \frac{1}{4}. \tag{2.36}$$

Now, we set $n = 1$. After simple calculations, the following conditions for oscillation of (2.31), i.e.,
$$\frac{a}{1-a}\ln c > \frac{1}{e}, \tag{2.37}$$
$$ac^a > \frac{1}{4}, \tag{2.38}$$

$$a\Big(1+\frac{c^a(c^a-1)}{1-a}\Big) > \frac{1}{4}, \tag{2.39}$$

$$\frac{(a-1)W(\frac{a}{a-1}\ln c)}{\ln c} > \frac{1}{4}, \quad \text{where } \frac{a}{a-1}\ln c \le 1/e, \tag{2.40}$$

result from Corollaries 2.11, 2.14, 2.16 and 2.19, respectively. A comparison of the effectiveness of the above-mentioned criteria in terms of the required value c for a given coefficient $a = 0.23$ is shown in the Table 1.

Table 1. Comparison of the strength of criteria (2.32)–(2.40) for a given $a = 0.23$

criterion	required c
(2.32)	inapplicable
(2.33)	4.950436
(2.34)	2.274700
(2.35)	1.459467
(2.36)	1.395881
(2.37)	3.426695
(2.38)	1.436966
(2.39)	1.304194
(2.40)	1.292806

On the other hand, if we set $a = 0.19$ and $c = 2$ in (2.31), then it is easy to verify that all criteria (2.33)−(2.40) fail. In such a case, it is interesting to compare the length of the iteration process in particular cases corresponding to Corollaries 2.11-2.19. As can be seen from Table 2, 13 iteration steps are necessary when applying Corollary 2.11, Corollary 2.14 requires 7 steps, while Corollaries 2.16 and 2.19 ensure the oscillation of (2.31) after the same number of iterations (6 steps).

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References

[1] R. P. Agarwal, S. R. Grace, D. O'Regan; *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic, Dordrchet, 2002.

[2] R. P. Agarwal, S. R. Grace, D. O'Regan; *Oscillation Theory for Second Order Dynamic Equations*, Taylor & Francis, London and New York, 2003.

[3] B. Baculíková; Oscillatory behavior of the second order functional differential equations, *Applied Mathematics Letters* **72** (2017): 35-41.

[4] E. Braverman, G. E. Chatzarakis, I. P. Stavroulakis; Iterative oscillation tests for differential equations with several non-monotone arguments, *Advances in Difference Equations* **2016.1** (2016): 87.

[5] J. Džurina; Oscillation of second order differential equations with advanced argument, *Mathematica Slovaca* **45.3** (1995): 263-268.

[6] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, D. E. Knuth; On the Lambert W function, *Advances in Computational mathematics*, **5(1),** (1996): 329–359.

[7] G. E. Chatzarakis, H. Péics; Differential equations with several non-monotone arguments: An oscillation result; *Applied Mathematics Letters*, (2016).

[8] L. E. Elsgolts, S. B. Norkin; *Introduction to the theory and application of differential equations with deviating arguments* Elsevier, 1973.

TABLE 2. Comparison of iterative processes for (2.31) resulting from Corollaries 2.11, 2.14, 2.16, 2.19, respectively.

n	crit. val. 1/e (Cor. 2.11)
1	0.162590
2	0.179813
3	0.191500
4	0.200467
5	0.208003
6	0.214830
7	0.221447
8	0.235846
9	0.244837
10	0.256514
11	0.273525
12	0.302947
13	0.372771

n	crit. val. 1/4 (Cor. 2.14)
1	0.216745
2	0.228721
3	0.235918
4	0.241002
5	0.245011
6	0.248452
7	0.251627

n	crit. val. 1/4 (Cor. 2.16)
1	0.231658
2	0.237998
3	0.24264
4	0.246414
5	0.249743
6	0.252893

n	crit. val. 1/4 (Cor. 2.19)
1	0.227666
2	0.235188
3	0.240441
4	0.244541
5	0.248028
6	0.251219
7	undefined

[9] L. H. Erbe, Q. Kong, B. G. Zhang; *Oscillation Theory for Functional Differential Equations*, Marcel Dekker Inc., New York, 1995.

[10] W. B. Fite; Properties of the solutions of certain functional-differential equations, *Transactions of the American Mathematical Society* **22** (1921): 311-319.

[11] I. Gyori, G. Ladas; *Oscillation Theory of Delay Differential Equations with Applications*, Clarendon Press, Oxford, 1991.

[12] P. Hartman; *Ordinary Differential Equations*, Wiley, New York, 1964.

[13] E. Hille; Nonoscillation theorems, *Trans. Amer. Math. Soc.*, **64** (1948), 234-252.

[14] J. Jaroš, I. P. Stavroulakis; Oscillation tests for delay equations, *Rocky Mountain J. Math.*, **29** (1999), 197-207.

[15] R. Koplatadze, G. Kvinikadze, I. P. Stavroulakis; Properties A and B of n-th order linear differential equations with deviating argument, *Georgian Math. J.* **6** (1999), 553-566.

[16] T. Kusano, M. Naito; Comparison theorems for functional differential equations with deviating arguments, *J. Math. Soc. Japan* **3** (1981), 509–533.

[17] G. Ladas, V. Lakhshmikantham, J. S. Papadakis; Oscillations of higher-order retarded differential equations generated by the retarded argument, *Delay and Functional Differential Equations and Their Applications, Academic Press, New York*, 1972, 219-231.

[18] G. S. Ladde, V. Lakshmikantham, B. G. Zhang; *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1987.

[19] W. Leighton; The detection of the oscillation of solutions of a second order linear differential equation, Duke J. Math. 17 (1950) 57-62.

[20] Ch. G. Philos; On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math.* **36**, (1981), 168-178.

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