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# GEVREY-SMOOTHNESS OF INVARIANT TORI FOR NEARLY INTEGRABLE SIMPLECTIC MAPPINGS

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Abstract. In this article, we propose a general normal form to prove the persistence and the Gevrey-smoothness of lower dimensional elliptic invariant tori of nearly integrable symplectic mappings under the Rüssmann non-degeneracy condition. Our results generalize the ones presented in the literature.

## 1. Introduction

The KAM theory for nearly integrable Hamiltonian systems has been developed extensively in the past decades (KAM theory was named after Andrey Kolmogorov, Vladimir Arnold and Jrgen Moser). Studies under different non-degeneracy conditions [1, 2, 6] generate various KAM theorems, among which the non-degeneracy condition proposed by Rüssmann [22, 23] sounds very useful and weaker. In the KAM theory, the regularity of KAM invariant tori is an important issue to consider, since small divisor may usually cause the loss of smoothness. Pöschel [18] proved that the persisting invariant tori are $C^\infty$-smooth in the frequency parameter. Later Popov [20] obtained the Gevrey-smoothness of invariant tori in their frequencies under the Kolmogorov non-degeneracy condition. Xu and You [29] extended this result to the case of the Rüssmann non-degeneracy condition. Zhang and Xu [30, 31] investigated the elliptic lower dimensional tori for Gevrey-smooth Hamiltonian systems under Rüssmann's non-degeneracy condition.

In addition to Hamiltonian systems, KAM theorems for mappings [4, 5, 8, 9, 10, 22, 27, 3, 7] have been proven ever since Moser's well known work [14, 15] on area-preserving mappings. As we have seen, many profound results for Hamiltonian systems can be generalized to symplectic mappings since the latter are discrete Hamiltonian systems. This is one of main motivations of our studies on the Gevrey-smoothness of elliptic lower dimensional KAM invariant tori for symplectic mappings.

Despite the fact that some results in symplectic mappings can be extended to Hamiltonian systems, there are still critical differences between symplectic mappings and Hamiltonian systems. Normal form is crucial for the study of the resonance relation between tangential frequencies and normal frequencies. Unlike the

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normal form in Hamiltonian systems which is unique, for symplectic mappings there is not a unique standard normal form and in some cases is not easy to be discovered. Even if the normal form can be discovered, it can cause much difficulty in the KAM iteration. Moreover, symplectic mappings are determined implicitly by the generating functions, which makes the KAM estimates more complicated.

Recently, Lu et al [7] found a normal form for the elliptic lower dimensional tori to prove the persistence of the invariant tori. In this article, we provide a more generic normal form to study the persistence and Gevrey-smoothness of KAM tori, which is parameter-dependent under the Rüssmann non-degeneracy condition

Consider a family of parameterized symplectic mappings

$$\Phi : (x, u, y, v) \in \mathbb{T}^n \times \mathcal{W} \times \mathcal{O} \times \mathcal{W} \to (\hat{x}, \hat{u}, \hat{y}, \hat{v}) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m,$$

which is implicitly defined by a generating function

$$H(x, u, \hat{y}, \hat{v}; \xi) = N + P, \tag{1.1}$$

with

$$\begin{aligned} \hat{x} &= \partial_{\hat{y}} H(x, u, \hat{y}, \hat{v}; \xi), \quad y = \partial_x H(x, u, \hat{y}, \hat{v}; \xi), \\ \hat{u} &= \partial_{\hat{v}} H(x, u, \hat{y}, \hat{v}; \xi), \quad v = \partial_u H(x, u, \hat{y}, \hat{v}; \xi), \end{aligned} \tag{1.2}$$

where

$$N(x, u, \hat{y}, \hat{v}; \xi) = \langle x + \omega(\xi), \hat{y} \rangle + \langle Au, \hat{v} \rangle + \frac{1}{2}\langle Bu, u \rangle + \frac{1}{2}\langle C\hat{v}, \hat{v} \rangle, \tag{1.3}$$

and $A, B, C$ are constant matrices. We suppose that $\xi \in \Pi$ is parameter and $\Pi \subset \mathbb{R}^n$ is a bounded closed connected domain.

If $P = 0$, $\Phi$ is expressed explicitly as

$$\begin{gathered} \hat{x} = x + \omega(y), \quad \hat{y} = y, \\ \hat{u} = (A - C(A^T)^{-1}B)u + C(A^T)^{-1}v, \quad \hat{v} = -(A^T)^{-1}Bu + (A^T)^{-1}v. \end{gathered} \tag{1.4}$$

Let

$$\Omega(A, B, C) = \begin{pmatrix} A - C(A^T)^{-1}B & C(A^T)^{-1} \\ -(A^T)^{-1}B & (A^T)^{-1} \end{pmatrix}_{2m\times 2m}.$$

Then $(\hat{u}, \hat{v})^T = \Omega(A, B, C)(u, v)^T$. It is easy to see that $\mathbb{T}^n \times \{0, 0, 0\}$ is a lower dimensional invariant torus with the rotational frequency $\omega(\xi)$.

We call the lower dimensional invariant torus to be elliptic if 1 is not an eigenvalue of $\Omega(A, B, C)$ and each eigenvalue has unit modulus; while hyperbolic if no eigenvalue has unit modulus. For simplicity, let

$$A = \operatorname{diag}(a_1, a_2, \dots, a_m), \quad B = \operatorname{diag}(b_1, b_2, \dots, b_m), \quad C = \operatorname{diag}(c_1, c_2, \dots, c_m).$$

The rest of this article is organized as follows. In Section 2, we present the related preliminary results on the Gevery-class $G^\mu(\mathcal{O})$ of index $\mu$ ($\mu \geq 1$) and state our main result. Section 3 is dedicated to the proof of our main result. Section 4 is an appendix.

## 2. Preliminaries

Before stating our main results, we introduce some preliminary results on assumptions, definitions and norm forms.

(H1) (Ellipticity condition) Suppose that $\Delta_l^2 - 4 < 0$, where $\Delta_l = \frac{a_l^2 - b_l c_l + 1}{a_l}$, $l = 1, 2, \dots, m$

**Remark 2.1.** Direct calculations show that the eigenvalues of $\Omega(A,B,C)$ are $\frac{\Delta_l \pm\sqrt{\Delta_l^2-4}}{2}$, $l=1,2,\dots,m$. If $\Delta_l^2-4<0$, we have

$$\Big|\frac{\Delta_l \pm \sqrt{\Delta_l^2-4}}{2}\Big| = 1.$$

Let $\theta=(\theta_1,\theta_2,\dots,\theta_m)$ such that $e^{\pm \mathrm{i}\theta_l}=\frac{\Delta_l\pm\sqrt{\Delta_l^2-4}}{2}$ and $0<|\theta_l|\leqslant\frac{\pi}{2}$, $l=1,2,\dots,m$, where $\mathrm{i}=\sqrt{-1}$ In this case, the lower dimensional invariant torus is elliptic. We call $\theta$ the normal frequency. If $\Delta_l^2-4>0$, we have

$$\Big|\frac{\Delta_l \pm \sqrt{\Delta_l^2-4}}{2}\Big| \neq 1,\ l=1,2,\dots,m.$$

This means that the lower dimensional invariant torus is hyperbolic.

**Remark 2.2.** If we choose $A,B,C$ such that $a_i=\sec\theta_i$, $b_i=c_i=\tan\theta_i$, and $0<|\theta|\leqslant\frac{\pi}{2}$, the generated function (1.1) reduces to the case described in [7]. Note that the normal form in [7] is unstable, which means that the normal form cannot remain after one KAM step, thus the normalization is necessary at every KAM step in [7] . In this study, we use the above normal form, which can persist under the KAM iteration.

- (H2) (Rüssmann's non-degeneracy condition) There exists an integer $\bar n\geqslant 1$ such that
$$\operatorname{rank}\{\partial_\xi^\beta\omega(\xi):1\leqslant|\beta|\le\bar n\}=n,\quad \forall\xi\in\Pi. \tag{2.1}$$

**Remark 2.3.** The non-degeneracy condition (2.1) is slightly different from that in Hamiltonian systems:

$$\operatorname{rank}\big\{\partial_\xi^\beta\omega(\xi):|\beta|\le\bar n\big\}=n,\quad \forall\xi\in\Pi.$$

- (H3) (Non-resonance conditions) Suppose that for $k\in Z^n$ with $|k|\neq0$, $i,j,w\in\mathbb{Z}$ and $1\le i,j\le m$, $\omega(\xi)$ satisfies
$$|\langle k,\omega(\xi)\rangle-2\pi w|\ge\frac{2\alpha}{(2+|k|)^\tau}, \tag{2.2}$$
$$|\langle k,\omega(\xi)\rangle-\theta_i(\xi)-2\pi w|\ge\frac{2\alpha}{(2+|k|)^\tau}, \tag{2.3}$$
$$|\langle k,\omega(\xi)\rangle+\theta_i(\xi)\pm\theta_j(\xi)-2\pi w|\ge\frac{2\alpha}{(2+|k|)^\tau},\ |k|+|i-j|\neq0. \tag{2.4}$$

**Definition 2.4.** Let $\mathcal{O}\subset\mathbb{R}^n$ be a bounded, closed, and connected domain. A function $F:\mathcal{O}\to\mathbb{R}$ is said to belong to the Gevery-class $G^\mu(\mathcal{O})$ of index $\mu$ ($\mu\ge1$), provided that $F$ is $C^\infty(\mathcal{O})$-smooth and there exists a constant $M$ such that for all $p\in\mathcal{O}$, it holds

$$|\partial_p^\beta F(p)|\le cM^{|\beta|+1}\beta!^\mu,$$

where $|\beta|=\beta_1+\beta_2+\dots+\beta_n$ and $\beta!^\mu=\beta_1!\beta_2!\dots\beta_n!$ for $\beta=(\beta_1,\beta_2,\dots,\beta_n)\in\mathbb{Z}_+^n$

**Remark 2.5.** From the definition, it is easy to see that the class $G^1$ of Gevery-smooth functions coincides with the class of analytic functions, and it also satisfies

$$G^1\subset G^{\mu_1}\subset G^{\mu_2}\subset C^\infty,$$

for $1<\mu_1<\mu_2<\infty$.

Set

$$\mathcal{T}_s = \{x \in \mathbb{C}^n/2\pi\mathbb{Z}^n : |\operatorname{Im} x|_\infty \le s\}, \quad \mathcal{B}_r = \{y \in \mathbb{C}^n : |y|_1 \le r^2\},$$
$$\mathcal{W}_r = \{w \in \mathbb{C}^m : \ |w|_2 \le r\}.$$

Denote

$$\mathcal{D}(s,r) = \mathcal{T}_s \times \mathcal{W}_r \times \mathcal{B}_r \times \mathcal{W}_r,$$
$$|x|_\infty = \max_{1\le j\le n} |x_j|, \quad |y|_1 = \sum_{1\le j\le n} |y_j|, \quad |w|_2 = \Big(\sum_{1\le j\le m} |w_j|^2\Big)^{1/2}.$$

Let

$$\Pi = \{\xi \in \mathcal{O} : \operatorname{dist}(\xi, \partial\mathcal{O}) \geqslant h\}, \quad \Pi_h = \{\xi \in \mathbb{C}^n : \operatorname{dist}(\xi, \Pi) \leqslant h\}.$$

**Remark 2.6.** By definition, $f \in G^{1,\mu}(\mathcal{D}(s,r)\times\Pi)$ which implies $f(x,y,u,v;\xi) \in C^\infty(\mathcal{D}(s,r)\times\Pi)$ and $f(x,y,u,v;\xi)$ is analytic with respect to $(x,y,u,v)$ on $\mathcal{D}(s,r)$ and $G^\mu$-smooth in $\xi$ on $\Pi_h$

If $P(x;\xi)$ is analytic on $\mathcal{T}_s\times\Pi$, we can expand $P(x;\xi)$ as the Fourier series

$$P(x;\xi) = \sum_{k\in\mathbb{Z}^n} P_k(\xi) e^{\mathrm{i}\langle \mathrm{k},\mathrm{x}\rangle}.$$

We define

$$\|P\|_s = \sum_{k\in\mathbb{Z}^n} |P_k|_\Pi e^{s|k|}, \quad |P_k|_\Pi = \max_{\xi\in\Pi} |P_k(\xi)|.$$

When $P(x,u,\hat{y},\hat{v};\xi)$ is analytic on $\mathcal{D}(s,r)\times\Pi$, we let

$$P(x,u,\hat{y},\hat{v};\xi) = \sum_{k\in\mathbb{Z}^n} P_k(u,\hat{y},\hat{v};\xi) e^{\mathrm{i}\langle \mathrm{k},\mathrm{x}\rangle}, \quad P_k(u,\hat{y},\hat{v};\xi) = \sum_{l,i,j} P_{klij}(\xi)\hat{y}^l u^i \hat{v}^j.$$

We define

$$\|P\|_{D(s,r)\times\Pi} = \sum_{k\in\mathbb{Z}^n} |P_k|_r e^{s|k|},$$

where

$$|P_k|_r = \sup_{(u,\hat{y},\hat{v})\in\mathcal{W}_r\times\mathcal{B}_r\times\mathcal{W}_r} \sum_{i,j,l} \|P_{klij}\|_s \hat{y}^l u^i \hat{v}^j.$$

This norm is obviously stronger than the sup-norm. Moreover, the Cauchy estimates of analytic functions are also valid under this norm. Let

$$X_P = (-\partial_{\hat{y}}P, -\partial_{\hat{v}}P, \partial_x P, \partial_u P),$$

endowed with the corresponding weighed norm

$$\begin{aligned}&\|X_P\|_{r;\mathcal{D}(s,r)\times\Pi}\\ &= \|\partial_{\hat{y}}P\|_{D(s,r)\times\Pi} + \frac{1}{r}\|\partial_{\hat{v}}P\|_{D(s,r)\times\Pi} + \frac{1}{r^2}\|\partial_x P\|_{D(s,r)\times\Pi} + \frac{1}{r}\|\partial_u P\|_{D(s,r)\times\Pi},\end{aligned}$$

where

$$\|\partial_{\hat{x}}P\|_{D(s,r)\times\Pi} = \sum_j \|\partial_{\hat{x}_j}P\|_{D(s,r)\times\Pi}, \quad \|\partial_{\hat{y}}P\|_{D(s,r)\times\Pi} = \max_j \|\partial_{\hat{y}_j}P\|_{D(s,r)\times\Pi},$$
$$\|\partial_u P\|_{D(s,r)\times\Pi} = \Big(\sum_j (\|\partial_{u_j}P\|_{s,r})^2\Big)^{1/2}.$$

Now, we state our main result.

**Theorem 2.7.** *Consider the symplectic mapping $\Phi(\cdot;\xi)$ defined by (1.1). Suppose that*

$$\tau \geqslant n\bar{n}-1, \quad \max_{\xi\in\Pi_h}\{|\frac{\partial\omega(\xi)}{\partial\xi}|,|\frac{\partial\theta(\xi)}{\partial\xi}|\} \leqslant T,$$

*and conditions (H1)–(H3) hold. There exists a $\gamma>0$ such that for any $0<\alpha<1$, if*

$$\|X_P\|_{r;\mathcal{D}(s,r)\times\Pi_h} = \epsilon \leqslant \gamma^3\alpha^{2\bar{\nu}}\rho^{2\nu},$$

*where $\bar{\nu}=4(\bar{n}+1)$ and $\nu=4\tau(\bar{n}+1)+n+\bar{n}$, then the following two statements are true.*

*(i) There exist a non-empty Cantor-like subset $\Pi_*\subset\Pi$, parameterized symplectic mappings $\Psi_*(\cdot;\xi)\in G^{1,\mu}(D(s/2,r/2)\times\Pi_*)$, and parameterized functions $H_*\in G^{1,\mu}(D(s/2,r/2)\times\Pi_*)$ such that*

$$\|\partial_\xi^\beta(\Psi_*-id)\|_{r;D(\frac{s}{2},\frac{r}{2})\times\Pi_*} \le c\rho^\nu M^{|\beta|}\beta!^\mu\gamma^{\frac{9}{4(n+1)}}, \quad \forall\beta\in Z_n^+, \quad \forall\xi\in\Pi_*, \tag{2.5}$$

*where $M=\frac{2T+1}{\alpha}[\frac{4(\mu-1)(n+1)}{3}]^{\mu-1}$, and $H_*(\cdot;\xi)=N_*+P_*$ satisfies*

$$N_*(x,u,\hat{y},\hat{v};\xi) = \langle x+\omega_*,\hat{y}\rangle + \langle A_*u,\hat{v}\rangle + \frac{1}{2}\langle B_*u,u\rangle + \frac{1}{2}\langle C_*\hat{v},\hat{v}\rangle,$$

$$P_*(x,u,\hat{y},\hat{v};\xi) = \sum_{|i|+|j|+2|l|\geq 3} P_{lij}(x;\xi)\hat{y}^l u^i \hat{v}^j.$$

*Moreover, $\Phi_*(\cdot;\xi)=\Psi_*^{-1}\circ\Phi\circ\Psi_*$ is generated by $H_*(\cdot;\xi)=N_*+P_*$.*

*(ii) For $\xi\in\Pi_*$, the symplectic mapping $\Phi(\cdot;\xi)$ admits an invariant torus*

$$\{T_\xi = \Psi_*(T^n,0,0,0;\xi) : \xi\in\Pi_*\}$$

*whose tangential frequency $\omega_*$ and normal frequency $\theta_*$ satisfy*

$$|\partial_\xi^\beta(\omega_*(\xi)-\omega(\xi))|_{\Pi_*} \le c\rho^{2\nu}M^{|\beta|}\beta!^\mu\gamma^{\frac{9}{4(n+1)}}, \tag{2.6}$$

$$|\partial_\xi^\beta(\theta_*(\xi)-\theta(\xi))|_{\Pi_*} \le c\rho^{2\nu}M^{|\beta|}\beta!^\mu\gamma^{\frac{9}{4(n+1)}}. \tag{2.7}$$

*Moreover, for $i,j\in\mathbb{Z}$ and $1\le i,j\le m$, we have*

$$|\langle\omega_*(\xi),k\rangle - s_1\theta_{*i}(\xi) - s_2\theta_{*j}(\xi) - 2\pi w| \ge \frac{\alpha}{(2+|k|)^\tau}, \tag{2.8}$$

*where $\xi\in\Pi_*$, $0\neq k\in Z^n$, $0\leqslant|s_1|+|s_2|\leqslant 2$, and $s_d\in\mathbb{Z}$ $(d=1,2)$. In addition, we have*

$$\text{meas}(\Pi\setminus\Pi_*)\to 0, \quad \text{as } \alpha\to 0.$$

## 3. Proof of main result

3.1. **KAM-steps.** To prove our main result, we apply the idea for Hamiltonian systems [19, 29] as well as some technical lemmas.

**KAM iteration lemma:** For the symplectic mapping $\Phi(\cdot;\xi)$ defined by (1.1), when $\delta\in(0,1)$, let $\mu=\tau+\delta+2$, $\sigma=(\frac{3}{4})^{\frac{\delta}{\tau+1+\delta}}$, $0<E<1$, $0<\eta<\frac{1}{8}$ and $0<\rho=(1-\sigma)s/10<\frac{s}{5}$. Let

$$\max_{\xi\in\Pi_h}\{|\frac{\partial\omega(\xi)}{\partial\xi}|,|\frac{\partial\theta(\xi)}{\partial\xi}|\} \leqslant T, \quad h=\frac{\alpha}{(2+K)^{\tau+1}T},$$

where $K>0$ satisfies $\eta^2e^{-K\rho}=E$. Suppose that conditions (H1)–(H3) hold and $P$ satisfies

$$\|X_P\|_{r;D(s,r)\times\Pi_d} \leqslant \epsilon = \eta^2\alpha^{2\bar{\nu}}\rho^{2\nu}E$$

with $0 < \alpha < 1$, $\bar{\nu} = 4(\bar{n}+1)$ and $\nu = 4\tau(\bar{n}+1) + n + \bar{n}$. Then the following three statements are true.

(i) For $\xi \in \Pi_h$, there exists a symplectic diffeomorphism $\Psi(\cdot;\xi)$ with

$$\|\Psi - id\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi_h} \le \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu}}, \quad \|D\Psi - id\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi_h} \le \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu+1}},$$

such that the conjugate mapping $\Phi_+(\cdot;\xi) = \Psi^{-1}\circ\Phi\circ\Psi$ is generated by $H_+(\cdot;\xi) = N_+ + P_+$, where

$$N_+ = \langle x + \omega_+(\xi), \hat{y}\rangle + \langle A_+ u, \hat{v}\rangle + \frac{1}{2}\langle B_+ u, u\rangle + \frac{1}{2}\langle C_+\hat{v}, \hat{v}\rangle$$

and $P_+$ satisfies

$$\|X_P\|_{r_+;D(s_+,r_+)\times\Pi_d} \leqslant \eta_+^2\alpha_+^{2\bar{\nu}}\rho_+^{\nu}E_+ = \epsilon_+$$

with

$$s_+ = s - 5\rho, \quad \rho_+ = \sigma\rho, \quad \eta = E, \quad r_+ = \eta r, \quad E_+ = E^{\frac{4}{3}}, \quad \frac{\alpha}{2} \leqslant \alpha_+ \leqslant \alpha.$$

Let $e^{\pm i\theta_{+l}}$ be the eigenvalues of $\Omega(A_+, B_+, C_+)$, where $\theta_+ = (\theta_{+1}, \theta_{+2}, \dots, \theta_{+m})$ and $l = 1, 2, \dots, m$. We have

$$|\omega_+(\xi) - \omega(\xi)| \le \epsilon, \quad |\theta_+(\xi) - \theta(\xi)| \le c\epsilon, \quad \forall\xi \in \Pi_h. \tag{3.1}$$

(ii) Let $\alpha_+ = \alpha - (K+2)^{\tau+1}\epsilon$,

$$\bar{\Pi} = \Big\{\xi \in \Pi : |\langle\omega_+(\xi), k\rangle - s_1\theta_{+i}(\xi) - s_2\theta_{+j}(\xi) - 2\pi w| < \frac{2\alpha_+}{(2+|k|)^{\tau}},\ k \in Z^n,$$
$$K < |k| \le K_+,\ 0 \leqslant |s_1| + |s_2| \leqslant 2,\ s_d \in \mathbb{Z}\ (d = 1, 2)\Big\},$$

and $\Pi_+ = \Pi \setminus \bar{\Pi}$. Then for $\xi \in \Pi_+$, $\forall k \in Z^n$ and $0 < |k| \le K_+$, we have

$$|\langle\omega_+(\xi), k\rangle - s_1\theta_{+i} - s_2\theta_{+j} - 2\pi w| \geqslant \frac{2\alpha_+}{(2+|k|)^{\tau}}, \tag{3.2}$$

where $K_+ > 0$ satisfies $\frac{e^{-K_+\rho_+}}{\eta_+^2} = E_+$.

(iii) Let $T_+ = T + \frac{6\epsilon}{h}$ and $h_+ = \frac{\alpha_+}{2(K_++2)^{\tau+1}T_+}$. If $h_+ \le \frac{5}{6}h$, we have

$$\max_{\xi\in\Pi_{h_+}} \big\{|\frac{\partial\omega_+(\xi)}{\partial\xi}|,\ |\frac{\partial\theta_+(\xi)}{\partial\xi}|\big\} \le T_+,$$

where $\Pi_{h_+}$ is the complex $h_+$-neighborhood of $\Pi_+$

**A. Generating functions of conjugate mappings:** Let $p = (x, u)$ and $q = (y, v)$. The symplectic structure becomes $dp \wedge dq$ on $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$. Consider a symplectic mapping $\Phi : (p, q) \to (\hat{p}, \hat{q})$ generated by

$$\hat{p} = \partial_{\hat{q}}H(p, \hat{q}) = H_2(p, \hat{q}) \quad \text{and} \quad q = \partial_p H(p, \hat{}) = H_1(p, \hat{q}). \tag{3.3}$$

The generating function is $H(p, \hat{q}) = N(p, \hat{q}) + P(p, \hat{q})$, where $N$ represents the main term and $P$ is a small perturbation. Define a symplectic transformation $\Psi : (p_+, q_+) \to (p, q)$ by

$$q = q_+ + F_1(p, q_+) \quad \text{and} \quad p_+ = p + F_2(p, q_+). \tag{3.4}$$

The generating function is $\langle p, q_+\rangle + F(p, q_+)$ with $F$ being a small function. So $\Psi$ approaches to the identity. Then, we get a conjugate mapping

$$\Phi_+ = \Psi^{-1}\circ\Phi\circ\Psi : (p_+, q_+) \to (\hat{p}_+, \hat{q}_+)$$

implicitly by

$$\hat{p}_+ = H_2(p,\hat{q}) + F_2(\hat{p},\hat{q}_+) \quad \text{and} \quad q_+ = H_1(p,\hat{q}) - F_1(p,q_+). \tag{3.5}$$

From the following Lemma, $\Phi_+$ is generated by a function $H_+(p_+,\hat{q}_+)$.

**Lemma 3.1** ([7]). *The conjugate symplectic mapping $\Phi_+$ can be determined by $H_+(p_+,\hat{q}_+)$ through*

$$\hat{p}_+ = \partial_{\hat{q}_+} H_+(p_+,\hat{q}_+),\ q_+ = \partial_{p_+} H_+(p_+,\hat{q}_+), \tag{3.6}$$

*where*

$$\begin{aligned} H_+(p_+,\hat{q}_+) &= H(p,\hat{q}) + H_1(p,\hat{q})F_2(p,q_+) - H_2(p,\hat{q})F_1(\hat{p},\hat{q}_+) \\ &\quad + F(\hat{p},\hat{q}_+) - F(p,q_+) - F_1(p,q_+)F_2(p,q_+), \end{aligned} \tag{3.7}$$

*with $p,\hat{p},\hat{q},q_+$ depending on $(p_+,\hat{q}_+)$ as explained above. Moreover, if we set $z = (p_+,\hat{q}_+)$, then we have*

$$H_+(z) = H(z) + F(N_2(z),\hat{q}_+) - F(p_+,N_1(z)) + Q(z). \tag{3.8}$$

*The small term $Q(z)$ has the estimate*

$$\|X_Q\|_{r;\mathcal{D}(s-5\rho,r/16)\times\Pi} \le \frac{c\epsilon^2}{\alpha^{2\bar{\nu}}\rho^{2\nu}}, \tag{3.9}$$

*with $\bar{\nu} = 4(\bar{n}+1)$ and $\nu = \bar{n} + n + 4\tau(\bar{n}+1)$.*

**B. Truncation:** Let

$$P = R + (P-R), \tag{3.10}$$

where

$$\begin{aligned} R(p,\hat{q}) &= P_{000}(x) + \langle P_{100}(x),\hat{y}\rangle + \langle P_{010}(x),u\rangle + \langle P_{001}(x),\hat{v}\rangle \\ &\quad + \langle P_{011}(x)u,\hat{v}\rangle + \frac{1}{2}\langle P_{020}(x)u,u\rangle + \frac{1}{2}\langle P_{002}(x)\hat{v},\hat{v}\rangle, \end{aligned} \tag{3.11}$$

with

$$P_{lij} = \frac{\partial^{l+i+j}P}{\partial\hat{y}^l\partial u^i\partial\hat{v}^j}|_{u=0,\hat{y}=0,\hat{v}=0}, \quad 2|l|+|i|+|j| \le 2.$$

So we have

$$P - R = \sum_{2|l|+|i|+|j|\leqslant 2,k\geqslant K} P_{lij}\hat{y}^l u^i\hat{v}^j + \sum_{2|l|+|i|+|j|\geqslant 3} P_{lij}\hat{y}^l u^i\hat{v}^j.$$

**C. Extension of small divisor estimate:** For $\xi\in\Pi_h$, there exists a $\xi_0\in\Pi$ such that $|\xi-\xi_0|<h$ For $|k|\le K$, we have

$$\begin{aligned} &|\langle\omega(\xi)-\omega(\xi_0),k\rangle + s_1(\theta_i(\xi)-\theta_i(\xi_0)) + s_2(\theta_j(\xi)-\theta_j(\xi_0))| \\ &\leqslant |\langle\omega(\xi)-\omega(\xi_0),k\rangle| + |s_1||(\theta_i(\xi)-\theta_i(\xi_0))| + |s_2||(\theta_j(\xi)-\theta_j(\xi_0))| \\ &\leqslant (k+|s_1|+|s_2|)Th \\ &\leqslant (k+2)Th \\ &\leqslant \frac{\alpha}{(K+2)^\tau}. \end{aligned} \tag{3.12}$$

It follows from (2.2)–(2.4) and (3.12) that

$$|\langle\omega(\xi),k\rangle + s_1\theta_i(\xi) + s_2\theta_j(\xi) - 2\pi w| \geqslant \frac{\alpha}{(2+|k|)^\tau}, \tag{3.13}$$

where $h = \frac{\alpha}{(2+K)^{\tau+1}T}$, $0\leqslant|s_1|+|s_2|\leqslant 2$ and $s_d\in\mathbb{Z}$ $(d=1,2)$.

**D. Homological equations:**

Following the idea described in [19], we consider the homological equation:

$$N(p_+,\hat{q}_+)+R(p_+,\hat{q}_+)-F(p_+,N_p(p_+,\hat{q}_+))+F(N_q(p_+,\hat{q}_+),\hat{q}_+)=\bar{N}(p_+,\hat{q}_+),$$

where $F(p,\hat{q})$ possess the same form as (3.11). Just for simplicity, here and below we drop the subscripts '+' in $p_+$ and $\hat{q}_+$.

Let $x+\omega=\tilde{x}$. Denoting

$$\hat{p}=N_{\hat{q}}(p,\hat{p})=(\tilde{x},Au+Cv),\quad q=N_p(p,\hat{p})=(\hat{y},A\hat{v}+Bu),$$

we have

$$F(N_q(p,\hat{q}),\hat{q})-F(p,N_p(p,\hat{q}))=L_0+L_1+L_2,$$

where $L_0,L_1,L_2$ indicate the $ith$ ($i=0,1,2$) order terms of $u$ and $\hat{v}$ respectively:

$$\begin{aligned}
L_0&=(F_{000}(\tilde{x})-F_{000}(x))+\langle F_{100}(\tilde{x})-F_{100}(x),\hat{y}\rangle,\\
L_1&=\langle A^TF_{010}(\tilde{x})-F_{010}(x)-BF_{001}(x),u\rangle+\langle CF_{010}(\tilde{x})+F_{001}(\tilde{x})-AF_{001}(x),v\rangle,\\
L_2&=\langle\{F_{011}(\tilde{x})A-AF_{011}(x)+CF_{020}(\tilde{x})A-AF_{002}(x)B\}u,\hat{v}\rangle\\
&\quad+\frac{1}{2}\langle\{A^TF_{020}(\tilde{x})A-F_{020}(x)-BF_{002}(x)B-BF_{011}(x)-F_{011}^T(x)B\}u,u\rangle\\
&\quad+\frac{1}{2}\langle\{CF_{020}(\tilde{x})C+F_{002}(\tilde{x})-AF_{002}(x)A^T+F_{011}(\tilde{x})C+CF_{011}^T(\tilde{x})\}\hat{v},\hat{v}\rangle.
\end{aligned}$$

We consider the equations

$$\begin{gathered}
L_0=(R_{000}(x)-[R_{000}])+\langle R_{100}(x)-[R_{100}],\hat{y}\rangle,\\
L_1=\langle R_{010}(x),u\rangle+\langle R_{001}(x),\hat{v}\rangle,\\
L_2=\langle(R_{011}(x)-\hat{A})u,\hat{v}\rangle+\frac{1}{2}\langle(R_{020}(x)-\hat{B})u,u\rangle+\frac{1}{2}\langle(R_{002}(x)-\hat{C})\hat{v},\hat{v}\rangle,
\end{gathered}\tag{3.14}$$

where $\hat{A},\hat{B}$ and $\hat{C}$ are to be determined.

We start with the equation

$$F_{j00}(x+\omega)-F_{j00}(x)=R_{j00}(x)-[R_{j00}],\quad j=0,1,$$

by expanding $F_{j00}(x)$ and $R_{j00}(x)$ as the Fourier series:

$$F_{j00}(x)=\sum_{k\in\mathbb{Z}^n}F_{kj00}e^{\mathrm{i}\langle\mathrm{k},\mathrm{x}\rangle},\quad R_{j00}(x)=\sum_{k\in\mathbb{Z}^n}R_{kj00}e^{\mathrm{i}\langle\mathrm{k},\mathrm{x}\rangle}.$$

It follows that

$$F_{kj00}=\frac{1}{e_k-1}R_{kj00},\tag{3.15}$$

with $e_k=e^{\mathrm{i}\langle\mathrm{k}\ (\omega\rangle},k\neq0)$ By (3.13), we have the estimate

$$\|F_{j00}\|_{(s-\rho)\times\Pi}\leq\frac{c\|R_{j00}\|_s}{\alpha^{\bar{n}+1}\rho^{\bar{n}+n+\tau(\bar{n}+1)}}.\tag{3.16}$$

Next we solve the second equation of (3.14). Let $F_{010}=(F_{010}^1,\dots,F_{010}^m)$ and $F_{001}=(F_{001}^1,\dots,F_{001}^m)$ and expand $F_{0i'j'}^l(x)$ and $R_{0i'j'}^l(x)$ as the Fourier series:

$$F_{0i'j'}^l(x)=\sum_{k\in\mathbb{Z}^n}F_{k0i'j'}^le^{\mathrm{i}\langle\mathrm{k},\mathrm{x}\rangle},\quad R_{0i'j'}^l(x)=\sum_{k\in\mathbb{Z}^n}R_{k0i'j'}^le^{\mathrm{i}\langle\mathrm{k},\mathrm{x}\rangle}$$

with $l=1,2,\dots,m$ and $(i',j')=(0,1)$ or $(1,0)$

By the definition of $L_1$ and the second equation of (3.14), one can see the relation between $F^l_{0i'j'}(x)$ and $R^l_{0i'j'}(x)$:

$$M_l \cdot \begin{pmatrix} F^l_{k010}(x) \\ F^l_{k001}(x) \end{pmatrix} = \begin{pmatrix} R^l_{k010} \\ R^l_{k001} \end{pmatrix},$$

where

$$M_l = \begin{pmatrix} a_l e_k - 1 & -b_l \\ c_l e_k & e_k - a_l \end{pmatrix}$$

with $e_k = e^{\mathrm{i}\langle \mathrm{k},\omega\rangle}$. By a straightforward calculation, we have

$$\begin{aligned} \det(M_l) &= \Big(e_k - \frac{\Delta_l + \sqrt{\Delta_l^2 - 4}}{2}\Big)\Big(e_k - \frac{\Delta_l - \sqrt{\Delta_l^2 - 4}}{2}\Big) \\ &= -2\Big(\sin\frac{\langle k,\omega\rangle + \theta_l}{2} - \mathrm{i}\cos\frac{\langle k,\omega\rangle + \theta_l}{2}\Big)\sin\frac{\langle k,\omega\rangle - \theta_l}{2}, \end{aligned}$$

where $\theta_l$, $\Delta_l$ $(l = i, j)$, are defined in Remark 2.1. By (3.13) we know $|\det(M_l)| \geqslant \frac{\alpha^2}{(2+|k|)^{2\tau}}$. Note that

$$F^l_{k0i'j'} = \frac{\tilde{R}^l_{i'j'}}{|\det(M_l)|}$$

with $\tilde{R}^l_{i'j'} = c_1 R^l_{k010}(x) + c_2 R^l_{k001}(x)$ Then

$$\|F_{0i'j'}\|_{D(s-\rho,r)\times\Pi} \le \frac{c\|R_{0i'j'}\|_s}{\alpha^{2\bar{n}+2}\rho^{2\tau(\bar{n}+1)+\bar{n}+n}} \tag{3.17}$$

with $(i', j') = (0, 1)$ or $(1, 0)$.

Before solving the third equation of (3.14), let us consider the equation

$$L_2 = \langle R_{011}(x)u, \hat{v}\rangle + \frac{1}{2}\langle R_{020}u, u\rangle + \frac{1}{2}\langle R_{002}\hat{v}, \hat{v}\rangle. \tag{3.18}$$

Let $F_{0\,i'j'} = (F^{ij}_{0\,i'j'})_{1\le i,j\le m}$ with $(i', j') = (1, 1), (2, 0)$ or $(0, 2)$ We expand $F^{ij}_{0\,i'j'}$ and $R^{ij}_{0\,i'j'}$ as

$$F^{ij}_{0\,i'j'} = \sum_{k\in\mathbb{Z}^n} F_{k0\ i'j'} e^{\mathrm{i}\langle \mathrm{k},\mathrm{x}\rangle}, \quad R^{ij}_{0\,i'j'} = \sum_{k\in\mathbb{Z}^n} R_{k0\,i'j'} e^{\mathrm{i}\langle \mathrm{k},\mathrm{x}\rangle}.$$

From the definition of $L_2$ and (3.18), we have

$$N_{ij} \begin{pmatrix} F^{ji}_{k011} \\ F^{ij}_{k011} \\ F^{ij}_{k020} \\ F^{ij}_{k002} \end{pmatrix} = \begin{pmatrix} R^{ji}_{k011} \\ R^{ij}_{k011} \\ R^{ij}_{k020} \\ R^{ij}_{k002} \end{pmatrix},$$

where

$$N_{ij} = \begin{pmatrix} 0 & e_k a_j - a_i & e_k c_i a_j & -a_i b_j \\ e_k a_i - a_j & 0 & e_k a_i c_j & -b_i a_j \\ -b_j & -b_i & e_k a_i a_j - 1 & -b_i b_j \\ e_k c_i & e_k c_j & e_k c_i c_j & e_k - a_i a_j \end{pmatrix}.$$

A direct calculation gives $\det(N_{ij}) = S_4 e_k^4 + S_3 e_k^3 + S_2 e_k^2 + S_1 e_k + S_0$, where

$$\begin{gathered} S_4 = a_i^2 a_j^2, \quad S_0 = a_i^2 a_j^2, \\ S_3 = S_1 = a_i^3 a_j^3 - a_i^3 a_j b_j c_j - a_j^3 a_i b_i c_i + a_j a_i b_j b_i c_j c_i, \\ + a_j a_i^3 + a_j^3 a_i - a_j a_i b_i c_i - a_j a_i b_j c_j + a_j a_i, \end{gathered}$$

$$\begin{aligned} S_2 = {} & a_i^4 a_j^2 - a_i^2 a_j^4 + 2a_j^2 a_i^2 b_i c_i + 2a_j^2 a_i^2 b_j c_j - a_i^2 b_j^2 c_j^2 \\ & - a_j^2 b_i^2 c_i^2 - 2a_i^2 a_j^2 + 2a_i^2 b_j c_j + 2a_j^2 b_i c_i - a_i^2 - a_j^2. \end{aligned}$$

For $i, j = 1, 2, \dots, m$, we find

$$\det(N_{ij}) = \left(e_k - e^{\mathrm{i}\theta_i} e^{\mathrm{i}\theta_j}\right)\left(e_k - e^{-\mathrm{i}\theta_i} e^{-\mathrm{i}\theta_j}\right)\left(e_k - e^{\mathrm{i}\theta_i} e^{-\mathrm{i}\theta_j}\right)\left(e_k - e^{-\mathrm{i}\theta_i} e^{\mathrm{i}\theta_j}\right),$$

with $\theta_l$ $(l = i, j)$ given as in Remark 2.1. By (3.13), we have

$$|\det(N_{ij})| \geqslant \frac{\alpha^4}{(2 + |k|)^{4\tau}},$$

with $|k| + |i - j| \neq 0$. Thus we can solve the equation (3.18) in the case of $|k| + |i - j| \neq 0$ and get

$$F^{ij}_{k0i'j'} = \frac{\tilde{R}^{ij}}{|\det(N_{ij})|}, \tag{3.19}$$

with $\tilde{R}^{ij} = c_1 R^{ji}_{k011} + c_2 R^{ij}_{k011} + c_3 R^{ij}_{k020} + c_4 R^{ij}_{k002}$.

From (3.18) and (3.19), we consider the third equation of (3.14) by setting

$$\begin{aligned} &\hat{\omega} = \operatorname{diag}(\hat{\omega}_1, \dots, \hat{\omega}_n), \quad \hat{A} = \operatorname{diag}(\hat{A}_1, \dots, \hat{A}_m), \\ &\hat{B} = \operatorname{diag}(\hat{B}_1, \dots, \hat{B}_m), \quad \hat{C} = \operatorname{diag}(\hat{C}_1, \dots, \hat{C}_m), \end{aligned} \tag{3.20}$$

with

$$\hat{\omega}_j = [R^{jj}_{100}], \quad \hat{A}_j = [R^{jj}_{011}], \quad \hat{B}_j = [R^{jj}_{020}], \quad \hat{C}_j = [R^{jj}_{002}].$$

By a similar discussion as the above, one can deduce that

$$\|F_{0i'j'}\|_{D(s-\rho,r)\times\Pi} \leq \frac{c\|R_{0i'j'}\|_s}{\alpha^{4\bar{n}+4}\rho^{4\tau(\bar{n}+1)+\bar{n}+n}} \tag{3.21}$$

with $(i', j') = (1, 1)$, $(2, 0)$ or $(0, 2)$

It follows from (3.16), (3.17) and (3.21) that

$$\|X_F\|_{r;D(s-\rho,r)\times\Pi} \leq \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu}} \tag{3.22}$$

with $\bar{\nu} = 4(\bar{n} + 1)$ and $\nu = 4\tau(\bar{n} + 1) + n + \bar{n}$.

Let $\chi : (p, q) \to (-F_{y_+}, F_x)$ Since $\Psi = id + \chi$, we combine the estimate of $F$ in (3.22) and the Cauchy estimate to obtain

$$\begin{aligned} \|\Psi - id\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi} &\leq \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu}}, \\ \|D\Psi - id\|_{r;D(s-3\rho,\frac{r}{4})\times\Pi} &\leq \frac{c\epsilon}{\alpha^{\bar{\nu}}\rho^{\nu+1}}. \end{aligned}$$

**E. Choices of parameters in KAM iteration:** Set

$$0 < E < 1, \quad \eta = E, \quad \epsilon = \eta^2\alpha^{2\bar{\nu}}\rho^{2\nu}E, \quad \frac{e^{-K\rho}}{\eta^2} = E, \quad h = \frac{\alpha}{2(K+2)^{\tau+1}T}.$$

Let $\sigma \in (0, 1)$ We denote

$$\begin{gathered} \rho_+ = \sigma\rho, \quad s_+ = s - 5\rho, \quad r_+ = \eta r, \\ \alpha_+ = \alpha - (K+2)^{\tau+1}\epsilon, \quad \epsilon_+ = c\eta\epsilon, \quad E_+ = cE^{\frac{4}{3}}. \end{gathered}$$

From the equality

$$P - R = \sum_{|2l|+|i|+|j|\leqslant 2, k\geqslant K} P_{lij}\hat{y}^l u^i \hat{v}^j + \sum_{2|l|+|i|+|j|\geqslant 3} P_{lij}\hat{y}^l u^i \hat{v}^j,$$

we get

$$\|X_{P-R}\|_{\eta r;\mathcal{D}(s-5\rho,\eta r)\times\Pi} \leqslant c\cdot\epsilon\Big(\eta+\frac{e^{-K\rho}}{\eta^2}\Big). \tag{3.23}$$

By (3.9) and (3.23), we have

$$\begin{aligned}\|X_{P_+}\|_{\eta r;D(s-5\rho,\eta r)\times\Pi_+} &\le c\cdot\epsilon\Big(\eta+\frac{e^{-K\rho}}{\eta^2}\Big)+\frac{c\epsilon^2}{\eta^2\alpha^{2\bar\nu}\rho^{2\nu}}\\ &\le c\eta\epsilon = c\alpha^{2\overline{\nu}}\rho^{2\nu}E^4\\ &\le \alpha_+^{2\bar\nu}\rho_+^{2\nu}E_+^3.\end{aligned}$$

Setting $\epsilon_+=\alpha_+^{2\bar\nu}\rho_+^{2\nu}E_+^3$, so we arrive at

$$\|X_{P_+}\|_{r_+;\mathcal{D}(s_+,r_+)\times\Pi_+}\le\epsilon_+,$$

Given the choice of $\alpha_+$, for $\xi\in\Pi_+$ and $0\neq k\le K$, we have

$$\begin{aligned}&|\langle k,\omega_+(\xi)\rangle-2\pi w|\\ &\geqslant |\langle k,\omega(\xi)\rangle+2\pi w|-|\langle k,\omega_+(\xi)-\omega(\xi)\rangle|\\ &\ge \frac{2}{(2+|k|)^\tau}[\alpha-(2+K)^{\tau+1}\epsilon].\end{aligned}$$

Similarly, for sufficiently large $K$ we have

$$|\langle k,\omega_+(\xi)\rangle+s_1\theta_{+i}(\xi)+s_1\theta_{+j}(\xi)-2\pi w|\geqslant\frac{2}{(2+|k|)^\tau}[\alpha-(2+K)^{\tau+1}\epsilon],$$

with $0<|s_1|+|s_2|\leqslant 2,\ s_d\in\mathbb{Z},\ (d=1,2),\ \xi\in\Pi_+$ and $0\neq k\le K$ In view of $\alpha_+=\alpha-(2+K)^{\tau+1}\epsilon$, we have

$$|\langle\omega_+(\xi),k\rangle-s_1\theta_{+i}(\xi)-s_2\theta_{+j}(\xi)-2\pi w|\geqslant\frac{2\alpha_+}{(2+|k|)^\tau},$$

where $\xi\in\Pi_+$ for all $k\in\mathbb{Z}^n$ $(0<|k|\le K_+)$, $0\leqslant|s_1|+|s_2|\leqslant 2$, and $s_d\in\mathbb{Z}$ $(d=1,2)$.

Given the choice of $T_+$, we suppose that $h_+\le\frac{5}{6}h$ For $\xi\in\Pi_{h_+}^+$, it follows the Cauchy estimate that

$$|\partial(\omega_+(\xi)-\omega(\xi))/\partial\xi|_{h_+}\le\frac{|\omega_+(\xi)-\omega(\xi)|_h}{h-h_+}\le\frac{6\epsilon}{h}.$$

Letting $T_+=T+\frac{6\epsilon}{h}$ and $h_+=\frac{\alpha_+}{T_+(2+K_+)^{\tau+1}}$, we obtain

$$\max_{\xi\in\Pi_{h_+}}|\partial\omega_+/\partial\xi|\leqslant\max_{\xi\in\Pi_{h_+}}|\partial(\omega_+-\omega(\xi))/\partial\xi|+\max_{\xi\in\Pi_{h_+}}|\partial\omega/\partial\xi|\le T_+,$$

and

$$\max_{\xi\in\Pi_{h_+}}|\partial\theta_+/\partial\xi|\le T_+.$$

3.2. **Iteration.** Set

$$\begin{gathered}s_0=s,\quad \rho_0=(1-\sigma)s/10,\quad r_0=r,\quad \alpha_0=\alpha,\\ \eta_0=E_0,\quad \epsilon_0=\alpha_0^{2\bar\nu}\rho_0^{2\nu}E_0\eta_0^2,\quad \frac{e^{-K_0\rho_0}}{\eta_0^2}=E_0.\end{gathered}$$

Let

$$\omega_0(\xi)=\omega(\xi),\quad \theta_0(\xi)=\big(\theta_{01}(\xi),\theta_{02}(\xi),\dots,\theta_{0m}(\xi)\big)$$

$$\Pi_0 = \Big\{\xi \in \Pi : |\langle\omega_0(\xi), k\rangle - s_1\theta_{0i}(\xi) - s_2\theta_{0j}(\xi) - 2\pi w| \geqslant \frac{2\alpha}{(1+|k|)^\tau}, \\ k \in Z^n,\ 0 < |k| \le K_0,\ 0 \leqslant |s_1| + |s_2| \leqslant 2,\ s_d \in \mathbb{Z},\ d = 1,2\Big\}.$$

Let

$$T_0 = T = \max_{\xi\in\Pi_h}\{|\frac{\partial\omega(\xi)}{\partial\xi}|, |\frac{\partial\theta(\xi)}{\partial\xi}|\}, \quad h_0 = \frac{\alpha_0}{(2+K_0)^{\tau+1}T_0}.$$

Assume that $\rho_j, s_j, r_j, E_j, \alpha_j, T_j$ are well-defined for the $j$-th step. Then we define $\eta_j, K_j, \epsilon_j, h_j$ as follows:

$$\eta_j = E_j, \quad \epsilon_j = \alpha_j^{2\bar\nu}\rho_j^{2\nu}E_j\eta_j^2, \tag{3.24}$$

$$\frac{e^{-K_j\rho_j}}{\eta_j^2} = E_j, \quad h_j = \frac{\alpha_j}{(1+K)_j^{\tau+1}T_j}. \tag{3.25}$$

Define the inductive sequences:

$$\rho_{j+1} = \sigma\rho_j, \quad s_{j+1} = s_j - 5\rho, \quad r_{j+1} = \eta_j r_j, \tag{3.26}$$

$$\alpha_{j+1} = \alpha_j - (1+K_j)^{\tau+1}\epsilon_j, \quad E_{j+1} = cE_j^{\frac{4}{3}}, \quad T_{j+1} = T_j + \frac{6\epsilon_j}{d_j}. \tag{3.27}$$

Let

$$\Pi_{j+1} = \Big\{\xi \in \Pi_j : |\langle\omega_{j+1}(\xi), k\rangle - s_1\theta_{j+1i}(\xi) - s_2\theta_{j+1z}(\xi) - 2\pi w| \geqslant \frac{2\alpha_{j+1}}{(|k|+2)^\tau}, \\ K_j < |k| \le K_{j+1},\ 0 \leqslant |s_1| + |s_2| \leqslant 2,\ s_d \in \mathbb{Z},\ d = 1,2\Big\}$$

and

$$\Pi_{j+1_{h_{j+1}}} = \Big\{\xi \in C^n : \mathrm{dist}(\xi, \Pi_{j+1}) \leqslant h_{j+1}\Big\}.$$

The proofs of the following two Lemmas are similar to the idea described in [20, 29]. To make the paper self-contained, we present our proofs in the Appendix.

**Lemma 3.2.** *In view of definitions of parameters in* (3.24)-(3.27)*, we have*

$$h_{j+1} \le \frac{5}{6}h_j, \quad \max_{\xi\in\Pi_{h_{j+1}}}\Big\{|\frac{\partial\omega_{j+1}(\xi)}{\partial\xi}|, |\frac{\partial\theta_{j+1}(\xi)}{\partial\xi}|\Big\} \le T_{j+1}, \tag{3.28}$$

$$T_0 \le T_j \le T_0 + 1, \quad \frac{1}{2}\alpha_j \le \alpha_{j+1} \le \alpha_j. \tag{3.29}$$

**Remark 3.3.** By the KAM iteration theory and Lemma 3.2, the KAM step can iterate infinitely times.

We now provide some useful estimates on the Gevrey-smoothness and convergence of the iteration. Let

$$D_j^\beta = \frac{c\alpha_{j-1}^{\bar\nu}\rho_{j-1}^{\nu}E_{j-1}^3\beta!}{h_j^{|\beta!|}} \quad \text{and} \quad J_j^\beta = \frac{c\epsilon_{j-1}\beta!}{h_j^{|\beta!|}}. \tag{3.30}$$

Then a straightforward calculation can lead to the following result.

**Lemma 3.4.** *If $D_j^\beta$ and $J_j^\beta$ are defined by* (3.30)*, then*

$$D_j^\beta \le c\rho_j^\nu M^{|\beta|}\beta!^\mu E_j^{\frac{9}{4(n+1)}},$$

$$J_j^\beta \le c\rho_j^{2\nu} M^{|\beta|}\beta!^\mu E_j^{\frac{9}{4(n+1)}},$$

*where $M = \frac{2T+1}{\alpha}[\frac{4(\mu-1)(n+1)}{3}]^{\mu-1}, \mu = \tau + \delta$ and $c$ only depends on $n, \alpha$ and $\mu$.*

Using the generating functions $\langle p, q_+\rangle + F_j(p, q_+)$ to define $\{\Psi_j(\cdot;\xi)\}$, the Cauchy estimate gives

$$\|\Psi_j - id\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \frac{c\epsilon_j}{\alpha_j^{\bar\nu}\rho_j^{\nu}},\quad \|D\Psi_j - Id\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \frac{c\epsilon_j}{\alpha_j^{\bar\nu}\rho_j^{\nu+1}}.$$

Let $\Psi^j = \Psi_1\circ\Psi_2\circ\cdots\circ\Psi_j$. Then we have $\{\Phi_{j+1}(\cdot;\xi) = (\Psi^j)^{-1}\circ\Phi_j\circ\Psi^j\}$, generated by $H_{j+1}(\cdot;\xi) = N_{j+1} + P_{j+1}$, where

$$N_{j+1} = \langle x + \omega_{j+1}(\xi), \hat y\rangle + \langle A_{j+1}u, \hat v\rangle + \frac{1}{2}\langle B_{j+1}u, u\rangle + \frac{1}{2}\langle C_{j+1}\hat v, \hat v\rangle$$

with

$$|\omega_{j+1} - \omega_j| \le \epsilon_j,\quad |\theta_{j+1} - \theta_j| \le c\epsilon_j,\quad \forall j \geqslant 1,$$
$$\|X_{P_{j+1}}\|_{r_{j+1};\mathcal{D}(s_{j+1},r_{j+1})\times\Pi_{h_{j+1}}} \le \epsilon_{j+1}.$$

3.3. **Convergence of the KAM iteration.** Following [29, 30, 31], we have

$$\|\Psi^j - \Psi^{j-1}\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le c\alpha_{j-1}^{\bar\nu}\rho_{j-1}^{\nu}E_{j-1}^3,$$
$$\|D(\Psi^j - \Psi^{j-1})\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_{h_j}} \le \alpha_{j-1}^{\bar\nu}\rho_{j-1}^{\nu+1}E_{j-1}^3.$$

By the Cauchy estimate and Lemma 3.4, we have

$$\|\partial_\xi^\beta(\Psi^j - \Psi^{j-1})\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_j} \le \rho_j^{\nu}M^{|\beta|}\beta!^{\mu}E_j^{\frac{9}{4(n+1)}},$$
$$\|\partial_\xi^\beta D(\Psi^j - \Psi^{j-1})\|_{r_j;D(s_j-3\rho_j,r_j)\times\Pi_j} \le \rho_j^{\nu}M^{|\beta|}\beta!^{\mu}E_j^{\frac{9}{4(n+1)}},$$
$$\|\partial_\xi^\beta(\omega_j - \omega_{j-1})\|_{\Pi_j} \le \rho_j^{2\nu}M^{|\beta|}\beta!^{\mu}E_j^{\frac{9}{4(n+1)}},$$
$$\|\partial_\xi^\beta(\theta_j - \theta_{j-1})\|_{\Pi_j} \le \rho_j^{2\nu}M^{|\beta|}\beta!^{\mu}E_j^{\frac{9}{4(n+1)}}.$$

Since $s_j \to s/2$, $r_j \to 0$, and $h_j \to 0$ as $j \to \infty$, we define

$$D_* = D(\frac{s}{2}, 0),\quad \Pi_* = \cap_{j\ge 0}\Pi_j \quad\text{and}\quad \Psi_* = \lim_{j\to\infty}\Psi^j.$$

So we have $\partial_\xi^\beta\Psi^j \to \partial_\xi^\beta\Psi^*$ on $D(\frac{s}{2}, \frac{r}{2})$ and

$$\|\partial_\xi^\beta(\Psi_* - id)\|_{\frac{r}{2};D(\frac{s}{2},\frac{r}{2})\times\Pi_*} \le c\rho_0^{\nu}M^{|\beta|}\beta!^{\mu}E_0^{\frac{9}{4(n+1)}}$$

for $\beta \in Z_n^+$. Thus, we arrive at (2.5).

Let $\omega_* = \lim_{j\to\infty}\omega_j$ and $\theta_* = \lim_{j\to\infty}\theta_j$ We then have

$$|\partial_\xi^\beta(\omega_*(\xi) - \omega(\xi))|_{\Pi_*} \le c\rho_0^{2\nu}M^{|\beta|}\beta!^{\mu}E_0^{\frac{9}{4(n+1)}},$$
$$|\partial_\xi^\beta(\theta_*(\xi) - \theta(\xi))|_{\Pi_*} \le c\rho_0^{2\nu}M^{|\beta|}\beta!^{\mu}E_0^{\frac{9}{4(n+1)}}$$

for all $\beta \in Z_n^+$. Thus, we arrive at the desired result (2.6) and (2.7) Moreover, we have

$$|\langle\omega_*(\xi), k\rangle - s_1\theta_{*i}(\xi) - s_2\theta_{*j}(\xi) - 2\pi w| \ge \frac{\alpha_*}{(2+|k|)^\tau},$$

where $\xi \in \Pi_*$, $0 \ne k \in Z^n$, $0 \leqslant |s_1| + |s_2| \leqslant 2$, $s_d \in \mathbb{Z}$ $(d = 1, 2)$, $\alpha_* = \lim_{j\to\infty}\alpha_j$ with $\frac{\alpha_0}{2} < \alpha_* < \alpha_0$ This implies that (2.8) holds.

3.4. **Estimate of measure.** We consider the measure of the subset $\Pi_*$ such that the small divisor conditions (2.2)-(2.4) hold for all $\omega_j, \theta_j, \alpha_j$ and $j \geq 1$

Recalling (2.1), we know that the frequency $\omega_j(\xi)$ satisfies (2.1). Thus, we can follow the same approach as in [28, 29] to obtain the estimate for $\Pi_*$. Here we omit the details.

## 4. Appendix

*Proof of Lemma 3.2.* From the definitions of $E_j$ and $\rho_j$, we have $E_j \leq (cE_0)^{(4/3)^j}$ Letting $x_j = K_j\rho_j = -\ln E_j^3$, we have

$$\frac{K_{j+1}}{K_j} = \frac{1}{2}\frac{\ln c}{\ln E_j} + \frac{4}{3\sigma}.$$

Let $E_0$ be small enough such that

$$-\frac{\ln c}{\ln E_j} \leq (1-\sigma)\frac{4}{3}.$$

Then we get

$$\frac{4}{3} \leq \frac{K_{j+1}}{K_j} \leq \frac{4}{3\rho}.$$

Moreover, for a sufficiently small $E_0$, we have that $24 < K_j < K_{j+1}$ Then we have

$$\frac{h_{j+1}}{h_j} = \frac{\alpha_{j+1}}{\alpha_j} \cdot \frac{T_j}{T_{j+1}} \cdot \frac{(2+K_j)^\tau}{(2+K_{j+1})^\tau} \leq \frac{5}{6}.$$

Clearly, $h_{j+1} \leq \frac{5}{6}h_j$ and so the assumption $h_+ \leq \frac{5}{6}h$ holds. Suppose that

$$\max_{\xi\in\Pi_{h_j}} \Big\{|\frac{\partial\omega_j(\xi)}{\partial\xi}|, |\frac{\partial\theta_j(\xi)}{\partial\xi}|\} \leq T_j.$$

From (3.27), we know that $T_{j+1} = T_j + \frac{6\epsilon_j}{d_j}$. Since $h_{j+1} \leq \frac{5}{6}h_j$ and $|\omega_{j+1} - \omega_j| \leqslant \epsilon$, we have

$$\begin{aligned}
|\frac{\partial\omega_{j+1}}{\partial\xi}| &= |\frac{\partial(\omega_{j+1} - \omega_j + \omega_j)}{\partial\xi}| \\
&\leq |\frac{\partial(\omega_{j+1} - \omega_j)}{\partial\xi}| + |\frac{\partial\omega_j}{\partial\xi}| \leq T_{j+1}
\end{aligned}$$

and similarly,

$$|\frac{\partial\theta_{j+1}}{\partial\xi}| \leqslant T_{j+1}.$$

Consequently, by mathematical induction we obtain the desired result (3.28).

From the definitions of $T_j, h_j$ and $\epsilon_j$, we have

$$\begin{aligned}
T_{j+1} &= T_j + \frac{6\epsilon_j}{d_j} \\
&= T_0 + \sum_{i=0}^{j}\frac{6\epsilon_i}{h_i} \\
&= T_0 + 6\sum_{i=0}^{j}(x_i)^{2\nu}e^{-x_i}T_i.
\end{aligned}$$

Let $E_0$ be sufficiently small such that

$$\sum_{i=0}^{j}(x_i)^{2\nu}e^{-x_i}T_i \le \frac{1}{6},$$

then we have $T_0 \le T_j \le T_0+1$.

Note that $\alpha_j^{2\bar{\nu}} \leqslant \alpha_j$ and $(2+K_j)^{\tau+1} \leqslant (3K_j)^{2\nu}$. Then we have

$$\begin{gathered}\alpha_j^{2\bar{\nu}}\rho_j^{2\nu}(2+K_j)^{2\nu}E_j^3 \leqslant \alpha_j(3\rho_jK_j)^{2\nu}E_j^3 \leqslant \alpha_j(3x_j)^{2\nu}e^{-x_j},\\ \alpha_{j+1}=\alpha_j-(2+K_j)^{\tau+1}\epsilon_j \geqslant \alpha_j\left(1-(3x_j)^{2\nu}e^{-x_j}\right).\end{gathered}$$

If $E_0$ is sufficiently small, then it gives

$$\prod_{j=1}^{\infty}(1-(3x_j)^{2\nu}e^{-x_j}) = 1-O(x_0^{-1}) > \frac{1}{2}.$$

Thus, we obtain

$$\frac{1}{2}\alpha_j \le \alpha_{j+1} \le \alpha_j.$$

□

*Proof of Lemma 3.4.* By the choices of parameters, we have

$$\begin{aligned}\rho_{j+1}x_{j+1}^{\frac{\delta}{\tau+1}} &= \rho_{j+1}K_{j+1}^{\frac{\delta}{\tau+1}}\rho_{j+1}^{\frac{\delta}{\tau+1}}\\ &\geqslant \Big(\frac{4}{3}\Big)^{\frac{\delta}{\tau+1}}\sigma^{\frac{\delta+\tau+1}{\tau+1}}\rho_j\rho_j^{\frac{\delta}{\tau+1}}K_j^{\frac{\delta}{\tau+1}}\\ &= \Big(\frac{4}{3}\Big)^{\frac{\delta}{\tau+1}}\sigma^{\frac{\delta+\tau+1}{\tau+1}}\rho_jx_j^{\frac{\delta}{\tau+1}}.\end{aligned}$$

Choosing $\sigma=(\frac{3}{4})^{\frac{\delta}{\delta+\sigma+1}}$, we get $\left(\frac{4}{3}\right)^{\frac{\delta}{\tau+1}}\sigma^{\frac{\delta+\tau+1}{\tau+1}} \geqslant 1$ Since $\rho_0x_0^{\frac{\delta}{\tau+1}} \ge 1$, we have $\rho_jx_j^{\frac{\delta}{\tau+1}} \ge 1$ for all $j \geqslant 1$, and hence $\frac{1}{\rho_j} \le x_j^{\frac{\delta}{\tau+1}}$ So we have

$$K_j = \frac{x_j}{\rho_j} \le x_j^{1+\frac{\delta}{\tau+1}},$$

which implies that $K_j^{\tau+1} \le x_j^{\tau+1+\delta}$ In view of $h_j=\frac{\alpha_j}{(K+2)_j^{\tau+1}T_j}$, $T_j<T+1$, $\frac{1}{2}\alpha \le \alpha_j$ and $E_{j-1}=E_j^{\frac{3}{4}}=e^{-\frac{x_j}{4}}$, we have

$$\begin{aligned}D_j^{\beta} &\le c\alpha^{\bar{\nu}}\rho_j^{\nu}\beta!\Big(\frac{T+1}{\frac{\alpha}{2}}\Big)^{|\beta|}\left(x_j^{\tau+1+\delta}\right)^{|\beta|}e^{-3x_j/4}\\ &\le c\rho_j^{\nu}\Big(\frac{2(T+1)}{\alpha}\Big)^{|\beta|}\beta!e^{-\frac{3x_j}{4}\frac{1}{n+1}}\left[x_j^{\beta_1}e^{-\frac{3x_j}{4}\frac{1}{(\tau+\delta)(n+1)}}\dots x_j^{\beta_n}e^{-\frac{3x_j}{4}\frac{1}{(\tau+\delta)(n+1)}}\right]^{\tau+\delta}\\ &\le c\rho_j^{\nu}M^{|\beta|}\beta!^{\mu}E_j^{\frac{9}{4(n+1)}},\end{aligned}$$

where $M=\frac{2T+1}{\alpha}[\frac{4(\mu-1)(n+1)}{3}]^{\mu-1}$, $\mu=\tau+\delta$ and $c$ only depends on $n,\alpha$ and $\mu$.

In an analogous manner, we can derive

$$J_j^{\beta} \le c\rho_j^{2\nu}M^{|\beta|}\beta!^{\mu}E_j^{\frac{9}{4(n+1)}}.$$

□

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## References

[1] V. Arnold; *Proof of A.N. Kolmogorov's theorem on conservation of conditionally periodic motions under small perturbations of the hamiltonian function*, Uspeki Matematicheskii Nauk, (18) 1963, 13–40.

[2] A. D. Bruno; *Analytic form of differential equations*, Trudy Moskovskogo Matematicheskogo Obshchestva, 26 (1972), 199-239.

[3] Q.-Y. Bi, J.-X. Xun; *Persistence of Lower Dimensional Hyperbolic Invariant Tori for Nearly Integrable Symplectic Mappings*, Qualitative Theory of Dynamical Systems, 13 (2014), 269-288.

[4] C.-Q. Cheng, Y.-S. Sun; *Existence of invariant tori in three-dimensional measure-preserving mappings*, Celestial Mechanics and Dynamical Astronomy, 47 (1989), 275-292.

[5] R. de la Llave, J. M. James; *Parameterization of invariant manifolds by reducibility for volume preserving and symplectic maps*, Discrete Contin. Dynam. Systems, 32 (2012), 4321-4360.

[6] A. N. Kolmogorov; *On preservation of conditionally periodic motions under a small change in the Hamiltonian function*, Dokl. Akad. Nauk SSSR, 98 (1954), 527-530.

[7] X.-Z. Lu, J. Li, J.-X. Xu; *A KAM Theorem for a Class of Nearly Integrable Symplectic Mappings*, Journal of Dynamics and Differential Equations, (2015), 1-24.

[8] H. R. Dullin, J. D. Meiss; *Resonances and twist in volume-preserving mappings*, SIAM Journal on Applied Dynamical Systems, 11 (2012), 319-349.

[9] A. M. Fox, J. D. Meiss; *Greene's residue criterion for the breakup of invariant tori of volume-preserving maps*, Physica D: Nonlinear Phenomena, 243(2013), 45-63.

[10] V. Gelfreich, C. Simó, A. Vieiro; *Dynamics of 4D symplectic maps near a double resonance*, Physica D: Nonlinear Phenomena, 243 (2013), 92-110.

[11] S. M. Graff; *On the conservation of hyperbolic invariant tori for Hamiltonian systems*, Journal of Differential Equations, 15 (1974), 1-69.

[12] V. K. Melnikov; *On some cases of conservation of conditionally periodic motions under a small change of the Hamiltonian function*, Soviet Math. Doklady, 6 (1965), 1592-1596.

[13] V. K. Melnikov; *A family of conditionally periodic solutions of a Hamiltonian system*, Soviet Math. Doklady, 9(1968), 822-886.

[14] J. Moser; *On invariant curves of area-preserving mappings of an annulus*, Vandenhoeck & Ruprecht, 1962.

[15] J. Moser; *Convergent series expansions for quasi-periodic motions*, Mathematische Annalen, 169 (1967), 136-176.

[16] H. Poincaré, R. Magini; *Les méthodes nouvelles de la mcanique céleste*, Il Nuovo Cimento (1895-1900), 1 0(1899), 128-130.

[17] H. Poincaré; *Sur un théoreme de géométrie*, Rendiconti del Circolo Matematico di Palermo (1884-1940), 33(1912), 375-407.

[18] J. Pöschel; *Integrability of Hamiltonian systems on Cantor sets*, Communications on Pure and Applied Mathematics, 35(1982), 653-696.

[19] J. Pöschel; *On elliptic lower dimensional tori in Hamiltonian systems*, Mathematische Zeitschrift, 202(1989), 559-608.

[20] G. Popov; *Invariant Tori, Effective Stability, and Quasimodes with Exponentially Small Error Terms I Birkhoff Normal Forms*, Annales Henri Poincaré, Birkhäuser Verlag, 1(2000), 223-248.

[21] G. Popov; *KAM theorem for Gevrey Hamiltonians*, Ergodic Theory and Dynamical Systems, 24 (2004), 1753-1786.

[22] H. Rüssmann; *On the existence of invariant curves of twist mappings of an annulus*, Geometric dynamics, Springer Berlin Heidelberg, 677-718 (1983).

[23] H. Rüssmann; *On twist Hamiltonian*, Talk on the Colloque International, Mécanique céleste et systemes hamiltoniens, Marseille, 1990.

[24] Z.-J. Shang; *A note on the KAM theorem for symplectic mappings*, Journal of Dynamics and Differential Equations, 12(2000), 357-383.

[25] S.-P. Wang, D.-F. Zhang, J.-X. Xu; *On the persistence of elliptic lower-dimensional tori in Hamiltonian systems under the first Melnikov condition and Rüssmann's non-degeneracy condition*, Nonlinear Analysis: Theory, Methods Applications, 66(2007), 1675-1685.

[26] H. Whitney; *Analytic extensions of differentiable functions defined in closed sets*, Transactions of the American Mathematical Society, 36 (1934), 63-89.

[27] Z.-H. Xia; *Existence of invariant tori in volume-preserving diffeomorphisms*, Ergodic Theory and Dynamical Systems, 12 (1992), 621-631.

[28] J.-X. Xu, J.-G. You, Q.-J.Qiu; *Invariant tori for nearly integrable Hamiltonian systems with degeneracy*, Mathematische Zeitschrift, 226 (1997), 375-387.

[29] J.-X. Xu, J.-G. You; *Gevrey-smoothness of invariant tori for analytic nearly integrable Hamiltonian systems under Rüssmann's non-degeneracy condition*, Journal of Differential Equations, 235 (2007), 609-622.

[30] D.-F. Zhang, J.-X. Xu; *On elliptic lower dimensional tori for Gevrey-smooth Hamiltonian systems under Rüssmann's non-degeneracy condition*, Discrete Contin. Dyn. Syst. Ser. A, 16 (2006), 635-655.

[31] D.-F. Zhang, J.-X. Xu; *Gevrey-smoothness of elliptic lower-dimensional invariant tori in Hamiltonian systems under Rüssmanns non-degeneracy condition*, J. Math. Anal. Appl. 322(2006), 293-312

[32] W. Zhu, B. Liu, Z. Liu; *The hyperbolic invariant tori of symplectic mappings*. Nonlinear Analysis: Theory, Methods Applications, 68(2008), 109-126.

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