Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 10, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

CRITICAL EXPONENT FOR THE ASYMPTOTIC BEHAVIOR OF RESCALED SOLUTIONS TO THE POROUS MEDIUM EQUATION

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Abstract. In this article, we find that $\mu_c \equiv 2N/(N(m-1)+2)$ is the critical exponent for the asymptotic behavior of rescaled solutions $t^{\mu/2}u(t^\beta x, t)$ for the porous medium equation.

1. Introduction

In this article, we consider the asymptotic behavior of solutions to the Cauchy problem of the porous medium equation

$$\frac{\partial u}{\partial t} - \Delta u^m = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

$$u(x,t) = u_0(x) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Here the initial value satisfies

$$u_0 \in C_0^+(\mathbb{R}^N) \equiv \{\varphi \in C(\mathbb{R}^N); \lim_{|x|\to\infty} \varphi(x) = 0 \text{ and } \varphi(x) \geq 0\}$$

and $m > 1$ is a physical constant.

Asymptotic behavior of solutions for the porous medium equation has attracted much attention of mathematicians for a long time and many interesting results have been obtained, see [2, 8, 9, 10, 12, 13, 14, 15, 18, 19, 20, 21].

Friedman and Kamin [9] first revealed the fact that if the nonnegative initial value $u_0 \in L^1(\mathbb{R}^N)$, then the solution $u(x,t)$ of problem (1.1)–(1.2) satisfies

$$\lim_{t\to\infty} t^{\frac{N}{N(m-1)+2}} \|u(\cdot,t) - U_M(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = 0,$$

where $U_M(x,t)$ is the source-type solution with the same mass M as that of u_0; see also [10, 13].

This result means that if $0 \leq u_0 \in L^1(\mathbb{R}^N)$, then the ω-limit set of rescaled solutions $t^{\mu/2}u(t^\beta x, t)$ with $\mu = \frac{2N}{N(m-1)+2}$ and $\beta = \frac{1}{N(m-1)+2}$ contains one point; that is, the rescaled solutions $t^{\frac{N}{N(m-1)+2}} u(t^{\frac{1}{N(m-1)+2}} x, t)$ possess the simple asymptotic behavior (KV point in Figure 1). However, for $u_0 \in L^\infty(\mathbb{R}^N)$, in 2002, Vázquez and Zuazua [14] found that the ω-limit set of the rescaled solutions $t^{\mu/2}u(t^\beta x, t)$

2010 *Mathematics Subject Classification.* 35B40, 35K65.
Key words and phrases. Complexity; asymptotic behavior; porous medium equation.

Submitted November 17, 2016. Published January 10, 2017.

of problem (1.1)–(1.2) with $\mu = 0$ and $\beta = 1/2$ may contain infinite points, i.e., $u(t^{1/2}\cdot, t)$ (VZ point in Figure 1) possess complicated asymptotic behavior.

Such phenomena that the different exponents of rescaled solutions $t^{\mu/2}u(t^{\beta}x, t)$ show different asymptotic behaviors for the porous medium equation have been studied in [2, 14, 15, 20, 21], for other evolution equations, one can see [3, 4, 5, 6, 7, 11].

For the ω-limit set of the rescaled solutions $t^{\mu/2}u(t^{\beta}\cdot, t)$ of problem (1.1)–(1.2) in $C_0(\mathbb{R}^N)$, we showed in our previous paper [20] that if $(\mu, \beta) \in$ I ($0 < \mu < \frac{2N}{N(m-1)+2}$ and $\beta > \beta(\mu) = \frac{2-\mu(m-1)}{4}$, then there exists $u_0 \in C_0^+(\mathbb{R}^N)$ such that this ω-limit set contains infinite points; see Figure 1). In another paper [21], we revealed that if μ and β in the line segment $\beta(\mu) = \frac{2-\mu(m-1)}{4}$ ($0 < \mu < \frac{2N}{N(m-1)+2}$, see Figure 1), then there also exists $u_0 \in C_0^+(\mathbb{R}^N)$ such that this ω-limit set contains infinite points. While in this paper, we will reveal the different fact that if $(\mu, \beta) \in II$ ($\mu \geq \frac{2N}{N(m-1)+2}$, $\beta > 0$, then for any $u_0 \in C_0^+(\mathbb{R}^N)$, this ω-limit set contains at most one point, see Figure 1), i.e., the complicated asymptotic behavior of the rescaled solutions cannot happen.

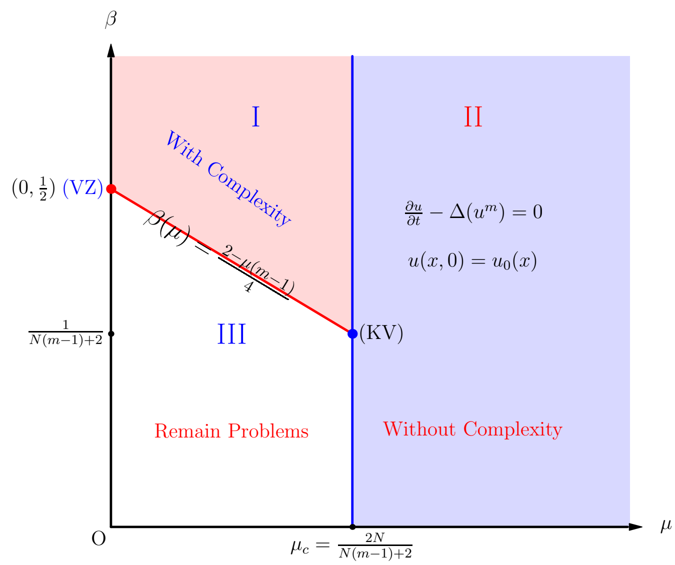

FIGURE 1. The μ-β Parameters Plane

Remark 1.1. From the above results, we can find that $\mu_c = 2N/(N(m-1)+2)$ is the critical exponent of μ on the asymptotic behavior of the rescaled solutions $t^{\mu/2}u(t^{\beta}x, t)$. It is not clear whether the rescaled solutions $t^{\mu/2}u(t^{\beta}x, t)$ with $(\mu, \beta) \in$ III ($0 < \mu < 2N/(N(m-1)+2)$ and $0 < \beta < (2-\mu(m-1))/4$, see Figure 1) possess complicated asymptotic behavior, so the problem of the critical exponent for β still has not been solved.

The rest of this article is organized as following. In the next section, we introduce some definitions and concepts to give a series of lemmas. In the last of this paper, we give and prove our results.

2. Preliminaries

Before introducing the main results of this paper, we give some concepts as in [1, 16, 17]. For $f \in L^1_{\rm loc}(\mathbb{R}^N)$ and $r > 0$, let

$$|||f|||_r = \sup_{R\geq r} R^{-\frac{N(m-1)+2}{m-1}} \int_{|x|\leq R} |f(x)|dx.$$

Then we define the space $X = X(\mathbb{R}^N)$ by

$$X \equiv \{f \in L^1_{\rm loc}(\mathbb{R}^N); |||f|||_1 < \infty\},$$

and equip this space with the norm $|||\cdot|||_1$. Hence it is a Banach space, and any norm $|||\cdot|||_r$, $r > 0$, is an equivalent norm. For $f \in X$, we define

$$\ell(f) = \lim_{r\to\infty} |||f|||_r.$$

The space $X_0 = X_0(\mathbb{R}^N)$ is defined by

$$X_0 \equiv \{f \in X; \ell(f) = 0\}.$$

Notice that $L^1(\mathbb{R}^N) \subset X_0 \subset X \subset L^1_{\rm loc}(\mathbb{R}^N)$ with continuous inclusions. Similarly, $L^\infty(\mathbb{R}^N) \subset X_0$ with continuous inclusion. We now give the definition of solutions for problem (1.1)–(1.2) with the initial value $u_0 \in X_0$.

Definition 2.1. A nonnegative measurable function $u = u(x,t)$ defined in $S_T = [0,T) \times \mathbb{R}^N$, $T > 0$, is a solution of (1.1)–(1.2) if

(I) $u \in C([0,T); L^1_{\rm loc}(\mathbb{R}^N)) \bigcap L^\infty(0,T;X)$;
(II) $u^m \in L^1((0,T) \times B_r(0))$ for any $B_r(0) \equiv \{x \in \mathbb{R}^N; |x| < r,\ r > 0\}$;
(III) for every test function $\phi \in C^{2,1}_c(S_T)$, it holds

$$\iint_{S_T} (u\phi_t + u^m\Delta\phi)\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\phi(x,0)dx = 0.$$

For any $u_0 \in X_0$, the existence and uniqueness of the solution is well established in [1, 16, 17]. Moreover, problem (1.1)–(1.2) generates a bounded continuous semigroup in the space X_0 given by

$$S(t) : u_0 \to u(x,t); \tag{2.1}$$

that is, $S(t)u_0 \in C([0,\infty); X_0)$, see [16, 17]. We now introduce the definitions of scalings and present the commutative relations between the semigroup operators and the dilation operators as in [20, 21]. For λ, μ, $\beta > 0$ and $u_0 \in X_0$, the space-time dilation $\Gamma^{\mu,\beta}_\lambda$ is defined as following:

$$\Gamma^{\mu,\beta}_\lambda[u_0](x) \equiv D^{\mu,\beta}_\lambda[S(\lambda^2 t)u_0(x)] = \lambda^\mu u(\lambda^{2\beta}x, \lambda^2 t),$$

where the dilation $D^{\mu,\beta}_\lambda$ is defined as

$$D^{\mu,\beta}_\lambda w(x) \equiv \lambda^\mu w(\lambda^{2\beta}x)$$

and $S(t)$ is the PME semigroup given by (2.1). From the definitions of $D^{\mu,\beta}_\lambda$ and $S(t)$, we can get the following commutative relations between the semigroup operators $S(t)$ and the dilation operators $D^{\mu,\beta}_\lambda$,

$$\Gamma^{\mu,\beta}_\lambda u_0(x) = D^{\mu,\beta}_\lambda[S(\lambda^2 t)u_0(x)] = S(\lambda^{2-4\beta-\mu(m-1)}t)[D^{\mu,\beta}_\lambda u_0](x).$$

In particular,

$$\Gamma^{\mu,\beta}_{\sqrt{t}} u_0(x) = S(t^{\frac{2-4\beta-\mu(m-1)}{2}})[D^{\mu,\beta}_{\sqrt{t}} u_0](x), \tag{2.2}$$

see details in [20, 21]. The set of functions

$$\omega^{\mu,\beta}(u_0) \equiv \{f \in C_0^+(\mathbb{R}^N); \exists t_n \to \infty \text{ s.t. } D^{\mu,\beta}_{\sqrt{t_n}}[S(t_n)u_0](\cdot) \xrightarrow{t_n\to\infty} f \text{ in } L^\infty(\mathbb{R}^N)\}$$

is called Ω-limit set. We also introduce the following symbol to denote the positive set of $u(x,t)$ at time t,

$$\Omega(t) \equiv \{x \in \mathbb{R}^N;\ u(x,t) > 0\}.$$

The ρ-neighborhood of the set $\Omega(t)$ is defined as

$$\Omega_\rho(t) \equiv \{x \in \mathbb{R}^N;\ d(x,\Omega(t)) \le \rho\},$$

where $d(x,\Omega(t))$ is the distance from x to $\Omega(t)$. We now list some important properties of the solutions.

Lemma 2.2 ([16]). *If $0 \le u_0 \in L^1(\mathbb{R}^N)$, then the solution $u(x,t)$ satisfies the L^1-L^∞ smoothing effect: for every $t > 0$,*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C_1\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N(m-1)+2}} t^{-\frac{N}{N(m-1)+2}},$$

where C_1 is a constant dependent on m and N.

The following lemma was proved in [20], we give here a different proof for the sake of completeness.

Lemma 2.3 ([20]). *Let $u(x,t)$ be a nonnegative solution of (1.1)–(1.2) with the initial value u_0 such that $0 \le u_0 \in L^1(\mathbb{R}^N)$. Then for any $0 \le t_1 < t_2 < \infty$,*

$$\Omega(t_2) \subset \Omega_{\rho(t_2-t_1)}(t_1),$$

where

$$\rho(t_2 - t_1) = C_2(t_2 - t_1)^{\frac{1}{N(m-1)+2}}\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}}$$

and C_2 is a constant dependent on m and N.

Proof. To prove this lemma, we need the fact that if $u(x,t)$ is a nonnegative solution of (1.1)–(1.2) with the initial data u_0 satisfying

$$0 \le u_0 \in L^\infty(\mathbb{R}^N),$$

then

$$\Omega(t_2) \subset \Omega_{\rho(t_2-t_1)}(t_1) \quad \text{for } 0 \le t_1 < t_2 < \infty, \tag{2.3}$$

where

$$\rho(t_2 - t_1) = C(t_2 - t_1)^{1/2}\|u_0\|_{L^\infty(\mathbb{R}^N)}^{\frac{m-1}{2}}.$$

In fact, for any given $x_0 \in \mathbb{R}^N$ with $d(x_0) > 0$, if $R \ge d(x_0)$, then

$$\begin{aligned}
R^{-\frac{N(m-1)+2}{m-1}} \int_{B_R(x_0)} u_0(y)dy &\le C\|u_0\|_{L^\infty(\mathbb{R}^N)} R^{-\frac{N(m-1)+2}{m-1}} R^N \\
&= C\|u_0\|_{L^\infty(\mathbb{R}^N)} R^{-\frac{2}{m-1}} \\
&\le C\|u_0\|_{L^\infty(\mathbb{R}^N)} d(x_0)^{-\frac{2}{m-1}};
\end{aligned}$$

or if $R < d(x_0)$, then

$$\int_{B_R(x_0)} u_0(y)dy = 0,$$

where $B_R(x_0) = \{y; |x_0 - y| < R\}$. So

$$B(x_0) \equiv \sup_{R\geq d(x_0)} R^{-\frac{N(m-1)+2}{m-1}} \int_{B_R(x_0)} u_0(y)dy \leq C\|u_0\|_{L^\infty(\mathbb{R}^N)} d(x_0)^{-\frac{2}{m-1}}. \tag{2.4}$$

The condition $0 \leq u_0 \in L^\infty(\mathbb{R}^N) \subset X_0$ implies that if $|x| \leq R$ and $r \leq R$, then

$$u(x,t) \leq Ct^{-\frac{N}{N(m-1)+2}} R^{\frac{2}{m-1}} |||u_0|||_r^{\frac{2}{N(m-1)+2}} \quad \text{for } 0 < t < \infty,$$

see [1, 16]. This result and (2.4) imply that

$$u(x_0,t) = 0 \text{ for all } 0 \leq t \leq C\|u_0\|_{L^\infty(\mathbb{R}^N)}^{-(m-1)} d(x_0)^2.$$

This implies $\Omega(t) \subset \Omega_{\rho(t)}(0)$, where

$$\rho(t) = C\|u_0\|_{L^\infty(\mathbb{R}^N)}^{\frac{m-1}{2}} t^{1/2}.$$

From this, we can get the desired result.

We now discuss the case that $0 \leq u_0 \in L^1(\mathbb{R}^N)$ to complete the proof. Without loss of generality, we can restrict our consideration to the case of $t_1 = 0$. For any $0 < t < \infty$, we select a sequence of times

$$t_k = 2^{-k}t \to 0 \quad \text{as } k \to \infty.$$

We then consider the evolution in the time intervals $I_k = [t_k, t_{k-1}]$; that is, we will estimate the increase of the support in these time intervals. From the L^1-L^∞ smoothing effect, at each initial time $t = t_k$, we have

$$\|u(t_k)\|_{L^\infty(\mathbb{R}^N)} \leq C(p,N)\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N(m-1)+2}} t_k^{-\frac{N}{N(m-1)+2}}. \tag{2.5}$$

Therefore, we can deduce from (2.3) that

$$\Omega(t_{k-1}) \subset \Omega_{\rho(t_{k-1}-t_k)}(t_k),$$

where $\rho(t_{k-1} - t_k) = C\|u(t_k)\|_{L^\infty(\mathbb{R}^N)}^{\frac{m-1}{2}} (t_{k-1} - t_k)^{1/2}$. Iterating, we have

$$\Omega(t) \subset \Omega_{\rho(t)}(0),$$

where

$$\begin{aligned}
\rho(t) &= C\sum_{k=1}^{\infty} \|u(t_k)\|_{L^\infty(\mathbb{R}^N)}^{\frac{m-1}{2}} (t_{k-1} - t_k)^{1/2} \leq C\sum_{k=1}^{\infty} \|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}} t_k^{\frac{1}{N(m-1)+2}} \\
&= C\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}} t^{\frac{1}{N(m-1)+2}} \sum_{k=1}^{\infty} 2^{-\frac{k}{N(m-1)+2}} \leq C\|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{m-1}{N(m-1)+2}} t^{\frac{1}{N(m-1)+2}}.
\end{aligned}$$

Here we have used the estimates (2.5). The proof is complete. □

The next lemma is called Aleksandrov's reflection (see [16]). We introduce some notation to give this principle. Any H, hyperplane of $\mathbb{R}^N$, divides $\mathbb{R}^N$ into two half spaces $\Omega_1(H)$ and $\Omega_2(H)$. We denote by $\pi = \pi_H$ the specular symmetry that maps a point $x \in \Omega_1(H)$ into its symmetric image with respect to H, $\pi_H(x) \in \Omega_2(H)$.

Lemma 2.4 (Aleksandrov's Reflection Principle [16]). *Let $u \geq 0$ be a solution of problem* (1.1)–(1.2) *with initial value $u_0 \in X_0$. Suppose that for a given hyperplane H and all $x \in \Omega_1(H)$,*

$$u_0(\pi_H(x)) \leq u_0(x).$$

Then, for all times $0 \le t < \infty$,

$$u(\pi_H(x), t) \le u(x,t), \quad x \in \Omega_1(H).$$

The following lemma depends on Lemma 2.3 and 2.4.

Lemma 2.5. *Suppose* $u(x,t)$ *is a non-negative solution of* (1.1)–(1.2) *with initial-value* $u_0 \in C_0^+(\mathbb{R}^N)$ *and* $u_0 \not\equiv 0$. *Let*

$$M(t) = \int_{|x| \le t^{\frac{1}{2N(m-1)+4}}} u_0(x)dx.$$

Then there exists a $0 < t_0 < \infty$ *such that for* $t \ge t_0$,

$$u(0,t) \ge Ct^{-\frac{N}{N(m-1)+2}} M(t)^{\frac{2}{N(m-1)+2}}.$$

Proof. Since the nonnegative initial value $u_0 \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, then there exist constants $t_1, C_3 > 0$ such that

$$\int_{B_{t_1}} u_0(x)\mathrm{d}x \ge C_3.$$

Now let

$$t_2 = C_2^{-\frac{2}{N(m-1)+2}} C_3^{-2m+2},$$
$$t_3 = (2^{N+1}C_1|B_1|)^{\frac{2N(m-1)+4}{N}} C_3^{-2m+2}$$

where C_1, C_2 are the constants given in Lemma 2.2 and Lemma 2.3 respectively. Let $t_0 = \max(t_1, t_2, t_3)$. Then for any $t \ge t_0$, using comparison principle, we can suppose that u_0 is supported in the ball $B_t = \{x; |x| \le t^{\frac{1}{2N(m-1)+4}}\}$. In fact, for general u_0, suppose $\eta_t(x)$ is a cut-off function compactly supported in B_t and less than one with

$$\int_{B_t} \eta_t(x)u_0(x)dx \ge \frac{1}{2}M(t),$$

then $u_0\eta_t$ is lesser than u_0. Therefore, if v is the solution with initial data $u_0\eta_t$, then

$$v(x,s) \le u(x,s) \quad \text{for all} \quad s > 0.$$

Hence, if this lemma holds for $v(x,t)$, then

$$u(0,t) \ge v(0,t) \ge C(\frac{1}{2}M(t))^{\frac{2}{N(m-1)+2}} t^{-\frac{N}{N(m-1)+2}}.$$

Therefore, in the next part of this proof, we assume that $\operatorname{supp} u_0 \subset B_t$. So,

$$M(t) = \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x \ge C_3.$$

The L^1-L^∞ smoothing effect implies that for any $s > 0$,

$$0 \le u(x,s) \le C_1 M(t)^{\frac{2}{N(m-1)+2}} s^{-\frac{N}{N(m-1)+2}}.$$

The conservation of mass means that for all $s \ge 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} u_0(x)dx &= \int_{\mathbb{R}^N} u(x,s)dx \\ &= \int_{|x| \ge 2t^{\frac{1}{2N(m-1)+4}}} u(x,s)dx + \int_{|x| \le 2t^{\frac{1}{2N(m-1)+4}}} u(x,s)dx,\end{aligned}$$

the last term can be estimated as

$$\int_{|x|\le 2t^{\frac{1}{2N(m-1)+4}}} u(x,s)dx \le 2^N C_1|B_1|M(t)^{\frac{2}{N(m-1)+2}} s^{-\frac{N}{N(m-1)+2}} t^{\frac{N}{2N(m-1)+4}}, \quad (2.6)$$

where $|B_1|$ is the measure of the unit ball B_1 in $\mathbb{R}^N$. Since $\operatorname{supp} u_0 \subset B_t$, then Lemma 2.3 indicates that for all $s>0$,

$$\operatorname{supp} u(x,s) \subset B_{R_1(s)},$$

where $R_1(s) = t^{\frac{1}{2N(m-1)+4}} + C_2 M(t)^{\frac{m-1}{N(m-1)+2}} s^{\frac{1}{N(m-1)+2}}$. Let $s=t$ and

$$R(t) = 4C_2 M(t)^{\frac{m-1}{N(m-1)+2}} t^{\frac{1}{N(m-1)+2}}.$$

Notice that $t \ge t_0 \ge t_2 = C_2^{-\frac{2}{N(m-1)+2}} C_3^{-2m+2}$ and $M(t) \ge C_3$. So

$$R(t) > 2R_1(t) \ge 4t^{\frac{1}{2N(m-1)+4}}. \quad (2.7)$$

The hypothesis $\operatorname{supp} u_0 \subset B_t$ implies, via the Aleksandrov reflection principle (Lemma 2.4), that for all $|x| \ge 2t^{\frac{1}{2N(m-1)+4}}$ and $s \ge 0$,

$$u(0,s) \ge u(x,s).$$

So, from (2.7), we have

$$\begin{aligned}
u(0,t)R(t)^N &\ge u(0,t)(R(t)^N - 2^N t^{\frac{N}{2N(m-1)+4}})\\
&= \frac{1}{|B_1|}\int_{2t^{\frac{1}{2N(m-1)+4}} \le |x| \le R(t)} u(0,t)dx\\
&\ge \frac{1}{|B_1|}\int_{2t^{\frac{1}{2N(m-1)+4}} \le |x| \le R(t)} u(x,t)dx\\
&= \frac{1}{|B_1|}\int_{|x|\ge 2t^{\frac{1}{2N(m-1)+4}}} u(x,t)dx\\
&= \frac{1}{|B_1|}\int_{\mathbb{R}^N} u(x,t)dx - \frac{1}{|B_1|}\int_{|x|<2t^{\frac{1}{2N(m-1)+4}}} u(x,t)dx.
\end{aligned}$$

Now using estimate (2.6) and $t \ge t_0 \ge t_3$, we obtain

$$u(0,t)R(t)^N \ge \frac{1}{|B_1|}[M(t) - 2^N C_1|B_1|M(t)^{\frac{2}{N(m-1)+2}} t^{-\frac{N}{2N(m-1)+4}}] \ge \frac{1}{2|B_1|}M(t).$$

It follows from the definition of $R(t)$ that

$$u(0,t) \ge Ct^{-\frac{N}{N(m-1)+2}} M(t)^{\frac{2}{N(m-1)+2}}.$$

The proof is complete. □

3. Results and their proofs

Theorem 3.1. *Let $u_0 \in C_0^+(\mathbb{R}^N)$, $u_0 \not\equiv 0$. If there exist $0 \not\equiv v \in C_0(\mathbb{R}^N)$, $\mu_0 \ge \frac{2N}{N(m-1)+2}$, $\beta_0 > 0$ and a sequence $\{t_n\}_{n=1}^\infty$ with $\lim_{n\to\infty} t_n = +\infty$ such that*

$$\Gamma^{\mu_0,\beta_0}_{\sqrt{t_n}} u_0 = t_n^{\frac{\mu_0}{2}}[S(t_n)u_0](t_n^{\beta_0}\cdot) \xrightarrow{t_n\to\infty} v \quad in \quad C_0(\mathbb{R}^N), \quad (3.1)$$

then

$$u_0 \in L^1(\mathbb{R}^N), \quad \mu_0 = \frac{2N}{N(m-1)+2},$$

$$\beta_0 = \frac{2-\mu_0[m-1]}{4} = \frac{1}{N(m-1)+2}.$$

In other words, if $\mu > \frac{2N}{N(m-1)+2}$*, or if* $\mu = \frac{2N}{N(m-1)+2}$ *and* $\beta \neq \frac{1}{N(m-1)+2}$*, then*

$$\omega(u_0) = \emptyset, \quad \textit{or} \quad \omega(u_0) = \{0\}.$$

Proof. It follows from (3.1) and Lemma 2.5 that if n sufficiently large, then

$$v(0)+1 \geq [\Gamma^{\mu_0,\beta_0}_{\sqrt{t_n}} u_0](0) = t_n^{\frac{\mu_0}{2}}[S(t_n)u_0](0) \geq C t_n^{\frac{\mu_0 - \frac{2N}{N(m-1)+2}}{2}} M(t_n)^{\frac{2}{N(m-1)+2}}. \tag{3.2}$$

Here $M(t)$ is given by Lemma 2.5. Letting $n \to \infty$, we conclude that

$$\mu_0 = \frac{2N}{N(m-1)+2}$$

and $u_0 \in L^1(\mathbb{R}^N)$. Notice also that $u_0 \geq 0$. This gives

$$D^{\frac{2N}{N(m-1)+2}, \frac{1}{N(m-1)+2}}_{\sqrt{t}} S(t)u_0(x) = t^{\frac{N}{N(m-1)+2}} u(t^{\frac{1}{N(m-1)+2}}x, t) \to U_M(x,1) \tag{3.3}$$

uniformly on $\mathbb{R}^N$ as $t \to \infty$. Here $U_M(x,t)$ is the source-type solution with the same mass as that of u_0, where $M = \int_{\mathbb{R}^N} u_0(x)\mathrm{d}x$, see [10, 13]. Therefore,

$$D^{\frac{2N}{N(m-1)+2}, \beta_0}_{\sqrt{t_n}} S(t_n)u_0(x) - U_M(x t_n^{\beta_0 - \frac{1}{N(m-1)+2}}, 1) \xrightarrow{n\to\infty} 0 \tag{3.4}$$

uniformly on $\mathbb{R}^N$. The expression of the source-type solution clearly means

$$\mathrm{supp}(U_M(x,1)) \subset \{x; |x| \leq C M^{\frac{m-1}{N(m-1)+2}}\},$$

so that if $\beta_0 > \frac{1}{N(m-1)+2}$, then

$$U_M(x t_n^{\beta_0 - \frac{1}{N(m-1)+2}}, 1) \to 0 \quad \text{for all } x \neq 0$$

as $t_n \to \infty$. Notice also that $v \not\equiv 0$, so (3.4) is compatible with (3.1) only if

$$\beta_0 \leq \frac{1}{N(m-1)+2}.$$

On the other hand, from (3.1) and (3.3) we deduce that

$$D^{\frac{2N}{N(m-1)+2}, \frac{1}{N(m-1)+2}}_{\sqrt{t_n}} S(t_n)u_0(x) - v(t_n^{\frac{1}{N(m-1)+2} - \beta_0} x) \to 0 \tag{3.5}$$

uniformly on $\mathbb{R}^N$ as $t_n \to \infty$. The hypothesis that $v \in C_0(\mathbb{R}^N)$ clearly implies that if $\beta_0 < \frac{1}{N(m-1)+2}$, then

$$v(t_n^{\frac{1}{N(m-1)+2} - \beta_0} x) \to 0 \quad \text{for all } x \neq 0$$

as $t_n \to \infty$. Recall that $u_0 \not\equiv 0$, so $U_M \not\equiv 0$. Therefore, (3.5) is compatible with (3.3) only if

$$\beta_0 \geq \frac{1}{N(m-1)+2}.$$

Hence

$$\beta_0 = \frac{1}{N(m-1)+2}.$$

So that $\omega^{\mu,\beta}(u_0) = \emptyset$ if $\mu > \frac{2N}{N(m-1)+2}$, or if $\mu = \frac{2N}{N(m-1)+2}$ and $\beta \neq \frac{1}{N(m-1)+2}$. This completes the proof. □

Theorem 3.2. *Let*

$$\mu = \frac{2N}{N(m-1)+2}, \quad \textit{and} \quad \beta = \frac{1}{N(m-1)+2}.$$

If $u_0 \in C_0^+(\mathbb{R}^N)$*, then*

$$\omega^{\mu,\beta}(u_0) = \emptyset, \quad \textit{or} \quad \omega^{\mu,\beta}(u_0) = \{U_M(x,1)\},$$

where $U_M(x,t)$ *is source-type solution with the same mass* M *as that of* u_0*.*

Proof. If $u_0 \in C_0^+(\mathbb{R}^N)$, then $u_0 \in L^1(\mathbb{R}^N)$, or else $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$ with $\|u_0\|_{L^1(\mathbb{R}^N)} = \infty$. If $u_0 \in L^1(\mathbb{R}^N)$, then

$$\lim_{t\to\infty} t^{\frac{N}{N(m-1)+2}} u(t^{\frac{1}{N(m-1)+2}} x, t) = U_M(x,1) \quad \text{in } L^\infty(\mathbb{R}^N). \tag{3.6}$$

So

$$\omega^{\mu,\beta}(u_0) = \{U_M(x,1)\}.$$

If $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$ and $\|u_0\|_{L^1(\mathbb{R}^N)} = \infty$, approximating u_0 by an increasing sequence of integrable data u_{0n}, applying (3.6) and passing to the limit, we have

$$\lim_{t\to\infty} t^{\frac{N}{N(m-1)+2}} u(t^{\frac{1}{N(m-1)+2}} x, t) = \infty \quad \text{in} \quad L^\infty(\mathbb{R}^N).$$

Hence $\omega^{\mu,\beta}(u_0) = \emptyset$. The proof is complete. □

Remark 3.3. As we had showed in [20, 21] that for $0 < \mu < 2N/(N(m-1)+2)$, if $\beta = (2-\mu(m-1))/4$, then there exists an initial value $u_0 \in C_0^+(\mathbb{R}^N)$ such that the Ω-limit set $\omega^{\mu,\beta}(u_0)$ contains the set

$$S(1)C_0^+(\mathbb{R}^N) \equiv \{S(1)\varphi; \varphi \in C_0^+(\mathbb{R}^N)\},$$

or if $\beta > \frac{2-\mu(m-1)}{4}$, then there also exists an initial value $u_0 \in C_0^+(\mathbb{R}^N)$ such that the Ω-limit set $\omega^{\mu,\beta}(u_0)$ contains the set

$$C_0^{+,0}(\mathbb{R}^N) \equiv \{\varphi \in C_0^+(\mathbb{R}^N); \varphi(0) = 0\}.$$

Therefore,

$$\mu_c = \frac{2N}{N(m-1)+2}$$

is the critical exponent of μ on the asymptotic behavior of the rescaled solutions $t^{\mu/2}u(t^\beta\cdot, t)$.

Acknowledgements. This research was supported by the NSFC (11071099 and 11371153), Natural Science Foundation Project of CQ (cstc2016jcyjA0596), Scientific and Technological Research Program of Chongqing Municipal Education Commission (KJ1401003, KJ1601006), and Innovation Team Building at Institutions of Higher Education in Chongqing (CXTDX201601035).

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