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ROBUSTNESS OF MEAN-SQUARE EXPONENTIAL DICHOTOMIES FOR LINEAR STOCHASTIC EQUATIONS

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Abstract. We present the notion of mean-square exponential dichotomies for linear stochastic differential equations. We study the robustness of the mean-square exponential dichotomies, in the sense of the existence of a mean-square exponential dichotomy for a given linear stochastic equation persists under sufficiently small linear perturbations. As a special case, we consider mean-square exponential contractions.

1. Introduction

The notion of exponential dichotomies [20] plays an important role in the theory of differential equations and dynamical systems, particularly in what concerns the study of stable and unstable invariant manifolds, and therefore has attracted much attention during the last few decades. We refer to [3, 6, 18, 21] for details related to exponential dichotomies. Exponential dichotomy of stochastic cocycles was first introduced in [22]. Among those results concerning exponential dichotomies, the so-called robustness problem is very important and has a long history. We refer to [4, 5, 6] and the references therein for the study of robustness of exponential dichotomies.

Let I be any interval on $\mathbb{R}$ and $A(t) = (A_{ij}(t))_{n\times n}$, $G(t) = (G_{ij}(t))_{n\times n}$ be Borel-measurable, bounded functions. In this study, we will introduce the notion of mean-square exponential dichotomies for the nonautonomous linear stochastic differential equations (SDEs for short)

$$dx(t) = A(t)x(t)dt + G(t)x(t)\,d\omega(t), \quad t \in I, \tag{1.1}$$

and limit our attention to the robustness, which means that such a mean-square exponential dichotomy persists under sufficiently small linear perturbations. Precisely, we consider the perturbed stochastic differential equation

$$dy(t) = (A(t) + B(t))y(t)dt + (G(t) + H(t))y(t)\,d\omega(t), \tag{1.2}$$

and we prove that (1.2) admits a mean-square exponential dichotomy for any arbitrary small perturbations B, H if the same happens for (1.1). We also explore the continuous dependence with the perturbation of the constants in the notion of

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dichotomies. Note that in (1.2) the perturbations appear in the "drift" as well as in the "volatility" and the proofs of the main results will become more complicated and difficult than those for linear determined equations.

Stochastic differential equations have been studied by many researchers on various problems because SDEs have important applications in many scientific area. We refer the reader to [1, 7, 9, 15, 17, 19] for more information about SDEs. Among those topics, the study of mean-square dynamical behavior of SDEs is an important and interesting one and has attracted many researchers [8, 11, 12, 13, 14]. Mean-square dynamical behavior are essentially deterministic with the stochasticity built into or hidden in the time-dependent state spaces. In [14], Kloeden and Lorenz provided a definition of mean-square random dynamical systems and studied the existence of pullback attractors (we refer to [2] for details on random dynamical systems). In [8, 16], the concept of mean-square almost automorphy for stochastic process is introduced and the existence, uniqueness and asymptotic stability of mean-square almost automorphic solutions of some linear and nonlinear stochastic differential equations are established. In [11], Higham provided a stochastic version of the theta method for mean-square asymptotic stability.

Now we introduce some notation. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a standard filtered probability space, i.e., $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, $\{\mathscr{F}_t\}_{t\geq 0}$ is a filtration with $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets. For a matrix or a vector A, we use A^T to denote its transpose. Let $\omega(t) = (\omega_1(t), \dots \omega_n(t))^T$ be an n-dimensional Brownian motion defined on the space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$ or operator norm. In addition, let $L^2_{\mathscr{F}_s}(\Omega, \mathbb{R}^n)$ denote the family of all $\mathscr{F}_s$-measurable $\mathbb{R}^n$-valued random variables, i.e., $\xi_s : \Omega \to \mathbb{R}^n$ such that $\mathbb{E}\|\xi_s\|^2 < \infty$ for all $s \geq 0$. Let $I^2_{\geq} := \{(t, s) \in I^2 : t \geq s\}$ and $I^2_{\leq} := \{(t, s) \in I^2 : t \leq s\}$.

The rest part of this article is organized as follows. In Section 2, we present the robustness of mean-square exponential contractions, and the robustness of mean-square exponential dichotomies is showed in Section 3.

2. Robustness of mean-square exponential contractions

In this section we consider the robustness of mean-square exponential contractions.

Definition 2.1. We say that (1.1) admits a mean-square exponential contraction if there exist positive constants M and α such that, for any solution $x(t) \in L^2_{\mathscr{F}_t}(\Omega, \mathbb{R}^n)$ of (1.1),

$$\mathbb{E}\|x(t)\|^2 \leq M e^{-\alpha(t-s)} \mathbb{E}\|x(s)\|^2, \quad \forall (t, s) \in I^2_{\geq}. \tag{2.1}$$

Lemma 2.2. *Let $\Phi(t)$ be a fundamental matrix solution of* (1.1). *Then* (1.1) *admits a mean-square exponential contraction if and only if*

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 \leq M e^{-\alpha(t-s)}, \quad \forall (t, s) \in I^2_{\geq}.$$

Proof. From [15, 17] it follows that $\Phi(t)$ of (1.1) is invertible with probability 1 for all $t \in I$. First, we have

$$\mathbb{E}\|x(t)\|^2 = \mathbb{E}[\|\Phi(t)\Phi^{-1}(s)x(s)\|^2] = \mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 \mathbb{E}\|x(s)\|^2,$$

where $\Phi(t)\Phi^{-1}(s)$ and $x(s)$ are independent, and therefore

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 = \frac{\mathbb{E}\|x(t)\|^2}{\mathbb{E}\|x(s)\|^2},$$

where $\mathbb{E}\|x(s)\|^2 \neq 0$; or else (1.1) admits a "trivial" solution due to (2.1), i.e., $\mathbb{E}\|x(t)\|^2 = 0$ for all $(t,s) \in I^2_{\geq}$. Thus we can obtain from (2.1) that

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 \leq Me^{-\alpha(t-s)}, \quad \forall (t,s) \in I^2_{\geq}.$$

the proof of the converse is very similar. □

The following variation of parameters formula will be essential to prove our main result of this section. The corresponding version of the nonlinear perturbation of (1.1) can be found in [17].

Lemma 2.3 ([15, Section 2.4.2])**.** *Let $\Phi(t)$ be a fundamental matrix of* (1.1)*. Then the solution of* (1.2) *is given as*

$$\begin{aligned} y(t) = \Phi(t)\Phi^{-1}(s)y(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)\big[B(\tau) - G(\tau)H(\tau)\big]y(\tau)d\tau \\ + \int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)y(\tau)\,d\omega(\tau), \end{aligned} \tag{2.2}$$

for all $(t,s) \in I^2_{\geq}$.

Theorem 2.4. *Assume that* (1.1) *admits a mean-square exponential contraction with* (2.1)*. Furthermore, assume that $B(t), G(t)$ and $H(t)$ are all Borel-measurable and there exist nonnegative constants β, g, h such that*

$$\|B(t)\| \leq \beta, \quad \|G(t)\| \leq g, \quad \|H(t)\| \leq h, \quad t \in I. \tag{2.3}$$

Then any solution $y(t)$ of (1.2) *satisfies*

$$\mathbb{E}\|y(t)\|^2 \leq 3Me^{(-\alpha+3MK)(t-s)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s) \in I^2_{\geq}, \tag{2.4}$$

where $K = 2\beta^2 + 2g^2h^2 + h^2$. In particular, (1.2) *also admits a mean-square exponential contraction if*

$$K < \frac{\alpha}{3M}. \tag{2.5}$$

Proof. Given any initial value $y(s)$ at time s, using Lemma 2.3, the solution of (1.2) can be expressed as (2.2) with $(t,s) \in I^2_{\geq}$.

Using conditions (2.3), the Hölder's inequality and the elementary inequality

$$\|\sum_{k=1}^m a_k\|^2 \leq m\sum_{k=1}^m \|a_k\|^2 \tag{2.6}$$

one can obtain that

$$\begin{aligned} \|y(t)\|^2 &\leq 3\|\Phi(t)\Phi^{-1}(s)\|^2\|y(s)\|^2 + 3\Big\|\int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)y(\tau)\,d\omega(\tau)\Big\|^2 \\ &\quad + 3\int_s^t \|\Phi(t)\Phi^{-1}(\tau)\|^2\|B(\tau) - G(\tau)H(\tau)\|^2\|y(\tau)\|^2d\tau \\ &\leq 3\|\Phi(t)\Phi^{-1}(s)\|^2\|y(s)\|^2 + 3\Big\|\int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)y(\tau)\,d\omega(\tau)\Big\|^2 \\ &\quad + 6(\beta^2 + g^2h^2)\int_s^t \|\Phi(t)\Phi^{-1}(\tau)\|^2\|y(\tau)\|^2d\tau. \end{aligned} \tag{2.7}$$

By (2.7) and

$$\mathbb{E}\Big[\Big(\int_s^t x(\tau)\,d\omega(\tau)\Big)^2\Big] = \mathbb{E}\Big[\int_s^t x^2(\tau)d\tau\Big], \quad x(\tau) \in L^2_{\mathscr{F}_\tau}(\Omega, \mathbb{R}^n) \text{ for } \tau \in [s,t] \tag{2.8}$$

we have

$$\begin{aligned}\mathbb{E}\|y(t)\|^2 &\leq 3\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2\mathbb{E}\|y(s)\|^2 + 3\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|^2\mathbb{E}\|H(\tau)y(\tau)\|^2 d\tau \\ &\quad + 6(\beta^2 + g^2h^2)\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\ &\leq 3Me^{-\alpha(t-s)}\mathbb{E}\|y(s)\|^2 + 3MK\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|y(\tau)\|^2 d\tau.\end{aligned} \tag{2.9}$$

Let

$$u(t) = e^{\alpha t}\mathbb{E}\|y(t)\|^2, \quad U(t) = 3Mu(s) + 3MK\int_s^t u(\tau)d\tau. \tag{2.10}$$

We can rewrite inequality (2.9) as

$$u(t) \leq U(t), \quad \text{for all } t \geq s.$$

On the other hand, $\frac{d}{dt}U(t) = 3MKu(t)$, and thus,

$$\frac{d}{dt}U(t) \leq 3MKU(t).$$

Integrating the above inequality from s to t and note that $U(s) = 3Mu(s)$, we obtain

$$u(t) \leq U(t) \leq 3Mu(s)e^{3MK(t-s)}, \quad \text{for all } (t,s) \in I^2_{\geq}.$$

Now the inequality (2.4) follows from (2.10) and the proof is complete. □

Remark 2.5. Assume that (1.1) and (1.2) have the same initial condition, that is, $x(s) = y(s)$. By using the Theorem 2.4, for β, g, h being sufficiently small, we have

$$\begin{aligned}\mathbb{E}\|y(t) - x(t)\|^2 &\leq 2K\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\ &\leq 6M^2K\int_s^t e^{-\alpha(t-\tau)}e^{(-\alpha+3MK)(\tau-s)}d\tau \\ &= 6M^2Ke^{-\alpha(t-s)}\frac{e^{3MK(t-s)} - 1}{3MK} \\ &\leq 2Me^{(-\alpha+3MK)(t-s)}.\end{aligned}$$

Thus for each β, g and h with (2.5), we have

$$\lim_{t\to+\infty}\frac{1}{t}\log(\mathbb{E}\|y(t) - x(t)\|^2) = -\alpha + 3MK < 0,$$

which means that the solution of the linear perturbation equation (1.2) is forward asymptotic to the solution of (1.1) in mean-square sense if they have the same initial data.

In the rest of this section, as a special case of (1.2), we consider

$$dy(t) = (A(t) + B(t))y(t)dt + G(t)y(t)\,d\omega(t), \tag{2.11}$$

in which the perturbed term only appears in the "drift". Of course, Theorem 2.4 can be applicable to (2.11). In the following, we will obtain another robustness result for (2.11), in which the constants can be improved slightly. In this case, the results are more similar to those for ordinary differential equations [10].

Lemma 2.6 ([15, Theorem 2.3.1])**.** *Let $\Phi(t)$ be a fundamental matrix of* (1.1)*. Then the matrix $\Phi^{-1}(t)$ is a fundamental matrix solution of the adjoint equation*

$$dx(t) = x(t)[-A(t) + G^2(t)]dt - x(t)G(t)\,d\omega(t), \quad t \in I. \tag{2.12}$$

As a special case of Lemma 2.3, we know that every solution of (2.11) can be written as

$$y(t) = \Phi(t)\big[\Phi^{-1}(s)y(s) + \int_s^t \Phi^{-1}(\tau)B(\tau)y(\tau)d\tau\big], \quad \tau \in I,$$

where $\Phi(t)$ is a fundamental matrix of (1.1).

Theorem 2.7. *Assume that the first inequality in* (2.3) *holds. Then any solution $y(t)$ of* (2.11) *satisfies*

$$\mathbb{E}\|y(t)\|^2 \le 2Me^{(-\alpha+2M\beta^2)(t-s)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s) \in I^2_{\ge}. \tag{2.13}$$

In particular, if $\beta < \sqrt{\alpha/(2M)}$, then (2.11) *also admits a mean-square exponential contraction.*

Proof. Given any initial value $y(s)$ at time s, the solution of (2.11) can be expressed as

$$y(t) = \Phi(t)\Phi^{-1}(s)y(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)B(\tau)y(\tau)d\tau,$$

for all $(t,s) \in I^2_{\ge}$, where $\Phi(t)$ is the fundamental matrix of (1.1).

Using the elementary inequality $\|a+b\|^2 \le 2(\|a\|^2+\|b\|^2)$, the Hölder's inequality, we obtain that

$$\|y(t)\|^2 \le 2\|\Phi(t)\Phi^{-1}(s)\|^2\|y(s)\|^2 + 2\beta^2\int_s^t \|\Phi(t)\Phi^{-1}(\tau)\|^2\|y(\tau)\|^2 d\tau. \tag{2.14}$$

It follows from (2.14) and (2.8) that

$$\begin{aligned} \mathbb{E}\|y(t)\|^2 &\le 2\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2\mathbb{E}\|y(s)\|^2 + 2\beta^2\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\ &\le 2Me^{-\alpha(t-s)}\mathbb{E}\|y(s)\|^2 + 2M\beta^2\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|y(\tau)\|^2 d\tau. \end{aligned} \tag{2.15}$$

Let

$$u(t) = e^{\alpha t}\mathbb{E}\|y(t)\|^2, \quad U(t) = 2Mu(s) + 2M\beta^2\int_s^t u(\tau)d\tau.$$

Then (2.15) can be rewritten as

$$u(t) \le U(t), \quad \text{for all } t \ge s.$$

On the other hand, $\frac{d}{dt}U(t) = 2M\beta^2 u(t)$, and thus

$$\frac{d}{dt}U(t) \le 2M\beta^2 U(t).$$

Integrating the above inequality from s to t and using the relation $U(s) = 2Mu(s)$, we obtain that

$$u(t) \le U(t) \le 2Mu(s)e^{2M\beta^2(t-s)}, \quad \text{for all } (t,s) \in I^2_{\ge},$$

which implies (2.13) and completes the proof.

Note that in Theorem 2.7 we do not impose any condition on $G(t)$.

Example 2.8 (Geometric Brownian motion [1, 17])**.** Consider the equation

$$dx(t) = -ax(t)dt + \sigma x(t)\, d\omega(t), \tag{2.16}$$

with initial data $x(0)$, where a, σ are constants satisfying $a > 0$ and $\sigma^2 < 2a$. Then the solution of (2.16) is given as

$$x(t) = x(0)\exp\big[\big(-a - \frac{\sigma^2}{2}\big)t + \sigma\omega(t)\big].$$

Further, we can obtain

$$\mathbb{E}\|x(t)\|^2 \le e^{(-2a+\sigma^2)(t-s)}\mathbb{E}\|x(s)\|^2$$

with $t \ge s$. Using Theorem 2.4 and Theorem 2.7, we know that

$$dx(t) = (-a+b)x(t)dt + (\sigma+\eta)x(t)\, d\omega(t),$$
$$dx(t) = (-a+b)x(t)dt + \sigma x(t)\, d\omega(t)$$

admits a mean-square exponential contraction if $|b|$ and $|\eta|$ are all sufficiently small.

3. Robustness of mean-square exponential dichotomies

In this section we consider the robustness of mean-square exponential dichotomies. We assume that the phase space $\mathbb{R}^n$ can be split as

$$\mathbb{R}^n = X_1 \oplus X_2,$$

where X_1 is a linear subspace of $\mathbb{R}^n$ and X_2 is the complementary subspace of X_1.

Definition 3.1. We say that (1.1) admit a mean-square exponential dichotomy if there exist positive constants M and α such that, for any solution $x(t)$ with initial data in X_1,

$$\mathbb{E}\|x(t)\|^2 \le Me^{-\alpha(t-s)}\mathbb{E}\|x(s)\|^2, \quad \forall (t,s) \in I^2_{\ge}, \tag{3.1}$$

and for any solution $x(t)$ with initial data in X_2,

$$\mathbb{E}\|x(t)\|^2 \le Me^{-\alpha(s-t)}\mathbb{E}\|x(s)\|^2, \quad \forall (t,s) \in I^2_{\le}. \tag{3.2}$$

The subspaces X_1 and X_2 are called the stable and instable spaces, respectively [19, 22]. Let $P(t)$ be the projections for each $t \in I$ such that

$$\Phi(t)\Phi^{-1}(s)P(s) = P(t)\Phi(t)\Phi^{-1}(s), \quad \forall (t,s) \in I^2_{\ge}$$

and

$$x(t) = \Phi(t)\Phi^{-1}(s)P(s)x(s)$$

for any solution $x(t)$ with initial data in X_1. Thus,

$$\mathbb{E}\|x(t)\|^2 = \mathbb{E}[\|\Phi(t)\Phi^{-1}(s)P(s)x(s)\|^2] = \mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2\mathbb{E}\|x(s)\|^2,$$

which is just

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 = \frac{\mathbb{E}\|x(t)\|^2}{\mathbb{E}\|x(s)\|^2}.$$

Hence from (3.1) we obtain

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 \le Me^{-\alpha(t-s)}, \quad \forall (t,s)\in I^2_{\geq}. \tag{3.3}$$

Similarly, we can obtain

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)Q(s)\|^2 \le Me^{-\alpha(s-t)}, \quad \forall (t,s)\in I^2_{\leq}, \tag{3.4}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$. We will use the estimates (3.1)-(3.2) as well as the equivalent formulation (3.3)-(3.4).

Theorem 3.2. *Assume that* (1.1) *admits a mean-square exponential dichotomy in* I *and*

$$K < \frac{\alpha}{10M}. \tag{3.5}$$

Then (1.2) *also admits a mean-square exponential dichotomy and for any solution* $y(t)$ *with initial data in* X_1*,*

$$\mathbb{E}\|y(t)\|^2 \le M_1 e^{-\sqrt{\alpha(\alpha-10MK)}(t-s)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s)\in I^2_{\geq},$$

and for any solution $y(t)$ *with initial data in* X_2*,*

$$\mathbb{E}\|y(t)\|^2 \le M_1 e^{-\sqrt{\alpha(\alpha-10MK)}(s-t)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s)\in I^2_{\leq},$$

where the positive constant M_1 *is given as*

$$M_1 = \max\Big\{\frac{5M\big(\alpha+\sqrt{\alpha(\alpha-10MK)}\big)}{\alpha+\sqrt{\alpha(\alpha-10MK)}-5MK}, 1\Big\}.$$

Proof. Firstly, we introduce the spaces

$$\mathscr{L}_c := \{\hat{\Phi} : I^2_{\geq} \to L^2_{\mathscr{F}_t}(\Omega,\mathbb{R}^n) : \|\hat{\Phi}\|_c < \infty\},$$

with the norm

$$\|\hat{\Phi}\|_c = \sup\Big\{(\mathbb{E}\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{P}(s)\|^2)^{\frac{1}{2}} : (t,s)\in I^2_{\geq}\Big\},$$

and

$$\mathscr{L}_d := \{\hat{\Phi} : I^2_{\leq} \to L^2_{\mathscr{F}_t}(\Omega,\mathbb{R}^n) : \|\hat{\Phi}\|_d < \infty\},$$

with the norm

$$\|\hat{\Phi}\|_d = \sup\Big\{(\mathbb{E}\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{Q}(s)\|^2)^{\frac{1}{2}} : (t,s)\in I^2_{\leq}\Big\},$$

where $\hat{\Phi}(t)$ is the fundamental matrix solution of (1.2), $\hat{P}(t)$ are projections for each $t\in I$ and $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ is the complementary projection. One can verify that both $(\mathscr{L}_c, \|\cdot\|_c)$ and $(\mathscr{L}_d, \|\cdot\|_d)$ are Banach spaces. $\square$

Next we show several auxiliary results.

Lemma 3.3. *For each* $(t,s)\in I^2_{\geq}$*, it holds*

$$\begin{aligned} y(t) = {}& \Phi(t)\Phi^{-1}(s)P(s)y(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau \\ &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &- \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &- \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau \in \mathscr{L}_c, \end{aligned} \tag{3.6}$$

and for each $(t,s) \in I^2_{\leq}$,

$$\begin{aligned} y(t) = \Phi(t)\Phi^{-1}(s)Q(s)y(s) &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau \\ &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau \in \mathscr{L}_d. \end{aligned} \tag{3.7}$$

Proof. Set

$$\begin{aligned} \hat{\xi}(t) = \Phi^{-1}(s)P(s)y(s) &+ \int_s^t \Phi^{-1}(\tau)P(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau \\ &+ \int_s^t \Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\,d\omega(\tau) - \int_t^\infty \Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &- \int_t^\infty \Phi^{-1}(\tau)Q(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau. \end{aligned}$$

We can obtain

$$d\hat{\xi}(t) = \Phi^{-1}(t)\big[B(t)-G(t)H(t)\big]y(t)dt + \Phi^{-1}(t)H(t)y(t)\,d\omega(t). \tag{3.8}$$

Let $y(t) = \Phi(t)\hat{\xi}(t)$. Using (3.8), Itô product rule and the following fact

$$d\Phi(t) = A(t)\Phi(t)dt + G(t)\Phi(t)\,d\omega(t),$$

we obtain that

$$\begin{aligned} dy(t) &= d\Phi(t)\hat{\xi}(t) + \Phi(t)d\hat{\xi}(t) + G(t)\Phi(t)\Phi^{-1}(t)H(t)y(t)dt \\ &= A(t)y(t)dt + G(t)y(t)\,d\omega(t) + \big[B(t)-G(t)H(t)\big]y(t)dt \\ &\quad + H(t)y(t)\,d\omega(t) + G(t)H(t)y(t)dt \\ &= (A(t)+B(t))y(t)dt + (G(t)+H(t))y(t)\,d\omega(t), \end{aligned}$$

which means that $y(t)$ given by (3.6) is a solution of (1.2). Now we show that $y(t)$ is unique in the space $(\mathscr{L}_c, \|\cdot\|_c)$. Set

$$\begin{aligned} \hat{\mathcal{T}}y(t) = \Phi(t)\Phi^{-1}(s)P(s)y(s) &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau \\ &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &- \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\,d\omega(\tau) \\ &- \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)[B(\tau)-G(\tau)H(\tau)]y(\tau)d\tau. \end{aligned}$$

Using (2.3), the Hölder's inequality and the inequality (2.6), we have

$\|\hat{\mathcal{T}}y(t)\|^2$

$$
\begin{aligned}
&\le 5\|\Phi(t)\Phi^{-1}(s)P(s)\|^2\|y(s)\|^2 + 10(\beta^2+g^2h^2)\int_s^t \|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\|y(\tau)\|^2 d\tau \\
&\quad + 5\Big\|\int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\,d\omega(\tau)\Big\|^2 \\
&\quad + 5\Big\|\int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\,d\omega(\tau)\Big\|^2 \\
&\quad + 10(\beta^2+g^2h^2)\int_t^\infty \|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\|y(\tau)\|^2 d\tau.
\end{aligned}
$$

Using (2.8) we can show that

$$
\begin{aligned}
\mathbb{E}\|\hat{\mathcal{T}}y(t)\|^2 &\le 5\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2\mathbb{E}\|y(s)\|^2 \\
&\quad + 10(\beta^2+g^2h^2)\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\
&\quad + 5\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|H(\tau)y(\tau)\|^2 d\tau \\
&\quad + 5\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\mathbb{E}\|H(\tau)y(\tau)\|^2 d\tau \\
&\quad + 10(\beta^2+g^2h^2)\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\
&\le 5Me^{-\alpha(t-s)}\mathbb{E}\|y(s)\|^2 + 5MK\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|y(\tau)\|^2 d\tau \\
&\quad + 5MK\int_t^\infty e^{-\alpha(\tau-t)}\mathbb{E}\|y(\tau)\|^2 d\tau.
\end{aligned}
$$

Note that $y(t)=\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{P}(s)y(s)$. Hence,

$$
\begin{aligned}
&\mathbb{E}\|\hat{\mathcal{T}}\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{P}(s)\|^2\mathbb{E}\|y(s)\|^2 \\
&\le 5MK\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|\hat{\Phi}(\tau)\hat{\Phi}^{-1}(s)\hat{P}(s)\|^2\mathbb{E}\|y(s)\|^2 d\tau \\
&\quad + 5MK\int_t^\infty e^{-\alpha(\tau-t)}\mathbb{E}\|\hat{\Phi}(\tau)\hat{\Phi}^{-1}(s)\hat{P}(s)\|^2\mathbb{E}\|y(s)\|^2 d\tau + 5Me^{-\alpha(t-s)}\mathbb{E}\|y(s)\|^2.
\end{aligned}
$$

Thus,

$$
\|\hat{\mathcal{T}}\hat{\Phi}\|_c^2 \le 5M + \frac{10MK}{\alpha}\|\hat{\Phi}\|_c^2 < \infty,
$$

and $\hat{\mathcal{T}}:\mathscr{L}_c\to\mathscr{L}_c$ is well-defined. Proceeding in the same procedure above, for any $\hat{\Phi}_1,\hat{\Phi}_2\in\mathscr{L}_c$, we have

$$
\|\hat{\mathcal{T}}\hat{\Phi}_1 - \mathcal{T}\hat{\Phi}_2\|_c \le \sqrt{\frac{10MK}{\alpha}}\|\tilde{\Phi}_1-\tilde{\Phi}_2\|_c.
$$

When condition (3.5) holds, $\hat{\mathcal{T}}$ is a contraction operator. Hence, there exist a unique $\hat{\Phi}\in\mathscr{L}_c$ such that $\hat{\mathcal{T}}\hat{\Phi}=\hat{\Phi}$. In a similar manner, we can also prove (3.7). $\square$

Lemma 3.4. *Let $x(t)$ be a bounded, continuous real-valued function such that*

$$
x(t) \le De^{-\alpha(t-s)}\zeta + \delta D\int_s^t e^{-\alpha(t-\tau)}x(\tau)d\tau + \delta D\int_t^\infty e^{-\alpha(\tau-t)}x(\tau)d\tau, \tag{3.9}
$$

where D, α, δ are all positive constants. If $\alpha > 2\delta D$, then

$$x(t) \le \tilde{K}\zeta e^{-\tilde{\alpha}(t-s)}, \quad (t,s) \in I^2_{\geq},$$

where

$$\tilde{\alpha} = \sqrt{\alpha(\alpha - 2\delta D)}, \quad \tilde{K} = \max\Big\{\frac{D(\alpha+\tilde{\alpha})}{\alpha+\tilde{\alpha}-\delta D}, 1\Big\}.$$

Proof. Let $\tilde{x}(t)$ be any bounded continuous function satisfying the integral equation

$$\tilde{x}(t) = De^{-\alpha(t-s)}\zeta + \delta D\int_s^t e^{-\alpha(t-\tau)}\tilde{x}(\tau)d\tau + \delta D\int_t^\infty e^{-\alpha(\tau-t)}\tilde{x}(\tau)d\tau, \tag{3.10}$$

with the initial condition $x(s) = \tilde{x}(s)$. It is easy to verify that

$$\tilde{x}'(t) = -\alpha De^{-\alpha(t-s)}\zeta - \alpha\delta D\int_s^t e^{-\alpha(t-\tau)}\tilde{x}(\tau)d\tau + \alpha\delta D\int_t^\infty e^{-\alpha(\tau-t)}\tilde{x}(\tau)d\tau,$$

and

$$\begin{aligned}\tilde{x}''(t) &= \alpha^2 De^{-\alpha(t-s)}\zeta + \alpha^2\delta D\int_s^t e^{-\alpha(t-\tau)}\tilde{x}(\tau)d\tau \\ &\quad + \alpha^2\delta D\int_t^\infty e^{-\alpha(\tau-t)}\tilde{x}(\tau)d\tau - 2\alpha\delta D\tilde{x}(t).\end{aligned}$$

Then it is easy to verify that $\tilde{x}(t)$ is a solution of differential equation

$$\tilde{x}'' = \alpha(\alpha - 2\delta D)\tilde{x}.$$

Note that $\alpha > 0$, $\alpha - 2\delta D > 0$ and $\tilde{x}(t)$ is a bounded continuous function, then

$$\tilde{x}(t) = \tilde{x}(s)e^{-\tilde{\alpha}(t-s)}.$$

In addition, setting $t = s$ in (3.10) gives

$$\tilde{x}(s) = D\zeta + \delta D\tilde{x}(s)\int_s^\infty e^{-(\alpha+\tilde{\alpha})(\tau-s)}d\tau.$$

Note that $\alpha + \tilde{\alpha} > 0$, we obtain that

$$\tilde{x}(s) \le \frac{D(\alpha+\tilde{\alpha})}{\alpha+\tilde{\alpha}-\delta D}\zeta.$$

Thus for any $(t,s) \in I^2_{\geq}$, it has

$$\tilde{x}(t) \le \tilde{K}\zeta e^{-\tilde{\alpha}(t-s)}.$$

Set $\Upsilon(t) = x(t) - \tilde{x}(t)$ for $(t,s) \in I^2_{\geq}$. It follows from (3.9) and (3.10) that

$$\Upsilon(t) \le \delta D\int_s^t e^{-\alpha(t-\tau)}\Upsilon(\tau)d\tau + \delta D\int_t^\infty e^{-\alpha(\tau-t)}\Upsilon(\tau)d\tau. \tag{3.11}$$

Let $\Upsilon = \sup\{\Upsilon(t) : (t,s) \in I^2_{\geq}\}$. Then Υ is finite. It follows from (3.11) that

$$\begin{aligned}\Upsilon &\le \delta D\Upsilon\sup_{t\ge s}\int_s^t e^{-\alpha(t-\tau)}d\tau + \delta D\Upsilon\sup_{t\ge s}\int_t^\infty e^{-\alpha(\tau-t)}d\tau \\ &\le \frac{2\delta D}{\alpha}\Upsilon.\end{aligned}$$

Since $\alpha > 2\delta D$, then $\Upsilon \le 0$ and thus $x(t) \le \tilde{x}(t)$ for $(t,s) \in I^2_{\geq}$, which means that

$$x(t) \le \tilde{K}\zeta e^{-\tilde{\alpha}(t-s)}, \quad (t,s) \in I^2_{\geq},$$

and the proof is complete. □

The proof of the following lemma is similar to that of Lemma 3.4, so we omit it.

Lemma 3.5. *Let $y(t)$ be a bounded, continuous real-valued function such that*

$$y(t) \le De^{-\alpha(s-t)}\zeta + \delta D\int_t^s e^{-\alpha(\tau-t)}y(\tau)d\tau + \delta D\int_{-\infty}^t e^{-\alpha(t-\tau)}y(\tau)d\tau,$$

where D, α, δ are all positive constants. If $\alpha > 2\delta D$, then

$$y(t) \le \tilde{K}\zeta e^{-\tilde{\alpha}(s-t)}, \quad (t,s) \in I^2_{\le}.$$

As in the proof for Theorem 3.2, we consider $\hat{\Phi} \in \mathscr{L}_c$. Then it follows from Lemma 3.3 that the unique solution of (1.2) in the space $(\mathscr{L}_c, \|\cdot\|_c)$ is given as (3.6). Then we have

$$\begin{aligned}
\mathbb{E}\|y(t)\|^2 &\le 5\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2\mathbb{E}\|y(s)\|^2 \\
&\quad + 10(\beta^2+g^2h^2)\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\
&\quad + 5\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|H(\tau)y(\tau)\|^2 d\tau \\
&\quad + 5\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\mathbb{E}\|H(\tau)y(\tau)\|^2 d\tau \\
&\quad + 10(\beta^2+g^2h^2)\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\mathbb{E}\|y(\tau)\|^2 d\tau \\
&\le 5Me^{-\alpha(t-s)}\mathbb{E}\|y(s)\|^2 + 5MK\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|y(\tau)\|^2 d\tau \\
&\quad + 5MK\int_t^\infty e^{-\alpha(\tau-t)}\mathbb{E}\|y(\tau)\|^2 d\tau.
\end{aligned} \tag{3.12}$$

Applying Lemma 3.4 to (3.12) and note the condition (3.5), we have

$$\mathbb{E}\|y(t)\|^2 \le M_1 e^{-\sqrt{\alpha(\alpha-10MK)}(t-s)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s) \in I^2_{\ge}.$$

Similarly, consider $\hat{\Phi} \in \mathscr{L}_d$, then from Lemma 3.3 it follows that the unique solution of (1.2) in the space $(\mathscr{L}_d, \|\cdot\|_d)$ is given as (3.7), and we have

$$\mathbb{E}\|y(t)\|^2 \le M_1 e^{-\sqrt{\alpha(\alpha-10MK)}(s-t)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s) \in I^2_{\le}.$$

Now the proof is complete. $\square$

The following theorem is equivalent to Theorem 3.2.

Theorem 3.6. *Assume that* (1.1) *admits a mean-square exponential dichotomy and condition* (3.5) *holds. Then there exist projections $\hat{P}(t)$ and $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ such that*

$$\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{P}(s)\| \le M_1 e^{-\sqrt{\alpha(\alpha-10MK)}(t-s)}, \quad \forall (t,s) \in I^2_{\ge}, \tag{3.13}$$

$$\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{Q}(s)\| \le M_1 e^{-\sqrt{\alpha(\alpha-10MK)}(s-t)}, \quad \forall (t,s) \in I^2_{\le}, \tag{3.14}$$

where $\hat{\Phi}(t)$ is the fundamental matrix solution of (1.2).

The following result is a direct consequence of Theorem 3.2.

Theorem 3.7. *Assume that* (1.1) *admits a mean-square exponential dichotomy in* I *and* $\beta^2 < \frac{\alpha}{20M}$. *Then* (2.11) *also admits a mean-square exponential dichotomy with*

$$\mathbb{E}\|y(t)\|^2 \le M_2 e^{-\sqrt{\alpha(\alpha-20M\beta^2)}(t-s)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s) \in I^2_{\geq},$$

for any solution $y(t)$ *with initial data from* X_1, *and*

$$\mathbb{E}\|y(t)\|^2 \le M_2 e^{-\sqrt{\alpha(\alpha-20M\beta^2)}(s-t)}\mathbb{E}\|y(s)\|^2, \quad \forall (t,s) \in I^2_{\leq},$$

for any solution $y(t)$ *with initial data from* X_2, *where*

$$M_2 = \max\Big\{\frac{5M(\alpha + \sqrt{\alpha(\alpha-20M\beta^2)})}{\alpha + \sqrt{\alpha(\alpha-20M\beta^2)} - 5M\beta^2}, 1\Big\}.$$

Next, we present an example to illustrate the robustness of mean-square exponential dichotomies.

Example 3.8. Consider the equation

$$\begin{gathered} dx(t) = -ax(t)dt + \sigma x(t)\, d\omega(t), \\ dy(t) = ay(t)dt + \sigma y(t)\, d\omega(t), \end{gathered} \tag{3.15}$$

with initial data $(x(0), y(0))$, where a, σ are constants satisfying $a > 0$ and $\sigma^2 < 2a$. Then the solution of (3.15) is given as

$$\begin{gathered} x(t) = x(0)\exp\big[\big(-a - \frac{\sigma^2}{2}\big)t + \sigma\omega(t)\big], \\ y(t) = y(0)\exp\big[\big(a - \frac{\sigma^2}{2}\big)t + \sigma\omega(t)\big]. \end{gathered}$$

It is easy to verify that

$$\begin{gathered} \mathbb{E}\|x(t)\|^2 \le e^{(-2a+\sigma^2)(t-s)}\mathbb{E}\|x(s)\|^2, \quad t \ge s, \\ \mathbb{E}\|y(t)\|^2 \le e^{-(2a+\sigma^2)(s-t)}\mathbb{E}\|y(s)\|^2, \quad s \ge t, \end{gathered}$$

and therefore (3.15) admits a mean-square exponential dichotomy. Using Theorem 3.2 and Theorem 3.7, we know that

$$\begin{gathered} dx(t) = (-a+b)x(t)dt + (\sigma+\eta)x(t)\, d\omega(t), \\ dy(t) = (a+b)y(t)dt + (\sigma+\eta)y(t)\, d\omega(t), \end{gathered}$$

and

$$\begin{gathered} dx(t) = (-a+b)x(t)dt + \sigma x(t)\, d\omega(t), \\ dy(t) = (a+b)y(t)dt + \sigma y(t)\, d\omega(t), \end{gathered}$$

also admit a mean-square exponential dichotomy if $|b|$ and $|\eta|$ are small enough.

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