Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 13, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

WELL-POSEDNESS OF WEAK SOLUTIONS TO ELECTRORHEOLOGICAL FLUID EQUATIONS WITH DEGENERACY ON THE BOUNDARY

HUASHUI ZHAN, JIE WEN

Communicated by Zhaosheng Feng

Abstract. In this article we study the electrorheological fluid equation

$$u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p(x)-2}\nabla u),$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance from the boundary, $p(x) \in C^1(\overline{\Omega})$, and $p^- = \min_{x\in\overline{\Omega}} p(x) > 1$. We show how the degeneracy of ρ^α on the boundary affects the well-posedness of the weak solutions. In particular, the local stability of the weak solutions is established without any boundary value condition.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, and $p(x)$ is a measurable function. The evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

comes from a new interesting type of fluids called electrorheological fluids [1, 10]. We consider an electromagnetic field with vector of magnetic density $\vec{B} = (0, 0, u(x,t))$, where $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$. Let $\vec{H} = (H_1, H_2, H_3)$ be a magnetic field intensity, $\vec{J} = (J_1, J_2, J_3)$ be a current density, $\vec{E} = (E_1, E_2, E_3)$ be an electrostatic field intensity and r be a resistivity. Review Maxwell's equations

$$\frac{\partial\vec{B}}{\partial t} + \operatorname{rot}\vec{E} = 0, \tag{1.2}$$

$$\vec{J} \approx \operatorname{rot}\vec{H}, \tag{1.3}$$

$$\vec{B} = \lambda\vec{H}, \tag{1.4}$$

$$\vec{E} = r\vec{J}, \tag{1.5}$$

2010 *Mathematics Subject Classification.* 35L65, 35K85, 35R35.
Key words and phrases. Electrorheological fluid equation; boundary degeneracy; Hölder's inequality; local stability.

Submitted August 12, 2016. Published January 12, 2017.

where $\lambda > 0$. By (1.4), we have $H_3 = u/\lambda$. Therefore,

$$\begin{aligned} J_1 &\approx \frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} = \frac{1}{\lambda}\frac{\partial u}{\partial x_2}, \\ J_2 &\approx \frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1} = -\frac{1}{\lambda}\frac{\partial u}{\partial x_1}, \\ J_3 &\approx \frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} = 0. \end{aligned} \tag{1.6}$$

Now, if we suppose that $r = r_0|\vec{J}|^{q(x)}$, taking into account (1.6) we know that

$$|\vec{J}| = \sqrt{J_1^2 + J_2^2 + J_3^2} = \frac{1}{\lambda}|\nabla u|,$$

where $r_0 > 0$ is a constant, $q(x)$ is a function which depends on the environment. If

$$a(x) = \frac{r_0}{\lambda^{q(x)+1}} > 0, \tag{1.7}$$

then

$$\vec{E} = r\vec{J} = a(x)|\nabla u|^{q(x)}(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}, 0),$$

as a generalization of Ohm's law (1.5). Hence the third coordinate of the vector $\operatorname{rot}\vec{E}$ is

$$\begin{aligned} (\operatorname{rot}\vec{E})_3 &= \frac{\partial E_2}{\partial x_1} - \frac{\partial E_1}{\partial x_2} \\ &= \frac{\partial}{\partial x_1}(-a(x)|\nabla u|^{q(x)}\frac{\partial u}{\partial x_1}) - \frac{\partial}{\partial x_2}(a(x)|\nabla u|^{q(x)}\frac{\partial u}{\partial x_2}) \\ &= -\operatorname{div}(a(x)|\nabla u|^{q(x)}\nabla u). \end{aligned}$$

Using (1.2), according to Mashiyev-Buhrii [9], letting $p(x) = q(x) + 2$, we have

$$u_t - \operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) = 0, \quad (x,t) \in Q_T, \tag{1.8}$$

with the initial value

$$u\big|_{t=0} = u_0(x), \quad x \in \Omega, \tag{1.9}$$

and the homogeneous boundary value

$$u\big|_{\Gamma_T} = 0, \quad (x,t) \in \Gamma_T = \partial\Omega \times (0,T), \tag{1.10}$$

which have been researched widely recently, one can refer to [2, 4, 8, 6].

If $r_0 = r(x)$ is a function, then $a(x)$ in (1.7) may be degenerate on the boundary. For example, if $r(x)\big|_{\Sigma_p} = 0$, where $\Sigma_p \subseteq \partial\Omega$, then the equation is degenerate on Σ_p. We will study the problem by taking a special but basic formula of the diffusion function $a(x) = \rho^\alpha(x)$, where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, and $\alpha > 0$. Then equation (1.8) becomes

$$u_t = \operatorname{div}(\rho^\alpha|\nabla u|^{p(x)-2}\nabla u), \quad (x,t) \in \Omega \times (0,T). \tag{1.11}$$

If $p(x) \equiv p$, the above equation becomes

$$u_t = \operatorname{div}(|\rho^\alpha\nabla u|^{p-2}\nabla u), \tag{1.12}$$

which was first studied by Yin-Wang [13]. They had proved the following results:

Theorem 1.1. *Let $p > 1$, and*

$$u_0 \in L^\infty(\Omega), \quad \rho^\alpha |\nabla u_0|^p \in L^1(\Omega). \tag{1.13}$$

If $\alpha < p - 1$, then there exists an unique solution of equation (1.12) *with the initial-boundary conditions* (1.9)-(1.10). *While, if $\alpha \geq p-1$, there exists an unique solution of* (1.12) *only with the initial value* (1.9). *In other words, if $\alpha \geq p - 1$, the stability of the solutions of* (1.12) *is true without any boundary condition.*

Inspired by [13], we studied (1.11) in a similar way as the one described in [15], and obtain a similar theorem.

Theorem 1.2. *Let $p > 1$, and*

$$u_0 \in L^\infty(\Omega), \quad \rho^\alpha |\nabla u_0|^{p(x)} \in L^1(\Omega). \tag{1.14}$$

If $\alpha < p^- - 1$, then there exists an unique solution of (1.11) *with the initial-boundary conditions* (1.9)-(1.10). *While, if $\alpha \geq p^+ - 1$, then there exists an unique solution of equation* (1.11) *with the initial value* (1.9).

We are interested in this problem because we would like to know how the degeneracy of the diffusion function ρ^α affects equation (1.11) essentially. To see that, we suppose that u and v are two classical solutions of (1.11) with the initial values $u(x,0)$ and $v(x,0)$ respectively. Then

$$\begin{aligned}
&\int_\Omega (u-v)(u-v)_t dx + \int_\Omega \rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \nabla(u-v)\, dx \\
&= \int_{\partial\Omega} \rho^\alpha (u-v)(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \vec{n} d\Sigma = 0,
\end{aligned}$$

where $\vec{n}$ is the outer unit normal vector of Ω. So

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u-v)^2 dx \leq 0,$$

$$\int_\Omega |u(x,t) - v(x,t)|^2 dx \leq \int_\Omega |u_0(x) - v_0(x)|^2 dx. \tag{1.15}$$

This implies that the classical solutions (if there are) of equation (1.11) are stable without any boundary value condition, only if that $\alpha > 0$. Certainly, since equation (1.11) is degenerate on the boundary and may be degenerate or singular at points where $|\nabla u| = 0$, it only has a weak solution generally, so whether the inequality (1.15) is true or not remains to be verified.

Obviously, since $p(x)$ is a function, there exists a gap if $p^- - 1 \leq \alpha < p^+ - 1$ in Theorem 1.2. In our paper, roughly speaking, only if $\alpha \geq p^- - 1$, we can establish the stability of the weak solutions of equation (1.11) without any boundary value condition. The conclusions not only make a supplementary of the results of [13, 15, 14], but also provide a new and more effective way to establish the stability of the solutions (see Theorems 2.6 and 2.7 below).

2. Basic functional spaces and main results

Throughout this article we assume that $1 < p(x) \in C^1(\overline{\Omega})$, and denote

$$p^+ = \max_{\bar{\Omega}} p(x), \quad 1 < p^- = \min_{\bar{\Omega}} p(x).$$

First of all, we introduce some basic functional spaces. The space

$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

is equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}}(\Omega) = \inf\{\lambda > 0 : \int_\Omega \big|\frac{u(x)}{\lambda}\big|^{p(x)}dx \le 1\}.$$

The space $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a separable, uniformly convex Banach space.

The space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

is endowed with the norm

$$\|u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We use $W_0^{1,p(x)}(\Omega)$ to denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$. Some properties of the function spaces $W^{1,p(x)}(\Omega)$ are quoted in the following lemma.

Lemma 2.1. (i) *The spaces* $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$, $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) *$p(x)$-Hölder's inequality. Let $q_1(x)$ and $q_2(x)$ be real functions with $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$ and $q_1(x) > 1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. And for any $u \in L^{q_1(x)}(\Omega)$ and $v \in L^{q_2(x)}(\Omega)$, we have*

$$\big|\int_\Omega uvdx\big| \le 2\|u\|_{L^{q_1(x)}(\Omega)}\|v\|_{L^{q_2(x)}(\Omega)}.$$

(iii)

$$\|u\|_{L^{p(x)}(\Omega)} = 1 \implies \int_\Omega |u|^{p(x)}dx = 1,$$

$$\|u\|_{L^{p(x)}}(\Omega) > 1 \implies |u|^{p^-}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}dx \le |u|^{p^+}_{L^{p(x)}},$$

$$\|u\|_{L^{p(x)}}(\Omega) < 1 \implies |u|^{p^+}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}dx \le |u|^{p^-}_{L^{p(x)}}.$$

(iv) *If $p_1(x) \le p_2(x)$, then $L^{p_1(x)}(\Omega) \supset L^{p_2(x)}(\Omega)$.*

(v) *If $p_1(x) \le p_2(x)$, then*

$$W^{1,p_2(x)}(\Omega) \hookrightarrow W^{1,p_1(x)}(\Omega).$$

(vi) *$p(x)$-Poincarés inequality. If $p(x) \in C(\Omega)$, then there is a constant $C > 0$, such that*

$$\|u\|_{L^{p(x)}}(\Omega) \le C\|\nabla u\|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies that $\|\nabla u\|_{L^{p(x)}}(\Omega)$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms of $W_0^{1,p(x)}$.

Zhikov [17] showed that

$$W_0^{1,p(x)}(\Omega) \ne \{v \in W_0^{1,p(x)}(\Omega) | v|_{\partial\Omega} = 0\} = \mathring{W}^{1,p(x)}(\Omega).$$

Hence, the property of the space is different from the case when p is a constant. This fact gives a general idea used in studying the well-posedness of the solutions to the evolutionary p-Laplacian equation which can not be used directly.

If the exponent $p(x)$ is required to satisfy logarithmic Hölder continuity condition

$$|p(x)-p(y)| \le \omega(|x-y|), \quad \forall x,y \in \Omega,\ |x-y|<\frac{1}{2},$$

with

$$\limsup_{s\to 0^+} \omega(s)\ln(\frac{1}{s}) = C < \infty,$$

then $W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega)$. In fact Antontsev-Shmarev [2] established the well-posedness of equation (1.8).

Now, we introduce some other Banach spaces used to define the weak solution of the equation. For every fixed $t \in [0,T]$, we define

$$V_t(\Omega) = \{u(x) : u(x) \in L^2(\Omega)\cap W_0^{1,1}(\Omega), |\nabla u(x)|^{p(x)} \in L^1(\Omega)\},$$
$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{p(x),\Omega},$$

and denote by $V_t'(\Omega)$ its dual, where $\|u\|_{2,\Omega} = \|u\|_{L^2(\Omega)}$, $\|\nabla u\|_{p(x),\Omega} = \|\nabla u\|_{L^{p(x)}(\Omega)}$. Also we use the Banach space

$$W(Q_T) = \{u : [0,T] \to V_t(\Omega) | u \in L^2(Q_T), |\nabla u|^{p(x)} \in L^1(Q_T), u = 0 \text{ on } \Gamma_T\},$$
$$\|u\|_{W(Q_T)} = \|\nabla u\|_{p(x),Q_T} + \|u\|_{2,Q_T}.$$

The space $W'(Q_T)$ is the dual of $W(Q_T)$ (the space of linear functionals over $W(Q_T)$): $w \in W'(Q_T)$ if and only if

$$w = w_0 + \sum_{i=1}^n D_i w_i,\ w_0 \in L^2(Q_T),\ w_i \in L^{p'(x,t)}(Q_T),$$
$$\forall \phi \in W(Q_T), \langle\langle w,\phi\rangle\rangle = \iint_{Q_T} \Big(w_0\phi + \sum_i w_i D_i\phi\Big)\,dx\,dt.$$

The norm in $W'(Q_T)$ is defined by

$$\|v\|_{W'(Q_T)} = \sup\{\langle\langle v,\phi\rangle\rangle : \phi \in W(Q_T), \|\phi\|_{W(Q_T)} \le 1\}.$$

Definition 2.2. A function $u(x,t)$ is said to be a weak solution of (1.11) with the initial value (1.9), if

$$u \in L^\infty(Q_T),\ u_t \in W'(Q_T), \quad \rho^\alpha |\nabla u|^{p(x)} \in L^1(Q_T), \tag{2.1}$$

and for any function $\varphi \in L^\infty(0,T;W_0^{1,p(x)}(\Omega)) \cap W(Q_T)$, it holds

$$\langle\langle u_t,\varphi\rangle\rangle + \iint_{Q_T} (\rho^\alpha |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi)\,dx\,dt = 0. \tag{2.2}$$

The initial value, as usual, is satisfied in the sense of that

$$\lim_{t\to 0}\int_\Omega u(x,t)\phi(x)dx = \int_\Omega u_0(x)\phi(x)dx, \forall \phi(x) \in C_0^\infty(\Omega). \tag{2.3}$$

The main result of this article is stated as follows.

Theorem 2.3. *Let* $1 < p^-$, $0 < \alpha$. *If*

$$u_0(x) \in L^\infty(\Omega), \quad \rho^\alpha |\nabla u_0|^{p(x)} \in L^1(\Omega), \tag{2.4}$$

then (1.11) *with initial value* (1.9) *has a weak solution* u *in the sense of Definition 2.2. If* $\alpha < p^- - 1$, *then* (1.11) *with initial-boundary values* (1.9)-(1.10) *has a weak solution* u. *The boundary value condition* (1.10) *is satisfied in the sense of trace.*

Theorem 2.4. *Let u and v be two weak solutions of equation* (1.11) *with initial values $u(x,0)$ and $v(x,0)$ respectively. If $p^- - 1 > \alpha > 0$,*

$$\int_\Omega \rho^{\alpha-1}|\nabla u|^{p(x)-1}dx < \infty, \quad \int_\Omega \rho^{\alpha-1}|\nabla u|^{p(x)-1}dx < \infty, \tag{2.5}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|dx \le \int_\Omega |u_0(x)-v_0(x)|dx. \tag{2.6}$$

The above theorem is a weaker version of [15, Theorem 1.4] when $\alpha < p^- - 1$. We can use it to prove the following Theorems.

Theorem 2.5. *Let u and v be two weak solutions of equation* (1.11) *with the different initial values $u(x,0), v(x,0)$ respectively, and the exponent $p(x)$ be required to satisfy logarithmic Hölder continuity condition. If $\alpha \ge p^- - 1$, u and v satisfy* (2.5)*, $u_t \in L^2(Q_T)$ and $v_t \in L^2(Q_T)$, then then the stability* (2.6) *is still true.*

Theorem 2.6. *Let $p > 1$ and $0 < \alpha < p^- - 1$. If u and v are two solutions of equation* (1.11) *with the differential initial values $u_0(x)$ and $v_0(x)$ respectively, then there exists a positive constant $\beta \ge \max\{\frac{p^+-\alpha}{p^--1}, 2\}$ such that*

$$\int_\Omega \rho^\beta |u(x,t)-v(x,t)|^2 dx \le c\int_\Omega \rho^\beta |u_0(x)-v_0(x)|^2 dx. \tag{2.7}$$

In particular, for any small enough constant $\delta > 0$, there holds

$$\int_{\Omega_\delta} |u(x,t)-v(x,t)|^2 dx \le c\delta^{-\beta}\int_\Omega |u_0(x)-v_0(x)|^2 dx. \tag{2.8}$$

Here, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \delta\}$, by the arbitrary of δ, we have the uniqueness of the solution. The inequality (2.7) shows the local stability of the solutions.

Theorem 2.7. *Let $p > 1$, $\alpha \ge p^- - 1$, $b_i(s)$ be a Lipschitz function, and the exponent $p(x)$ be required to satisfy logarithmic Hölder continuity condition. If u, v are two solutions of equation* (1.11) *with the different initial values $u_0(x), v_0(x)$ respectively, then the inequality* (2.7) *is true, which implies the uniqueness of the solution.*

The proof of the existence (Theorem 2.3) is quite different from that shown in [13, 15, 14]. There to prove the stability of solutions, the authors used two ways to deal with the cases $\alpha < p^- - 1$ and $\alpha \ge p^+ - 1$. In this article, we adopt a similar method to prove Theorems 2.4 and 2.5, and then develop it to prove Theorems 2.6 and 2.7. The methods used here seem to be more effective, and can be extended to the degenerate parabolic equation related to the $p(x)$-Laplacian directly.

3. Proof of Theorem 2.3

Following [3], we have the following lemma.

Lemma 3.1. *Let $q \ge 1$. If $u_\varepsilon \in L^\infty(0,T;L^2(\Omega)) \cap W(Q_T)$, $\|u_{\varepsilon t}\|_{W'(Q_T)} \le c$, and $\|\nabla(|u_\varepsilon|^{q-1}u_\varepsilon)\|_{p^-,Q_T} \le c$, then there is a subsequence of $\{u_\varepsilon\}$ which is relatively compactness in $L^s(Q_T)$ with $s \in (1,\infty)$.*

To study (1.11), let us consider the associated regularized problem

$$u_{\varepsilon t}-\operatorname{div}(\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon)=0,\quad (x,t)\in Q_T, \tag{3.1}$$

$$u_\varepsilon(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T), \tag{3.2}$$

$$u_\varepsilon(x,0)=u_{0\varepsilon}(x),\quad x\in\Omega. \tag{3.3}$$

where $\rho_\varepsilon=\rho*\delta_\varepsilon+\varepsilon$, $\varepsilon>0$, δ_ε is the usual mollifier, $u_{\varepsilon,0}\in C_0^\infty(\Omega)$ and $\rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^{p(x)}\in L^1(\Omega)$ is uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W_0^{1,p(x)}(\Omega)$. It is well-known that the above problem has a unique classical solution [5, 11].

Lemma 3.2. *There is a subsequence of u_ε (we still denote it as u_ε), which converges to a weak solution u of equation* (1.11) *with the initial value* (1.9).

Proof. By the maximum principle, there is a constant c dependent on $\|u_0\|_{L^\infty(\Omega)}$ and independent on ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)}\leqslant c. \tag{3.4}$$

Multiplying (2.1) by u_ε and integrating it over Q_T, we have

$$\frac12\int_\Omega u_\varepsilon^2dx+\iint_{Q_T}\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}|\nabla u_\varepsilon|^2\,dx\,dt=\frac12\int_\Omega u_0^2dx\le c. \tag{3.5}$$

For small enough $\lambda>0$, let $\Omega_\lambda=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\lambda\}$. Since $p^->1$, by (3.5) we have

$$\int_0^T\int_{\Omega_\lambda}|\nabla u_\varepsilon|\,dx\,dt\le c\Big(\int_0^T\int_{\Omega_\lambda}|\nabla u_\varepsilon|^{p^-}\,dx\,dt\Big)^{1/p^-}\le c(\lambda). \tag{3.6}$$

Now, for any $v\in W(Q_T)$, $\|v\|_{W(Q_T)}=1$, and

$$\langle u_{\varepsilon t},v\rangle=-\iint_{Q_T}\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\cdot\nabla v\,dx\,dt,$$

by Young's inequality, we can show that

$$|\langle u_{\varepsilon t},v\rangle|\le c\Big[\iint_{Q_T}\rho_\varepsilon^\alpha|\nabla u_\varepsilon|^{p(x)}\,dx\,dt+\iint_{Q_T}(|v|^{p(x)}+|\nabla v|^{p(x)})\,dx\,dt\Big]\le c,$$

then

$$\|u_{\varepsilon t}\|_{W'(Q_T)}\le c. \tag{3.7}$$

Now, let $\varphi\in C_0^1(\Omega)$, $0\le\varphi\le1$ such that $\varphi|_{\Omega_{2\lambda}}=1$ and $\varphi|_{\Omega\setminus\Omega_\lambda}=0$. Then

$$|\langle(\varphi u_\varepsilon)_t,v\rangle|=|\langle\varphi u_{\varepsilon t},v\rangle|\le|\langle u_{\varepsilon t},v\rangle|;$$

so we have

$$\|(\varphi(x)u)_{\varepsilon t}\|_{W'(Q_T)}\le\|u_{\varepsilon t}\|_{W'(Q_T)}\le c. \tag{3.8}$$

By (3.6),

$$\iint_{Q_T}|\nabla(\varphi u_\varepsilon)|^{p^-}\,dx\,dt\le c(\lambda)(1+\int_0^T\int_{\Omega_\lambda}|\nabla u_\varepsilon|^{p^-}\,dx\,dt)\le c(\lambda), \tag{3.9}$$

and so

$$\|\nabla(|\varphi u_\varepsilon)\|_{p^-,Q_T}\le c(\lambda). \tag{3.10}$$

By Lemma 3.1, φu_ε is relatively compactness in $L^s(Q_T)$ with $s\in(1,\infty)$. Then $\varphi u_\varepsilon\to\varphi u$ a.e. in Q_T. In particular, by the arbitraries of λ, it follows that $u_\varepsilon\to u$ a.e. in Q_T.

Hence, by (3.4), (3.5), (3.7), there exists a function u and the n-dimensional vector function $\overrightarrow{\zeta} = (\zeta_1, \cdots, \zeta_n)$ satisfying

$$u \in L^\infty(Q_T),\ u_t \in W'(Q_T), \quad |\overrightarrow{\zeta}| \in L^{\frac{p(x)}{p(x)-1}}(Q_T),$$

and

$$\begin{gathered} u_\varepsilon \rightharpoonup *u, \quad \text{in } L^\infty(Q_T), \\ \nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^{p(x)}_{\rm loc}(Q_T), \\ \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon \rightharpoonup \overrightarrow{\zeta} \quad \text{in } L^{\frac{p(x)}{p(x)-1}}(Q_T). \end{gathered}$$

To prove that u satisfies (1.11), we notice that for any function $\varphi \in C_0^\infty(Q_T)$, we have

$$\iint_{Q_T} [u_{\varepsilon t}\varphi + \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \cdot \nabla\varphi]\, dx\, dt = 0. \tag{3.11}$$

Then

$$\iint_{Q_T} (\frac{\partial u}{\partial t}\varphi + \vec{\zeta}\cdot \nabla\varphi)\, dx\, dt = 0. \tag{3.12}$$

Now, similar to [15, 14], we can prove that

$$\iint_{Q_T} \rho^\alpha |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\, dx\, dt = \iint_{Q_T} \overrightarrow{\zeta} \cdot \nabla\varphi\, dx\, dt \tag{3.13}$$

for any function $\varphi \in C_0^\infty(Q_T)$. Thus u satisfies (1.11).

Similarly, we can prove (1.9) as in [3] in the same manner. The proof is complete. □

Lemma 3.3. *If $\alpha < p^- - 1$, and let u be the solution of equation* (1.11) *with the initial value* (1.9)*, then the trace of u on the boundary $\partial\Omega$ can be defined in the traditional way.*

The above lemma was proved in [15, 14]. Note that Theorem 2.3 is the directly consequence of Lemmas 3.2 and 3.3.

4. Proof of Theorem 2.4

For small $\eta > 0$, let

$$S_\eta(s) = \int_0^s h_\eta(\tau) d\tau, \quad h_\eta(s) = \frac{2}{\eta}\big(1 - \frac{|s|}{\eta}\big)_+. \tag{4.1}$$

Obviously $h_\eta(s) \in C(\mathbb{R})$, and

$$\begin{gathered} h_\eta(s) \geq 0, \quad |sh_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1, \\ \lim_{\eta\to 0} S_\eta(s) = \operatorname{sgn}(s), \quad \lim_{\eta\to 0} sS'_\eta(s) = 0. \end{gathered} \tag{4.2}$$

Proof. If $\alpha < p^- - 1$, by Lemma 3.3, the weak solution of (1.11) can be defined by the trace on the boundary $\partial\Omega$ in the traditional way. Let u and v be two weak solutions of (1.11) with the initial values $u(x,0)$ and $v(x,0)$ respectively.

Let $\beta > 0$ and

$$\phi(x) = \rho^\beta(x). \tag{4.3}$$

Then, we can choose $S_\eta(\phi(u-v))$ as the test function, and find

$$\begin{aligned}
&\int_\Omega S_\eta(\phi(u-v))\frac{\partial(u-v)}{\partial t}dx \\
&+\int_\Omega \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\phi\nabla(u-v)S_\eta'(\phi(u-v))dx \\
&+\int_\Omega \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla\phi(u-v)S_\eta'(\phi(u-v))dx \\
&=0.
\end{aligned}\tag{4.4}$$

Thus, we have

$$\begin{aligned}
\lim_{\eta\to 0}\int_\Omega S_\eta(\phi(u-v))\frac{\partial(u-v)}{\partial t}dx &= \int_\Omega \operatorname{sgn}(\phi(u-v))\frac{\partial(u-v)}{\partial t}dx \\
&= \int_\Omega \operatorname{sgn}(u-v)\frac{\partial(u-v)}{\partial t}dx=\frac{d}{dt}\|u-v\|_1,
\end{aligned}\tag{4.5}$$

$$\int_\Omega \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)S_\eta'(\phi(u-v))\phi(x)dx\ge 0,\tag{4.6}$$

and

$$\begin{aligned}
&\Big|\int_\Omega \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla\phi(u-v)S_\eta'(\phi(u-v))dx\Big| \\
&\le c\int_{\{x:\rho^\beta|u-v|<\eta\}}|\rho^{\alpha-1}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)| \\
&\quad\times|\phi(u-v)S_\eta'(\phi(u-v))|dx,
\end{aligned}\tag{4.7}$$

which tends to 0 as $\eta\to 0$, because of (2.5) and

$$\lim_{\eta\to 0}\phi(u-v)S_\eta'(\phi(u-v))=0.$$

Now, let $\eta\to 0$ in (4.4). Then

$$\frac{d}{dt}\|u-v\|_1\leqslant c\|u-v\|_1.\tag{4.8}$$

This implies

$$\int_\Omega |u(x,t)-v(x,t)|dx\leqslant c(T)\int_\Omega |u_0-v_0|dx.\tag{4.9}$$

The proof is complete. □

5. Proof of Theorem 2.5

Proof. If $\alpha\ge p^- -1$, the weak solution of equation (1.11) lacks the regularity on the boundary, we can not define the trace on $\partial\Omega$. Denote

$$\Omega_\lambda=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\lambda\},\tag{5.1}$$

let $\beta>0$ and

$$\phi(x)=[\operatorname{dist}((x,\Omega\setminus\Omega_\lambda)]^\beta=d_\lambda^\beta.\tag{5.2}$$

Let u and v be two weak solutions of equation (1.11) with the initial values $u(x,0)$ and $v(x,0)$ respectively. We can choose $S_\eta(\phi(u_\varepsilon-v_\varepsilon))$ as the test function, where u_ε and v_ε are the mollified function of the solutions u and v respectively. Then

$$\begin{aligned} &\int_{\Omega_\lambda} S_\eta(\phi(u_\varepsilon - v_\varepsilon))\frac{\partial(u-v)}{\partial t}dx \\ &+\int_{\Omega_\lambda}\rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot\phi\nabla(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon))dx \\ &+\int_{\Omega_\lambda}\rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot\nabla\phi(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon))dx \\ &= 0. \end{aligned} \tag{5.3}$$

For any given $\lambda > 0$, denoting $Q_{T\lambda} = \Omega_\lambda \times (0,T)$, by (iii) of Lemma 2.1 and (2.1) in Definition 2.2, we know that $|\nabla u| \in L^{p(x)}(Q_{T\lambda})$, $|\nabla v|^{p(x)} \in L^{p(x)}(Q_{T\lambda})$. Thus according to the definition of the mollified functions u_ε and v_ε, the exponent $p(x)$ is required to satisfy the logarithmic Hölder continuity condition, by [4, 17, 8], we have

$$u_\varepsilon \in L^\infty(Q_T), \quad v_\varepsilon \in L^\infty(Q_T), \quad u_\varepsilon \to u, v_\varepsilon \to v, \quad \text{a.e. in } Q_T, \tag{5.4}$$

$$\begin{gathered} \||\nabla u_\varepsilon|^{p(x)}\|_{1,\Omega_\lambda} \le \||\nabla u|^{p(x)}\|_{1,\Omega_\lambda}, \quad \||\nabla v_\varepsilon|^{p(x)}\|_{1,\Omega_\lambda} \le \||\nabla v|^{p(x)}\|_{1,\Omega_\lambda}, \\ \nabla u_\varepsilon \to \nabla u, \quad \nabla v_\varepsilon \to \nabla v, \quad \text{in } L^{p(x)}(\Omega_\lambda). \end{gathered} \tag{5.5}$$

Since $0 \le S'_\eta(\phi(u_\varepsilon - v_\varepsilon)) \le \frac{2}{\eta}$, it follows that

$$|\nabla(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon))|_{L^{p(x)}(\Omega_\lambda)} \le c(\eta)|\nabla(u_\varepsilon - v_\varepsilon)|_{L^{p(x)}(\Omega_\lambda)} \le c(\eta),$$

For any $\varphi \in L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)$, it holds

$$\begin{aligned} &\int_{\Omega_\lambda}\nabla(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon))\varphi\,dx - \int_{\Omega_\lambda}\nabla(u-v)S'_\eta(\phi(u-v))\varphi\,dx \\ &= \int_{\Omega_\lambda}\nabla(u_\varepsilon - v_\varepsilon)[S'_\eta(\phi(u_\varepsilon - v_\varepsilon)) - S'_\eta(\phi(u-v))]\varphi\,dx \\ &\quad + \int_{\Omega_\lambda}[\nabla(u_\varepsilon - v_\varepsilon) - \nabla(u-v)]S'_\eta(\phi(u-v))\varphi\,dx \\ &= I_1 + I_2. \end{aligned} \tag{5.6}$$

Since $\nabla u_\varepsilon \to \nabla u$ and $\nabla v_\varepsilon \to \nabla v$ in $L^{p(x)}(\Omega_\lambda)$, it follows that

$$\lim_{\varepsilon\to 0} I_2 = 0, \tag{5.7}$$

while

$$\begin{aligned} &\lim_{\varepsilon\to 0} I_1 \\ &\le \lim_{\varepsilon\to 0}\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^{p(x)}(\Omega_\lambda)}\|[S'_\eta(\phi(u_\varepsilon - v_\varepsilon)) - S'_\eta(\phi(u-v))]\varphi\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)} \\ &\le \lim_{\varepsilon\to 0}\|\nabla(u-v)\|_{L^{p(x)}(\Omega_\lambda)}\|[S'_\eta(\phi(u_\varepsilon - v_\varepsilon)) - S'_\eta(\phi(u-v))]\varphi\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)} \\ &= 0, \end{aligned} \tag{5.8}$$

by the controlled convergent theorem. Thus we have

$$\nabla(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon)) \rightharpoonup \nabla(u-v)S'_\eta(\phi(u-v)), \quad \text{in } L^{p(x)}(\Omega_\lambda). \tag{5.9}$$

Since on Ω_λ, one has

$$|\rho^\alpha \phi(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)| \in L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)$$

by the weak convergency of (5.9) we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega_\lambda} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \phi\nabla(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon))\,dx \\ &= \int_{\Omega_\lambda} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \phi\nabla(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u-v))dx. \end{aligned} \tag{5.10}$$

At the same time, it is clear that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\Omega_\lambda} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \nabla\phi(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u_\varepsilon - v_\varepsilon))\,dx \\ &= \int_{\Omega_\lambda} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\cdot \nabla\phi(u_\varepsilon - v_\varepsilon)S'_\eta(\phi(u-v))dx, \end{aligned} \tag{5.11}$$

by the controlled convergent theorem. Also, since $u_t, v_t \in L^2(Q_T)$, by Hölder's inequality, we have

$$\iint_{Q_T} |\frac{\partial u}{\partial t}|\,dx\,dt < \infty, \quad \iint_{Q_T} |\frac{\partial v}{\partial t}|\,dx\,dt < \infty. \tag{5.12}$$

By the controlled convergent theorem, we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\lambda} S_\eta(\phi(u_\varepsilon - v_\varepsilon))\frac{\partial(u-v)}{\partial t}dx = \int_{\Omega_\lambda} S_\eta(\phi(u-v))\frac{\partial(u-v)}{\partial t}dx. \tag{5.13}$$

Now, we let $\varepsilon \to 0$, and then let $\lambda \to 0$, at last, let $\eta \to 0$ in (5.3). As the proof of (4.5)-(4.9), we arrive at the desired result. □

6. Uniqueness in the case $0 < \alpha < p^- - 1$

Proof. Let u and v be two solutions of equation (1.11) with initial values $u_0(x)$ and $v_0(x)$ respectively. According to the definition of $W(Q_T)$, $L^2(Q_T) \subset W(Q_T)$, when $\varphi \in L^2(Q_T)$, we have

$$\langle\langle (u-v)_t, \varphi\rangle\rangle = \iint_{Q_{\tau s}} \varphi\frac{\partial(u-v)}{\partial t}\,dx\,dt. \tag{6.1}$$

From the definition of the weak solution, we have

$$\iint_{Q_T} \varphi\frac{\partial(u-v)}{\partial t}\,dx\,dt = -\iint_{Q_T} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla\varphi\,dx\,dt, \tag{6.2}$$

for any $\varphi \in L^\infty(0,T;W_0^{1,p(x)}(\Omega)) \cap L^2(Q_T)$. By Lemma 3.3, $\alpha < p^- - 1$, then the trace of u on the boundary $\partial\Omega$ can be defined in the traditional way. For any fixed $\tau, s \in [0,T]$, $\chi_{[\tau,s]}$ is the characteristic function on $[\tau, s]$. Since $\beta \geq 2$, and

$$\chi_{[\tau,s]}(u-v)\rho^\beta \in L^2(Q_T) \cap L^\infty(0,T;W_0^{1,p(x)}(\Omega)) \tag{6.3}$$

we may choose it as a test function in the above equality. Thus, by denoting $Q_{\tau s}=\Omega\times[\tau,s]$, we have

$$\begin{aligned}
&\iint_{Q_{\tau s}}(u-v)\rho^{\beta}\frac{\partial(u-v)}{\partial t}\,dx\,dt\\
&=-\iint_{Q_{\tau s}}\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\nabla[(u-v)\rho^{\beta}]\,dx\,dt\\
&=\iint_{Q_{\tau s}}\rho^{\alpha+\beta}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\nabla(u-v)\,dx\,dt\\
&\quad+\iint_{Q_{\tau s}}\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)(u-v)\nabla\rho^{\beta}\,dx\,dt.
\end{aligned}\tag{6.4}$$

The first term on the right hand side of (6.4) satisfies

$$\iint_{Q_{\tau s}}\rho^{\alpha+\beta}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\nabla(u-v)\,dx\,dt\geq 0.\tag{6.5}$$

The second term on the right hand side of (6.4), by (iii) of Lemma 2.1, satisfies

$$\begin{aligned}
&\Big|\iint_{Q_{\tau s}}(u-v)\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\nabla\rho^{\beta}\,dx\,dt\Big|\\
&\leq\iint_{Q_{\tau s}}|u-v|\rho^{\alpha}(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1})|\nabla\rho^{\beta}|\,dx\,dt\\
&\leq c\int_{\tau}^{s}\|\rho^{\alpha\frac{p(x)-1}{p(x)}}(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1})\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\\
&\quad\times\|\rho^{\frac{\alpha}{p(x)}}|\nabla\rho^{\beta}(u-v)|\|_{L^{p(x)}(\Omega)}dt\\
&\leq c\int_{\tau}^{s}\|\rho^{\alpha\frac{p(x)-1}{p(x)}}(|\nabla u|^{p(x)-1}+|\nabla v|^{p(x)-1})\|_{L^{\frac{p(x)}{p(x)-1}}(\Omega)}\\
&\quad\times\|\rho^{\frac{\alpha}{p(x)}+(\beta-1)}|(u-v)|\|_{L^{p(x)}(\Omega)}dt\\
&\leq c\int_{\tau}^{s}\Big(\int_{\Omega}\rho^{\alpha}(|\nabla u|^{p(x)}+|\nabla v|^{p(x)})dx\Big)^{1/p_1'}\\
&\quad\times\Big(\int_{\Omega}\rho^{\alpha+p(x)(\beta-1)}|u-v|^{p(x)}dx\Big)^{1/p_1}dt\\
&\leq c\int_{\tau}^{s}\Big(\int_{\Omega}\rho^{\alpha+p(x)(\beta-1)}|u-v|^{p(x)}dx\Big)^{1/p_1}dt.
\end{aligned}\tag{6.6}$$

Here, we used that $|\nabla\rho|=1$ almost everywhere, that $p_1=p^{+}$ or p^{-}, and that $p'(x)=\frac{p(x)}{p(x)-1}$, where $p_1'=p'^{+}$ or p'^{-}.

Now, from $\beta\geq\frac{p^{+}-\alpha}{p^{-}-1}$, we have

$$\Big(\int_{\Omega}\rho^{\alpha+p(x)(\beta-1)}|u-v|^{p(x)}dx\Big)^{1/p_1}\leq c\Big(\int_{\Omega_1+\Omega_2}\rho^{\beta}|u-v|^{p(x)}dx\Big)^{1/p_1},\tag{6.7}$$

where $\Omega_1=\{x\in\Omega:p(x)\geq 2\}$, $\Omega_2=\{x\in\Omega:1<p(x)<2\}$. Then

$$\begin{aligned}
\Big(\int_{\Omega_1}\rho^{\beta}|u-v|^{p(x)}dx\Big)^{1/p_1}&\leq c\Big(\int_{\Omega_1}\rho^{\beta}|u-v|^{2}dx\Big)^{1/p_1}\\
&\leq c\Big(\int_{\Omega}\rho^{\beta}|u-v|^{2}dx\Big)^{1/p_1}.
\end{aligned}\tag{6.8}$$

In Ω_2, by Hölder's inequality and (iii) in Lemma 2.1, we obtain

$$\begin{aligned}\Big(\int_{\Omega_2}\rho^\beta|u-v|^{p(x)}dx\Big)^{1/p_1} &\le c\Big(\|\rho^{\frac{p(x)}{2}}|u-v|^{p(x)}\|_{L^{\frac{2}{p(x)}}(\Omega_2)}\Big)^{1/p_1}\\ &\le c\Big(\int_\Omega\rho^\beta|u-v|^2dx\Big)^q.\end{aligned}\tag{6.9}$$

where $q<1$. Also we have

$$\begin{aligned}&\iint_{Q_{\tau s}}(u-v)\rho^\beta\frac{\partial(u-v)}{\partial t}\,dx\,dt\\ &=\int_\Omega\rho^\beta[u(x,s)-v(x,s)]^2dx-\int_\Omega\rho^\beta[u(x,\tau)-v(x,\tau)]^2dx.\end{aligned}\tag{6.10}$$

From (6.4)–(6.10), it follows that

$$\begin{aligned}&\int_\Omega\rho^\beta[u(x,s)-v(x,s)]^2dx-\int_\Omega\rho^\beta[u(x,\tau)-v(x,\tau)]^2dx\\ &\le c\int_\tau^s\Big(\int_\Omega\rho^\beta|u(x,t)-v(x,t)|^2dx\Big)^q dt\\ &\le c\Big(\int_\tau^s\int_\Omega\rho^\beta|u(x,t)-v(x,t)|^2\,dx\,dt\Big)^q,\end{aligned}\tag{6.11}$$

where $q<1$. Let $\kappa(s)=\int_\Omega\rho^\beta[u(x,s)-v(x,s)]^2dx$. Then we deduce

$$\frac{\kappa(s)-\kappa(\tau)}{s-\tau}\le c\frac{\big(\int_\tau^s\kappa(t)dt\big)^q}{s-\tau}.$$

By the L'Hospital Rule,

$$\kappa'(\tau)\le c\lim_{s\to\tau}\frac{\kappa(s)}{\big(\int_\tau^s\kappa(t)dt\big)^{1-q}}=c\lim_{s\to\tau}\frac{\kappa'(s)}{\kappa(s)}\Big(\int_\tau^s\kappa(t)dt\Big)^q=0.\tag{6.12}$$

Thus, because τ is arbitrary, we have

$$\int_\Omega\rho^\beta|u(x,\tau)-v(x,\tau)|^2dx\le\int_\Omega\rho^\beta|u_0-v_0|^2dx.\tag{6.13}$$

The proof is complete. □

7. Uniqueness in the case $\alpha\ge p^- -1$

When $\alpha\ge p^- -1$, let u be a weak solution of equation (1.11) with the initial value (1.9). Generally, we can not define the trace of u on the boundary.

Proof. Let the constant $\beta\ge\max\{\frac{p^+-\alpha}{p^- -1},2\}$. Denote Ω_λ, $Q_{T\lambda}=\Omega_\lambda\times(0,T)$ as (5.1)-(5.4), and let $\xi_\lambda=d_\lambda^\beta$. Let u and v be two solutions of equation (1.11) with the initial values $u_0(x)$ and $v_0(x)$ respectively. We choose $\chi_{[\tau,s]}(u_\varepsilon-v_\varepsilon)\xi_\lambda$ as a test function, where u_ε and v_ε are the mollified function of the solutions u and v respectively. Then

$$\begin{aligned}&\langle\langle(u-v)_t,\chi_{[\tau,s]}(u_\varepsilon-v_\varepsilon)\xi_\lambda\rangle\rangle\\ &=\iint_{Q_{\tau s}}(u_\varepsilon-v_\varepsilon)\xi_\lambda\frac{\partial(u-v)}{\partial t}\,dx\,dt\\ &=-\iint_{Q_{\tau s}}\rho^\alpha(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\nabla[(u_\varepsilon-v_\varepsilon)\xi_\lambda]\,dx\,dt.\end{aligned}\tag{7.1}$$

Now, by the weak convergence of (5.4) and

$$|\rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)| \in L^{\frac{p(x)}{p(x)-1}}(\Omega_\lambda)$$

we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla(u_\varepsilon - v_\varepsilon)\,dx\,dt \\
&= \iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla(u - v)\,dx\,dt.
\end{aligned} \tag{7.2}$$

By (5.4)-(5.5) and the Lebesgue controlled convergence theorem, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(u_\varepsilon - v_\varepsilon)\nabla\xi_\lambda\,dx\,dt \\
&= \iint_{Q_{\tau s}} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(u - v)\nabla\xi_\lambda\,dx\,dt.
\end{aligned} \tag{7.3}$$

So

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla[(u_\varepsilon - v_\varepsilon)\xi_\lambda]\,dx\,dt \\
&= \iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla(u - v)\,dx\,dt \\
&\quad + \iint_{Q_{\tau s}} \rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)(u - v)\nabla\xi_\lambda\,dx\,dt.
\end{aligned} \tag{7.4}$$

At the same time, by Hölder's inequality, similar to (6.6)-(6.8), we have

$$\iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla(u - v)\,dx\,dt \geq 0, \tag{7.5}$$

and

$$\begin{aligned}
&\Big|\iint_{Q_{\tau s}} (u-v)\rho^\alpha(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla\xi_\lambda\,dx\,dt\Big| \\
&\leq c\int_\tau^s \Big(\int_{\Omega_\lambda} \rho^\alpha d_\lambda^{p(x)(\beta-1)}|u-v|^{p(x)}dx\Big)^{1/p_1} dt \\
&\leq c\int_\tau^s \Big(\int_{\Omega} \rho^{\alpha+p(x)(\beta-1)}|u-v|^{p(x)}dx\Big)^{1/p_1} dt \\
&\leq c\Big(\int_\tau^s\int_\Omega |u-v|^2\,dx\,dt\Big)^q,
\end{aligned} \tag{7.6}$$

where $q < 1$. From (5.12), since $(u_\varepsilon - v_\varepsilon)\xi_\lambda \in L^\infty(Q_T)$, we can use the Lebesgue controlled convergence theorem to deduce that

$$\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} (u_\varepsilon - v_\varepsilon)\xi_\lambda \frac{\partial(u-v)}{\partial t}\,dx\,dt = \iint_{Q_{\tau s}} (u - v)\xi_\lambda \frac{\partial(u-v)}{\partial t}\,dx\,dt.$$

Now, after letting $\varepsilon \to 0$ and $\lambda \to 0$ in (7.1), by a similar argument for (6.10)-(6.13), we arrive at the desired result. □

Acknowledgments. This research is supported by the NSF of China 11371297 and NSF of Fujian Province 2015J01592.

References

[1] E. Acerbi, G. Mingione; Regularity results for stationary electrorheological fluids, *Arch. Ration. Mech. Anal.*, **164** (2002), 213-259.

[2] S. Antontsev, S. Shmarev; Anisotropic parabolic equations with variable nonlinearity, *Publ. Mat.*, **53** (2009), 355-399.

[3] S. Antontsev, S. Shmarev; Parabolic equations with double variable nonlinearlities, *Math. and Compu. in Simulation*, **81** (2011), 2018-2032.

[4] X. L. Fan, D. Zhao; On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

[5] L. Gu; Second order parabolic partial differential equations (in chinese), *The Publishing Company of Xiamen University*, 2002.

[6] Y. Kim. L. Wang, C. Zhang; Global bifurcation for a class of degenerate elliptic equations with variables exponents, *J. Math. Anal. Appl.* **371** (2010), 624-637.

[7] S. Lian, W. Gao, H.Yuan, C. Cao; Existence of solutions to an initial Dirichlet problem of evolutional $p(x)$-Laplace equations, *Ann. I. H. Poincare -AN.*, **29** (2012), 377-399.

[8] O. Kovcik, J. Rkosnk; On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math.J.*, **41** (1991), 592-618.

[9] R. A. Mashiyev, O. M. Buhrii; Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity, *J. Math. Anal. Appl.* **377** (2011), 450-463.

[10] M. Ruzicka; Electrorheological Fluids: Modeling and Mathematical Theory, *Lecture Notes in Math.*, vol.1748, Springer, Berlin, 2000.

[11] E. Taylor Michael; Partial differential equations III, *Springer-Verlag*, 1999.

[12] F. Yao; Holder regularity for the general parabolic $p(x,t)$-Laplacian equations, *Nonlinear Differential Equations and Applications.*, DOI 10.1007/s00030-014-0277-y, 2014.

[13] J. Yin, C. Wang; Properties of the boundary flux of a singular diffusion process, *Chinese Annals of Mathematics, Ser.B*, **25** (2004), 175-182.

[14] H. Zhan; The boundary value condition of an evolutionary p(x)-Laplacian equation, *Boundary value problems*, **2015** (112) (2015), DOI 10.1186/s13661-015-0377-6.

[15] H. Zhan, J. Wen; Evolutionary $p(x)$-Laplacian equation free from the limitation of the boundary value, *Electron. J. Diffrential Equ.*, **143** (2016), 1-13.

[16] C. Zhang, S. Zhuo, X. Xue; Global gradient estimates for the parabolic $p(x,t)$-Laplacian equation, *Nonlinear Analysis.*, **105** (2014), 86-101.

[17] V. V. Zhikov; On the density of smooth functions in Sobolev-Orlicz spaces, *Otdel.Mat.Inst.Steklov.(POMI)*, **310** (2004), 67-81, translation in *J. Math. Sci. (NY)*, **132** (2006), 285-294.

Huashui Zhan

School of Applied Mathematics, Xiamen University of Technology, Xiamen, Fujian 361024, China

E-mail address: 2012111007@xmut.edu.cn

Jie Wen

School of Sciences, Jimei University, Xiamen, Fujian 361021, China

E-mail address: 1195103523@qq.com