OPTIMIZATION PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN

CHONG QIU, YISHENG HUANG, YUYING ZHOU

ABSTRACT. In this article we study rearrangement optimization problems related to boundary-value problems involving the fractional Laplacian. We establish the existence and uniqueness of a solution under suitable assumptions.

1. Introduction

A rearrangement optimization problem is referred to an optimization problem in which the admissible set consists of functions that are rearrangements of a prescribed function. The theory of rearrangement optimizations has been established by Burton [5, 6]. Since then, this topic has been widely studied by many authors in different aspects, see for example [7, 10, 11, 12, 16, 17, 18, 19]. Burton [6] proved that both the minimization and maximization problems for the boundary value problem involving the Laplacian have solutions. However, the results obtained in [6] can not be directly applied to the optimization problems for the boundary value problem involving $p$-Laplacian ($1 < p < \infty$). So by using a new approach, Cuccu et al [10] proved that the minimization problem has a solution for $1 < p < \infty$. But their approach is not efficient for the maximization problem. Marras [17] obtained the solvability of the maximization for $1 < p < \infty$ by using another method. While Cuccu et al [12] obtained a result of uniqueness for a class of $p$-Laplace equations under non-standard assumptions. Recently, Qiu et al [19] considered a rearrangement optimization problem related to the quasilinear elliptic boundary value problem, where under suitable assumptions, it is shown that both the minimization and maximization problems are solvable, which extends the results in [6, 10, 17].

It is worth to note that most of the rearrangement optimization problems considered in the above papers are related to boundary value problems involving the Laplacian or $p$-Laplacian. In recent years, fractional and nonlocal operators of elliptic type have been attracted a lot of interests since these operators appear in concrete applications in fields such as minimal surface [9], thin obstacle problem [8], anomalous diffusion [11], phase transition [21], hemivariational inequality [23, 25], shape optimization [13], optimal transportation theory [14] and so on. Interesting
In this article, we study several rearrangement optimization problems related to a class of boundary value problems involving the fractional Laplacian. We need to overcome the difficulties coming from both the fractional Laplacian and the rearrangement optimization problem. As our knowledge, this kind of problems has not been considered in literature.

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$ ($N \geq 2$) and let $k \in L^q(\Omega)$ with $1 \leq q \leq \infty$. We recall that a rearrangement of $k$ is an element of the set $\mathcal{R}(k)$ of all measurable functions $g$ on $\Omega$ satisfying

$$\text{meas}\{x \in \Omega : g(x) \geq a\} = \text{meas}\{x \in \Omega : k(x) \geq a\}, \quad \forall a \in \mathbb{R}.$$  

It is easy to prove that if $g \in \mathcal{R}(k)$, then $g \in L^q(\Omega)$ and $\|g\|_{L^q} = \|k\|_{L^q}$ (cf. Lemma 2.1).

Let $h(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $f \in L^\infty(\Omega)$. Under suitable assumptions we can show that the boundary value problem

$$-L^s_\theta u + h(x,u) = f(x) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega$$

has a unique solution $u_f \in H^s(\Omega)$ (cf. Proposition 3.1), where $L^s_\theta$ is the fractional Laplace type operator defined as

$$L^s_\theta u(x) = \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \theta(y)dy, \quad x \in \mathbb{R}^N$$

with $0 < s < 1$ and $\theta : \mathbb{R}^N \to (0, +\infty)$.

In particular, if $h \equiv 0$, then the boundary value problem (1.1) becomes

$$-L^s_\theta u = f(x) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

and it obviously has a unique solution $u_f$ (cf. Proposition 4.1), thus it deduces that (cf. Remark 4.2),

$$\sup_{v \in H^s(\Omega)} \left( \int_\Omega 2fv dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - v(y))^2}{|x-y|^{N+2s}} \theta(x-y)dx dy \right)$$

$$= \int_\Omega 2fu_f dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x-y|^{N+2s}} \theta(x-y)dx dy$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x-y|^{N+2s}} \theta(x-y)dx dy = \int_\Omega fu_f dx.$$  

Therefore, we can define the functional $\Phi$ on $L^\infty(\Omega)$ as

$$\Phi(f) = \int_\Omega fu_f dx$$

and we are able to consider the following two optimization problems:

(Opt1) Find $f_1 \in \mathcal{R}(f_0)$ such that $\Phi(f_1) = \sup_{f \in \mathcal{R}(f_0)} \Phi(f)$,

(Opt2) Find $f_2 \in \mathcal{R}(f_0)$ such that $\Phi(f_2) = \inf_{f \in \mathcal{R}(f_0)} \Phi(f)$,

where $f_0 \in L^\infty(\Omega)$ is a given function and $\mathcal{R}(f_0)$ is the set of all rearrangement of $f_0$. 

studies involving nonlocal fractional problems by variational methods can be found in [2, 3] and the references therein.
If \( h \neq 0 \), then it is difficult to consider the above optimization problems related to the boundary value problem (1.1) since the functional \( \Psi \) on \( L^\infty(\Omega) \), corresponding to the optimization problems, defined by

\[
\Psi(f) = \frac{1}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy \right) + \int_\Omega H(x, u_f) \, dx - \int_\Omega f u_f \, dx
\]

(1.6)
is hard to be reduced to a simple formula like (1.5). However, in the case of \( h \neq 0 \), we can consider the following minimization optimization problem related to (1.1):

\[
\text{(Opt*) Find } f_\ast \in \mathcal{H}(f_0) \text{ such that } \Psi(f_\ast) = \inf_{f \in \mathcal{H}(f_0)} \Psi(f),
\]

where \( f_0 \in L^\infty(\Omega) \) is a given function and \( \mathcal{H}(f_0) \) is the set of all rearrangement of \( f_0 \).

This article is organized as follows. In Section 2, we give some preliminaries. Section 3 is devoted to discussing the minimization problem (Opt*) in detail. After establishing the uniqueness result of solutions for the problem (1.1), we show that the minimization problem (Opt*) is solvable. In Section 4, we prove that both the maximization and minimization optimization problems (Opt1) and (Opt2) are solvable. To our best of knowledge, the results of this paper are new and nontrivial.

2. Preliminaries

Given \( 0 < s < 1 \), we define the fractional Sobolev space

\[ H^s(\Omega) = \{ u \in L^2(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega, \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy < \infty \} \]

with the inner product

\[ \langle u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dx \, dy \quad \forall u, v \in H^s(\Omega). \]

Then the norm of \( u \) is

\[ \| u \| = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{1/2}. \]

Throughout this article, we denote by \( \| u \| \) and \( \| u \|_p \) the usual norm in spaces \( H^s(\Omega) \) and \( L^p(\Omega) \) \((1 \leq p \leq \infty)\), respectively. As usual, “\( \hookrightarrow \)” and “\( \rightharpoonup \)” denote the strong and weak convergence.

We now list some lemmas which are useful in the proof of our main results.

Lemma 2.1 ([20 Lemma 8]). \( H^s(\Omega) \hookrightarrow L^r(\mathbb{R}^N), \) for \( 1 \leq r \leq \frac{2N}{N-2s} \), and the embedding is compact if \( 1 \leq r < \frac{2N}{N-2s} \).

Lemma 2.2 ([6 Lemma 2.2]). Assume that \( 1 \leq r < \infty \) and for \( f \in L^r(\Omega) \) denote by \( \mathcal{H}(f) \) the weak closure of \( \mathcal{H}(f) \) in \( L^r(\Omega) \). Then \( \mathcal{H}(f) \) is convex and weakly compact in \( L^r(\Omega) \).

Lemma 2.3 ([6 Lemma 2.9] or [11 Lemma 2.1]). Let \( f, g : \Omega \to \mathbb{R} \) be measurable functions and suppose that for each \( t \in \mathbb{R} \), the level set of \( g \) at \( t \), i.e., \( \{ x \in \Omega : g(x) = t \} \), has zero measure. Then there exists an increasing (decreasing) function \( \varphi \) such that \( \varphi \circ g \) is a rearrangement of \( f \) where \( \varphi \circ g \) denotes a composite function defined by

\[ (\varphi \circ g)(x) = \varphi(g(x)), \forall x \in \Omega. \]
Lemma 2.4 ([6, Lemma 2.4] or [11, Lemma 2.2]). For any \( 1 \leq r < \infty \) define 
\[ r' = \frac{r}{r-1} \]  
if \( r > 1 \) and 
\[ r' = \infty \]  
if \( r = 1 \). Let \( f \in L^r(\Omega) \) and \( g \in L^{r'}(\Omega) \). Suppose that there exists an increasing (decreasing) function \( \varphi : \mathbb{R} \to \mathbb{R} \) such that \( \varphi \circ g \in \mathcal{A}(f) \). Then \( \varphi \circ g \) is the unique maximizer (minimizer) of the linear functional \( \int_{\Omega} h g dx \), relative to \( h \in \mathcal{A}(f) \).

Lemma 2.5 ([15, Lemma 2.3]). Suppose that \( f \in L^r(\Omega) \) and \( g \in L^{r'}(\Omega) \). Then there exists \( \hat{f} \in \mathcal{A}(f) \) which maximizes (minimizes) the linear functional \( \int_{\Omega} h g dx \), relative to \( h \in \mathcal{A}(f) \).

As in the proof of [22, Lemma 2.1] we have the following result.

Lemma 2.6. Given \( u \in H^{1/2}(\Omega) \), there exists a unique extension \( v \in H^1(\mathbb{R}^{N+1}) \) of \( u \) such that

\[ -\Delta v(x,y) = 0, \quad \text{for } x \in \mathbb{R}^N, y > 0, \]

\[ v(x,0) = u(x), \quad \text{for } x \in \mathbb{R}^N. \]

Moreover,

\[ -\partial_y v(x,0) = (-\Delta)^{1/2} u(x) \]

in the sense that

\[ -\int_{\mathbb{R}^N \times \{0\}} \frac{\partial v}{\partial y} \varphi dx = \int_{\mathbb{R}^N} \varphi (-\Delta)^{1/2} ud\]

for every \( \varphi \in H^{1/2}(\mathbb{R}^N) \).

3. Minimization related to (1.1)

Recall that the energy functional \( I : H^s(\Omega) \to \mathbb{R} \) corresponding to (1.1) is

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \theta(x-y) dx dy + \int_{\Omega} H(x,u) dx - \int_{\Omega} f u dx, \]

where \( H(x,u) = \int_0^u h(x,t) dt \).

We use the following hypotheses on the functions \( h : \Omega \times \mathbb{R} \to \mathbb{R} \) and \( \theta : \mathbb{R}^N \to (0, +\infty) \):

(H1) \( h(x,t) \) is Carathéodory and is non-decreasing with respect to the second variable for almost all \( x \in \Omega \).

(H2) There exist \( a(x), b(x) \in L^\infty(\Omega), 0 < l < 1 \), such that \( |h(x,t)| \leq a(x) + b(x) |t|^l \), for all \( t \in \mathbb{R} \), a.e. \( x \in \Omega \).

(H3) \( \theta(x) = \theta(-x) \) for any \( x \in \mathbb{R}^N \setminus \{0\} \).

(H4) \( \theta \in L^\infty(\mathbb{R}^N) \) and there exists \( \theta_0 \in \mathbb{R}_+ \) such that \( \theta(x) \geq \theta_0 \), for any \( x \in \mathbb{R}^N \).

Proposition 3.1. Suppose that (H1)–(H4) hold. Then (1.1) has a unique solution \( u_f \in H^s(\Omega) \) and \( I(u_f) = \inf_{v \in H^s(\Omega)} I(v) \).

Proof. First, we show that the problem (1.1) has a solution. Let \( C \) be a positive constant. From (H2), (H4), and the Sobolev embedding inequality it follows that

\[ \left| \int_{\Omega} H(x,u) dx \right| \leq \int_{\Omega} \left| \int_0^u h(x,v) dv \right| dx \leq C \|u\| + C \|u\|^{l+1}, \]

\[ \left| \int_{\Omega} f u dx \right| \leq \|f\|_\infty \|u\|_1 \leq C \|u\|, \]
\begin{equation}
\theta_0 u^2 \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy, \quad \forall u \in H^s(\Omega). \tag{3.4}
\end{equation}
Hence we deduce from (3.1), (3.2), (3.3) and (3.4) that

\[ I(u) \geq \frac{\theta_0}{2} \|u\|^2 - C\|u\|^{l+1} - C\|u\| \to \infty \]

as \( \|u\| \to \infty \), which shows that the functional \( I \) is coercive. We will prove that the functional \( I \) is weakly lower semi-continuous. To do this, let \( v_n \to v \) in \( H^s(\Omega) \) as \( n \to \infty \), noting both the embeddings \( H^s(\Omega) \hookrightarrow L^1(\Omega) \) and \( H^s(\Omega) \hookrightarrow L^{1+}(\Omega) \) are compact, we see that \( v_n \to v \) in \( L^{1+}(\Omega) \) and \( L^1(\Omega) \) as \( n \to \infty \). Therefore, \( H(x,v_n) \to H(x,v) \) in \( L^1(\Omega) \) as \( n \to \infty \) by the continuity of the operator \( u \mapsto H(x,u) \) from \( L^{1+}(\Omega) \) to \( L^1(\Omega) \), which implies that

\[ \int_{\Omega} H(x,v_n) \, dx \to \int_{\Omega} H(x,v) \, dx \tag{3.5} \]
as \( n \to \infty \). Then we have

\[
\lim_{n \to \infty} I(v_n) = \lim_{n \to \infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v_n(x) - v_n(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy + H(x,v_n) - f v_n \right) \, dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy + \int_{\Omega} H(x,v) \, dx - \limsup_{n \to \infty} \int_{\Omega} f v_n \, dx \\
\geq I(v) - \limsup_{n \to \infty} \|f\| \|v_n - v\|_1 \\
= I(v).
\]

Thus the functional \( I \) is weakly lower semi-continuous (which we will denote by w.l.s.c for short). So that the functional \( I \) has a minimizer \( u_f \in H^s(\Omega) \) with \( I(u_f) = \inf_{v \in H^s(\Omega)} I(v) \). By assumptions (H1), (H2), and using a standard argument [24, Lemma 2.16], we can easily show that \( I \in C^1(H^s(\Omega), \mathbb{R}) \), therefore \( u_f \) is a critical point of \( I \), i.e.,

\[
I'(u_f)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy \\
+ \int_{\Omega} (h(x,u_f)v - f v) \, dx = 0, \quad \forall v \in H^s(\Omega). \tag{3.6}
\]

By the definition of the fractional Laplace type operator \( L_\theta^s \) (see (1.2)), we have

\[
\int_{\mathbb{R}^N} -L_\theta^s u(x)v(x) \, dx \\
= - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} \theta(y)v(x) \, dy \, dx \\
= - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x+y) - u(x)}{|y|^{N+2s}} \theta(y)v(x) \, dy \, dx \\
- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x-y) - u(x)}{|y|^{N+2s}} \theta(y)v(x) \, dy \, dx. \tag{3.7}
\]
Let $z = x + y$, $t = x - y$. By using (H3) we obtain

$$
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x + y) - u(x)}{|y|^{N+2s}} \theta(y)v(x) \, dy \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{N+2s}} \theta(x-z)v(x) \, dz \, dx,
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|y|^{N+2s}} \theta(y)v(x) \, dy \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(t)}{|x-t|^{N+2s}} \theta(x-t)v(x) \, dt \, dx.
$$

(3.8)

Obviously,

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x-z|^{N+2s}} \theta(x-z)v(x) \, dz \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(t)}{|x-t|^{N+2s}} \theta(x-t)v(x) \, dt \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \theta(x-y)v(x) \, dy \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{N+2s}} \theta(x-y)v(y) \, dy \, dx.
$$

(3.9)

It follows from (3.7), (3.8) and (3.9) that

$$
\int_{\mathbb{R}^n} -L_\theta^s u(x)v(x) \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \theta(x-y)v(x) \, dy \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x-y|^{N+2s}} \theta(x-y)v(y) \, dy \, dx
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y) \, dy \, dx.
$$

This, combined with (3.6) yields

$$
\int_{\mathbb{R}^n} -L_\theta^s u_f(x)v(x) \, dx + \int_{\Omega} (h(x, u_f(x)) - f(x))v(x) \, dx = 0, \quad \forall v \in H^s(\Omega).
$$

Thus, $u_f$ is a solution of problem (1.1).

Next, we show that $u_f$ is the unique solution of (1.1). Assume that $w \in H^s(\Omega)$ is another solution of (1.1) and $u_f \neq w$. Then

$$
||u_f - w|| > 0.
$$

(3.10)

Since $h(x, \cdot)$ is non-decreasing,

$$
\int_{\Omega} (h(x, u_f) - h(x, w))(u_f - w) \, dx \geq 0.
$$

(3.11)

From (3.6) we obtain that for every $v \in H^s(\Omega),

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_f(x) - u_f(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y) \, dy \, dx + \int_{\Omega} h(x, u_f)v \, dx = \int_{\Omega} f \, v \, dx,
$$

(3.12)

$$
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(w(x) - w(y))(v(x) - v(y))}{|x-y|^{N+2s}} \theta(x-y) \, dy \, dx + \int_{\Omega} h(x, w)v \, dx = \int_{\Omega} f \, v \, dx.
$$

(3.13)
From these two equalities, we obtain that for every \( v \in H^s(\Omega) \),
\[
\int_\Omega (h(x, u_f) - h(x, w))v dx = \int_\mathbb{R}^N \int_\mathbb{R}^N \frac{(w(x) - u_f(x) - w(y) + u_f(y))(v(x) - v(y))}{|x - y|^{N+2s}} \theta(x - y) dx dy.
\]

Let \( v = u_f - w \) we have
\[
\int_\Omega (h(x, u_f) - h(x, w))(u_f - w) dx = \int_\mathbb{R}^N \int_\mathbb{R}^N \frac{-(v(x) - v(y))}{|x - y|^{N+2s}} \theta(x - y) dx dy \\
\leq -\theta_0 \|v\|^2 < 0,
\]
the last inequality above comes from (3.10). So, (3.14) contradicts (3.11). Therefore, we have proved that \( u_f \) is the unique solution of (1.1). \( \square \)

Let \( u_f \) be the unique solution of (1.1). Recall that \( \Psi(f) \) is defined by (1.6). Considering the optimization problem \( \text{(Opt*)} \), we have the following result.

**Theorem 3.2.** Suppose that (H1–(H4) hold. Then for a fixed nonnegative function \( f_0 \in L^\infty(\Omega) \) there exists \( f_* \in \mathcal{A}(f_0) \) which solves the minimization optimization problem \( \text{(Opt*)} \), i.e.,
\[
\Psi(f_*) = \inf_{f \in \mathcal{A}(f_0)} \Psi(f).
\]

**Proof.** Let \( A = \inf_{f \in \mathcal{A}(f_0)} \Psi(f) \), then \( A \) is well-defined. Indeed, for each \( f \in \mathcal{A}(f_0) \), we have
\[
\Psi(f) = \frac{1}{2} \left( \int_\mathbb{R}^N \int_\mathbb{R}^N \frac{(u_f(x) - u_f(y))^2}{|x - y|^{N+2s}} \theta(x - y) dx dy \\
+ (H(x, u_f)dx - \int_\Omega f u_f dx \right)
\geq \frac{\theta_0}{2} \|u_f\|^2 - C(\|f\|_\infty \|u_f\| + \|u_f\|^{l+1}).
\]
Noting that \( \|f\|_\infty = \|f_0\|_\infty \) and \( l + 1 < 2 \), we deduce that \( A \) must be finite.

Let \( \{f_i\} \subset \mathcal{A}(f_0) \) be such that \( \Psi(f_i) \rightarrow A \) as \( i \rightarrow \infty \) and we denote \( u_i = u_{f_i} \). It follows from (3.15) that \( \{u_i\} \) is bounded in \( H^s(\Omega) \), then it has a subsequence (still denoted \( \{u_i\} \)) which converges weakly to \( u \in H^s(\Omega) \) and strongly to \( u \) in \( L^2(\Omega) \).

On the other hand, since \( \|f_i\|_{L^\infty} = \|f_0\|_{L^\infty} \), \( \{f_i\} \) is bounded in \( L^2(\Omega) \), it must contain a subsequence (still denoted \( \{f_i\} \)) converging weakly to some \( f \in \mathcal{A}(f_0) \), the weak closure of \( \mathcal{A}(f_0) \) in \( L^2(\Omega) \). Then
\[
| \int_\Omega (f_i - f) u dx | \rightarrow 0 \quad \text{as} \quad i \rightarrow \infty,
\]
since \( u \in L^2(\Omega) \). It follows from the Hölder inequality that
\[
| \int_\Omega (f_i u_i - f u) dx | \leq | \int_\Omega f_i (u_i - u) dx | + | \int_\Omega (f_i - f) u dx | \\
\leq \|f_i\|_2 \|u_i - u\|_2 + | \int_\Omega (f_i - f) u dx | \rightarrow 0
\]
as \( i \to \infty \). By (3.5), (3.16) and the weak lower semi-continuity of the norm in \( H^s(\Omega) \), we obtain

\[
A = \lim_{i \to \infty} \Psi(f_i) \\
\geq \frac{1}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy \right) + \int_{\Omega} H(x,u) \, dx - \int_{\Omega} \tilde{f} \, u \, dx. 
\] (3.17)

From Lemma 2.5 we infer the existence of \( \hat{f} \in \mathcal{A}(f_0) \) which maximizes the linear functional \( \int_{\Omega} h \, u \, dx \), relative to \( h \in \mathcal{A}(f_0) \). As a consequence,

\[
\int_{\Omega} \tilde{f} \, u \, dx \leq \int_{\Omega} \hat{f} \, u \, dx.
\]

Combining this with (3.17), we obtain

\[
A \geq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy + \int_{\Omega} H(x,u) \, dx - \int_{\Omega} \tilde{f} \, u \, dx. 
\] (3.18)

By Proposition 3.1,

\[
\Psi(\hat{f}) = \inf_{v \in H^s(\Omega)} \left( \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy + \int_{\Omega} (H(x,v) - \hat{f} \, v) \, dx \right) \\
\leq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy + \int_{\Omega} (H(x,u) - \tilde{f} \, u) \, dx.
\] (3.19)

It follows from (3.18) and (3.19) that \( \Psi(\hat{f}) \leq A \).

On the other hand, recall that \( A = \inf_{f \in \mathcal{A}(f_0)} \Psi(f) \) and \( \hat{f} \in \mathcal{A}(f_0) \), we must have \( A \leq \Psi(\hat{f}) \). So that \( A = \Psi(\hat{f}) \). We complete the proof by letting \( f_* = \hat{f} \). \( \square \)

4. MAXIMIZATION AND MINIMIZATION RELATED TO (1.3)

In this section, we consider two optimization problems (Opt1) and (Opt2) related to (1.3). The energy functional of (1.3) is

\[
J(u) = \frac{1}{2} \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy \right) - \int_{\Omega} f(x) u(x) \, dx.
\]

It is easy to see \( J \in C^1(H^s(\Omega), \mathbb{R}) \) and

\[
J'(u)v = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)][v(x) - v(y)]}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy - \int_{\Omega} f(x)v(x) \, dx.
\]

\( u \in H^s(\Omega) \) is a solution of the problem (1.3) if and only if \( J'(u)v = 0, \forall v \in H^s(\Omega) \).

By proposition 3.1 we have the following result.

**Proposition 4.1.** Assume that \( \theta \) satisfies (H3) and (H4). Then for each \( f \) in \( L^\infty(\Omega) \), (1.3) has a unique solution \( u_f \in H^s(\Omega) \), and \( J(u_f) = \inf_{v \in H^s(\Omega)} J(v) \).

**Remark 4.2.** Since \( J'(u_f)u_f = 0 \),

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_f(x) - u_f(y))^2}{|x - y|^{N+2s}} \theta(x - y) \, dx \, dy = \int_{\Omega} f(x) u_f(x) \, dx. 
\] (4.1)
Recall that $\Phi(f) = \int_\Omega f u_f dx$, we have

$$\Phi(f) = -2 \left( \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(uf(x) - uf(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy - \int_\Omega f(x) u_f(x) \, dx \right)$$

$$\quad = -2 \inf_{v \in H^s(\Omega)} \left( \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy - \int_\Omega f(x) v(x) \, dx \right)$$

$$\quad = -2 J(u_f).$$

**Theorem 4.3.** Assume that $\theta$ satisfies (H3) and (H4). Then for each non-negative function $f_0 \in L^\infty(\Omega)$, there exists $f_1 \in \mathcal{A}(f_0)$ which solves the maximization optimization problem (Opt1), i.e.,

$$\Phi(f_1) = \int_\Omega f_1 u_1 \, dx = \sup_{f \in \mathcal{A}(f_0)} \int_\Omega f u_f \, dx = \sup_{f \in \mathcal{A}(f_0)} \Phi(f)$$

where $u_1 = u_{f_1}$. Moreover, if $s = \frac{1}{2}$ then there exists an increasing function $\phi$ such that $f_1 = \phi(u_1)$ almost everywhere in $\Omega$.

**Proof.** Let

$$M = \sup_{f \in \mathcal{A}(f_0)} \int_\Omega f u_f \, dx.$$

We first show that $M$ is finite. Let $f \in \mathcal{A}(f_0)$. From (H4), (4.1) and Hölder’s inequality we find

$$\theta_0 \|u_f\|^2 \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(uf(x) - uf(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy$$

$$\quad = \int_\Omega f u_f \, dx \leq \|f\|_{\infty} \|u_f\|_1. \quad (4.2)$$

Since $\|f\|_{\infty} = \|f_0\|_{\infty}$, it follows from (4.2) and Lemma 2.1 that $M$ is finite.

Let $\{f_i\}$ be a maximizing sequence and let $u_i = u_{f_i}$. From (4.2) it is clear that $\{u_i\}$ is bounded in $H^s(\Omega)$, hence it has a subsequence (still denoted $\{u_i\}$) that converges weakly to $u \in H^s(\Omega)$. We also infer that $\{u_i\}$ converges strongly to $u$ in $L^2(\mathbb{R}^N)$. On the other hand, since $\{f_i\}$ is bounded in $L^\infty(\Omega)$, it must contain a subsequence (still denoted $\{f_i\}$) converging weakly to $\tilde{f} \in L^2(\Omega)$. Note that $\tilde{f} \in \mathcal{A}(f_0)$, the weak closure of $\mathcal{A}(f_0)$ in $L^2(\Omega)$. Thus, using the weak lower semi-continuity of the $H^s(\Omega)$ norm and (3.2) we obtain

$$M = \lim_{i \to \infty} \int_\Omega f_i u_i \, dx$$

$$\quad = \lim_{i \to \infty} \left( \int_\Omega 2f_i u_i \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_i(x) - u_i(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy \right)$$

$$\quad \leq \int_\Omega 2f u \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy. \quad (4.3)$$

Note that from Lemma 2.3 we infer the existence of $\hat{f} \in \mathcal{A}(f_0)$ that maximizes the linear functional $\int_\Omega h u \, dx$, relative to $h \in \mathcal{A}(f_0)$. As a consequence we obtain

$$\int_\Omega \hat{f} u \, dx \leq \int_\Omega \tilde{f} u \, dx. \quad (4.4)$$
Applying (4.1), (4.3) and (4.4) we find
\[ M \leq \int_{\Omega} 2\hat{f}u \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy \]
\[ \leq \int_{\Omega} 2\hat{f}u \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\hat{u}(x) - \hat{u}(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy \]
\[ = \int_{\Omega} \hat{f}u \, dx \leq M \]

where \( \hat{u} = u \). Thus let \( f_1 = \hat{f} \) we complete the first part of the proof.

We then show that \( f_1 \) is also a maximizer of the functional \( \int_{\Omega} hu_1 \, dx \), relative to \( h \in \mathcal{R}(f_0) \). In fact, we notice that for each \( g \in \mathcal{R}(f_0) \),
\[ \int_{\Omega} f_1 u_1 \, dx \geq \int_{\Omega} gu_1 \, dx \]
\[ \geq \int_{\Omega} 2gu_1 \, dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_1(x) - u_1(y))^2}{|x-y|^{N+2s}} \theta(x-y) \, dx \, dy \]
\[ = \int_{\Omega} 2gu_1 \, dx - \int_{\Omega} f_1 u_1 \, dx, \]

which implies that
\[ \int_{\Omega} f_1 u_1 \, dx \geq \int_{\Omega} gu_1 \, dx, \quad \forall g \in \mathcal{R}(f_0). \quad (4.5) \]

If \( g \in \mathcal{R}(f_0) \) then we may choose a sequence \( \{g_n\} \subset \mathcal{R}(f_0) \) such that \( \{g_n\} \) converge weakly to \( g \) in \( L^2(\Omega) \). By (4.5), we obtain
\[ \int_{\Omega} f_1 u_1 \, dx \geq \int_{\Omega} g_n u_1 \, dx \rightarrow \int_{\Omega} gu_1 \, dx \]
as \( n \rightarrow \infty \). So that
\[ \int_{\Omega} f_1 u_1 \, dx \geq \int_{\Omega} gu_1 \, dx, \quad \forall g \in \mathcal{R}(f_0) \quad (4.6) \]

and our claim is valid.

Let \( E = \{ x \in \Omega : f_1(x) = 0 \} \) and define \( S = \sup_{x \in E} u_1(x) \), we claim that \( u_1(x) \geq S \) on \( E^c \) almost everywhere. Arguing by contradiction suppose the claim is false. Therefore there exists a number \( S_1 < S \) and a subset \( A \) of \( E^c \) with \( |A| > 0 \) such that \( u_1(x) < S_1 \) on \( A \) almost everywhere. Now let \( S_1 < S_2 < S \). We can find a set \( D \) of positive measure contained in \( E \) such that \( u_1(x) > S_2 \) on \( D \) almost everywhere. We can assume \( |A| = |D| \). Next, consider a measure-preserving map \( T : A \rightarrow D \). Using \( T \) we define a particular rearrangement of \( f_1 \), denoted by \( \overline{f} \), as follows:
\[
\overline{f} = \begin{cases}
  f_1(Tx) & x \in A, \\
  f_1(T^{-1}x) & x \in D, \\
  f_1(x) & x \in \Omega \setminus (A \cup D).
\end{cases}
\]

Thus
\[ \int_{\Omega} \overline{f} u_1 \, dx - \int_{\Omega} f_1 u_1 \, dx \]
\[ = \int_{A \cup D} \overline{f} u_1 \, dx - \int_{A \cup D} f_1 u_1 \, dx \]
\[
\int_A \mathcal{T} u_1 dx + \int_A (u_1 \ast T) f_1 dx - \int_A f_1 u_1 dx - \int_A (u_1 \ast T) f_1 dx
= \int_A (u_1 \ast T - u_1)(f_1 - \mathcal{T} f_1) dx > (S_2 - S_1) \int_A f_1 dx > 0.
\]

Therefore \( \int_\Omega \mathcal{T} u_1 dx > \int_\Omega f_1 u_1 dx \), which contradicts (4.6).

Since \( s = \frac{1}{2} \), by Lemma 2.6 we know that \( u_1 \) has no level set of positive measure on \( E^c \), then by Lemma 2.3 we infer the existence of an increasing function \( \phi_1 \) such that \( \phi_1(u_1) \) is a rearrangement of \( f_1 \) on \( E^c \). Now we define an increasing function \( \phi_2 \) as
\[
\phi_2(t) = \begin{cases} 
0 & t < S, \\
\phi_1(t) & t \geq S.
\end{cases}
\]

It is easy to check that \( \phi_2(u_1) \) is a rearrangement of \( f_1 \) on \( \Omega \). By (4.6) and Lemma 2.4 we infer that \( f_1 = \phi_2(u_1) \), so that we complete the proof. \( \square \)

**Theorem 4.4.** Suppose that \( f_0 \) is positive, \( \theta(x) \equiv m \geq \theta_0 \) and \( s = 1/2 \). Then there exists \( f_2 \in \mathcal{R}(f_0) \) which solves the minimization optimization problem (Opt2), i.e.,
\[
\Phi(f_2) = \int_\Omega f_2 u_2 dx = \inf_{f \in \mathcal{R}(f_0)} \int_\Omega f u_f dx = \inf_{f \in \mathcal{R}(f_0)} \Phi(f)
\]
where \( u_2 = u_{f_2} \).

We need some lemmas before we give the proof of the above theorem.

**Lemma 4.5.** The functional \( \Phi|_{\mathcal{R}(f_0)} \) is strictly convex.

**Proof.** Let \( g, h \in \mathcal{R}(f_0) \) and \( v \in H^s(\Omega) \), then for all \( t \in (0, 1) \), we have
\[
2 \int_\Omega (tg + (1 - t)h) v dx - m \|v\|^2
= t \left( \int_\Omega 2gv dx - m \|v\|^2 \right) + (1 - t) \left( \int_\Omega 2hv dx - m \|v\|^2 \right).
\]

By (1.4) and (1.5), and taking the superior relative to \( v \in H^s(\Omega) \) in both sides of the above equality, we obtain
\[
\Phi(tg + (1 - t)h) \leq t\Phi(g) + (1 - t)\Phi(h);
\]
that is, the convexity of \( \Phi \) has been proved. Now, suppose that equality holds in the above inequality for some \( t \in (0, 1) \). Then, denote by \( u_t \) the solution of the problem (1.3) corresponding to \( tg + (1 - t)h \), we have
\[
t \left( \int_\Omega 2g_u dx - m \|u_t\|^2 \right) + (1 - t) \left( \int_\Omega 2h_u dx - m \|u_t\|^2 \right)
= t \left( \int_\Omega 2g_u dx - m \|u_t\|^2 \right) + (1 - t) \left( \int_\Omega 2h_u dx - m \|u_t\|^2 \right).
\]
It follows that
\[
\int_\Omega 2g_u dx - m \|u_t\|^2 = \int_\Omega 2g_u dx - m \|u_t\|^2,
\]
\[
\int_\Omega 2h_u dx - m \|u_t\|^2 = \int_\Omega 2h_u dx - m \|u_t\|^2.
\]
By the uniqueness of the minimizer of the functional $J$, we must have $u_t = u_g = u_h$. Moreover, since

\[ m \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} g(x)v(x) \, dx, \quad \forall v \in H^s(\Omega), \]

\[ m \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_h(x) - u_h(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} h(x)v(x) \, dx, \quad \forall v \in H^s(\Omega), \]

if $u_g = u_h$, we must have $g(x) = h(x)$ a.e. in $\Omega$, and the strict concavity is proved.

\[ \square \]

**Lemma 4.6.** The functional $\Phi|_{\mathcal{A}(f_0)}$ is weakly continuous.

**Proof.** Suppose $\{g_i\} \subset \mathcal{A}(f_0)$ and $g_i \rightharpoonup g \in \mathcal{A}(f_0)$. If $u_i$ and $u_g$ are the corresponding solutions of (1.3) we have

\[ \Phi(g) + \int_{\Omega} 2(g_i - g)u_g \, dx = \int_{\Omega} 2g_i u_g \, dx - m\|u_g\|^2 \]

\[ \leq \Phi(g_i) \]

\[ = \int_{\Omega} 2g_i u_g \, dx - m\|u_i\|^2 + \int_{\Omega} 2(g_i - g)u_i \, dx \]

\[ \leq \Phi(g) + \int_{\Omega} 2(g_i - g)u_i \, dx. \]

We have

\[ \lim_{i \to \infty} \int_{\Omega} (g_i - g)u_g \, dx = 0. \]

We only need to prove that

\[ \lim_{i \to \infty} \int_{\Omega} (g_i - g)u_i \, dx = 0. \quad (4.7) \]

Since $J(u_i) \leq J(0) = 0$ and

\[ J(u_i) \geq \frac{m}{2}\|u_i\|^2 - C\|g_i\|\|u_i\|, \]

it follows that $\{u_i\}$ is bounded in $H^s(\Omega)$. Hence,

\[ |\int_{\Omega} (g_i - g)u_i \, dx| \leq C\|u_i\| \leq C. \]

Now we can choose a subsequence $\{u_{i_j}\}$ such that

\[ \lim_{j \to \infty} |\int_{\Omega} (g_{i_j} - g)u_{i_j} \, dx| = \limsup_{i \to \infty} |\int_{\Omega} (g_i - g)u_i \, dx|. \]

Noting that $\{u_{i_j}\}$ is also bounded in $H^s(\Omega)$, going if necessary to a subsequence, we may assume that $u_{i_j} \rightharpoonup u$ in $H^s(\Omega)$ and $u_{i_j} \to u$ in $L^2(\Omega)$ as $j \to \infty$. By the Hölder inequality, we obtain

\[ |\int_{\Omega} (g_{i_j} - g)u_{i_j} \, dx| \leq |\int_{\Omega} (g_{i_j} - g)(u_{i_j} - u) \, dx| + |\int_{\Omega} (g_{i_j} - g)u \, dx| \]

\[ \leq \|g_{i_j} - g\|_{L^2}\|u_{i_j} - u\|_{L^2} + |\int_{\Omega} (g_{i_j} - g)u \, dx| \to 0 \]
as \( j \to \infty \). So that
\[
0 \leq \liminf_{i \to \infty} \left| \int_{\Omega} (g_i - g)u_i \, dx \right| \leq \limsup_{i \to \infty} \left| \int_{\Omega} (g_i - g)u_i \, dx \right| \leq 0,
\]
which implies (4.7), and then \( \Phi(g_i) \to \Phi(g) \). \( \square \)

Proof of Theorem 4.4. Denote by \( \mathcal{R}(f_0) \) the weak closure of \( \mathcal{R}(f_0) \) in \( L^2(\Omega) \), then \( \mathcal{R}(f_0) \) is convex and weakly compact in \( L^2(\Omega) \) by Lemma 2.2. By using Lemma 4.5 and Lemma 4.6, one can prove easily that \( \Phi \) has a unique minimizer in \( \mathcal{R}(f_0) \).

Let \( g \in \mathcal{R}(f_0) \) be the minimizer of \( \Phi \). If \( g \in \mathcal{R}(f_0) \) and \( 0 < t < 1 \) we have
\[
g_t = tg + (1 - t)g \in \mathcal{R}(f_0).
\]
By the minimality condition, we have
\[
\Phi(g_t) \leq \Phi(g_t).
\]
(4.8)

If \( u_g, u_\frac{1}{2} \) and \( u_t \) are solutions of (1.3) corresponding to \( g, \frac{1}{2} \) and \( g_t \), respectively, we have
\[
u_t = tu_g + (1 - t)u_g.
\]
Therefore, by (1.4), (1.5) and (4.8), we have
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))^2}{|x - y|^{N+s}} \, dx \, dy
\]
\[
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{t(u_g(x) - u_g(y)) + (1 - t)(u_g(x) - u_g(y))}{|x - y|^{N+s}} \right)^2 \, dx \, dy.
\]
After simplification, we obtain
\[
(1 + t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))^2}{|x - y|^{N+s}} \, dx \, dy
\]
\[
\leq 2t \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))(u_g(x) - u_g(y))}{|x - y|^{N+s}} \, dx \, dy
\]
\[
+ (1 - t) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))^2}{|x - y|^{N+s}} \, dx \, dy.
\]
Letting \( t \to 1 \), we find
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))^2}{|x - y|^{N+s}} \, dx \, dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u_g(x) - u_g(y))(u_g(x) - u_g(y))}{|x - y|^{N+s}} \, dx \, dy.
\]
We can rewrite the latter inequality as
\[
\int_{\Omega} g \, dx = \int_{\Omega} g \, dx, \quad \forall g \in \mathcal{R}(f_0).
\]
Since \( f_0 \) is positive it is easy to deduce that \( g > 0 \) a.e. \( x \in \Omega \). By Lemma 2.6 we have that each level set of \( u_\frac{1}{2} \) has zero measure. By Lemmas 2.3 and 2.4 there exists a decreasing function \( \phi \) such that \( \phi \circ u_\frac{1}{2} \) is a rearrangement of \( \frac{1}{2} \) and the unique minimizer of the linear functional \( \int_{\Omega} gu_\frac{1}{2} \, dx \), related to \( g \in \mathcal{R}(f_0) \). So that
\[
g = \phi \circ u_\frac{1}{2} \in \mathcal{R}(f_0).
\]
We complete the proof by letting \( f_2 = g \). \( \square \)

Acknowledgements. This work was supported by Natural Science Foundation of China (11171247, 11371273, 11471235, 11501437) and the scientific research fund project of Taizhou University (TZXY2015QDXM033). The authors are grateful to the anonymous referee for the useful comments.
References


Chong Qiu
Department of Mathematics, Taizhou University, Taizhou 225300, China
E-mail address: qchauda@163.com
YISHENG HUANG  
DEPARTMENT OF MATHEMATICS, SOOCHOW UNIVERSITY, SUZHOU 215006, CHINA  
E-mail address: yishenh@suda.edu.cn

YUYING ZHOU (CORRESPONDING AUTHOR)  
DEPARTMENT OF MATHEMATICS, SOOCHOW UNIVERSITY, SUZHOU 215006, CHINA  
E-mail address: yuyings@suda.edu.cn