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MULTIPLE SOLUTIONS FOR CRITICAL ELLIPTIC PROBLEMS WITH FRACTIONAL LAPLACIAN

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Abstract. This article is devoted to the study of the nonlocal fractional equation involving critical nonlinearities

$$\begin{gathered}(-\Delta)^{\alpha/2}u = \lambda u + |u|^{2^*_\alpha - 2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2\alpha$, $\alpha \in (0,2)$, $\lambda \in (0,\lambda_1)$ and $2^*_\alpha = \frac{2N}{N-\alpha}$ is critical exponent. We show the existence of at least $\mathrm{cat}_\Omega(\Omega)$ nontrivial solutions for this problem.

1. Introduction

This article concerns the critical elliptic problem with the fractional Laplacian

$$\begin{gathered}(-\Delta)^{\alpha/2}u = \lambda u + |u|^{2^*_\alpha - 2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N > \alpha$, $\alpha \in (0,2)$ is fixed and $2^*_\alpha = \frac{2N}{N-\alpha}$ is the critical Sobolev exponent.

In a bounded domain $\Omega \subset \mathbb{R}^N$, the operator $(-\Delta)^{\alpha/2}$ can be defined as in [3, 6] as follows. Let $\{(\lambda_k, \varphi_k)\}_{k=1}^\infty$ be the eigenvalues and corresponding eigenfunctions of the Laplacian $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$ normalized by $\|\varphi_k\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in } \Omega; \quad \varphi_k = 0 \quad \text{on } \partial\Omega.$$

We define the space $H_0^{\alpha/2}(\Omega)$ by

$$H_0^{\alpha/2}(\Omega) = \{u = \sum_{k=1}^\infty u_k\varphi_k \in L^2(\Omega) : \sum_{k=1}^\infty u_k^2\lambda_k^{\frac{\alpha}{2}} < \infty\},$$

which is equipped with the norm

$$\|u\|_{H_0^{\alpha/2}(\Omega)} = \Big(\sum_{k=1}^\infty u_k^2\lambda_k^{\frac{\alpha}{2}}\Big)^{\frac{1}{2}}.$$

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For $u \in H_0^{\alpha/2}(\Omega)$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined by

$$(-\Delta)^{\alpha/2}u = \sum_{k=1}^{\infty} u_k \lambda_k^{\alpha/2} \varphi_k.$$

Problem (1.1) is the Brézis-Nirenberg type problem with the fractional Laplacian. Brézis and Nirenberg [4] considered the existence of positive solutions for problem (1.1) with $\alpha = 2$. Such a problem involves the critical Sobolev exponent $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and it is well known that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact even if Ω is bounded. Hence, the associated functional of problem (1.1) does not satisfy the Palais-Smale condition, and critical point theory cannot be applied directly to find solutions of the problem. However, it is found in [4] that the functional satisfies the $(PS)_c$ condition for $c \in (0, \frac{1}{N}S^{N/2})$, where S is the best Sobolev constant and $\frac{1}{N}S^{N/2}$ is the least level at which the Palais-Smale condition fails. So a positive solution can be found if the mountain pass value corresponding to problem (1.1) is strictly less than $\frac{1}{N}S^{N/2}$.

Problems with the fractional Laplacian have been extensively studied, see for example [2, 3, 5, 6, 7, 9, 10, 12, 13] and the references therein. In particular, the Brézis-Nirenberg type problem was discussed in [12] for the special case $\alpha = \frac{1}{2}$, and in [2] for the general case, $0 < \alpha < 2$, where existence of one positive solution was proved. To use the idea in [4] to prove the existence of one positive solution for the fractional Laplacian, the authors in [2, 12] used the following results in [10] (see also [3]): for any $u \in H_0^{\alpha}(\Omega)$, the solution $v \in H_{0,L}^1(\mathcal{C}_\Omega)$ of the problem

$$\begin{gathered} -\operatorname{div}(y^{1-\alpha}\nabla v) = 0, \quad \text{in } \mathcal{C}_\Omega = \Omega \times (0,\infty), \\ v = 0, \quad \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0,\infty), \\ v = u, \quad \text{on } \Omega \times \{0\}, \end{gathered} \tag{1.2}$$

satisfies

$$-\lim_{y\to 0^+} k_\alpha y^{1-\alpha} \frac{\partial v}{\partial y} = (-\Delta)^\alpha u,$$

where we use $(x, y) = (x_1, \dots, x_N, y) \in \mathbb{R}^{N+1}$, and

$$H_{0,L}^1(\mathcal{C}_\Omega) = \big\{ w \in L^2(\mathcal{C}_\Omega) : w = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w|^2 \,dx\,dy < \infty \big\}. \tag{1.3}$$

Therefore, the nonlocal problem (1.1) can be reformulated as the local problem

$$\begin{gathered} -\operatorname{div}(y^{1-\alpha}\nabla w) = 0, \quad \text{in } \mathcal{C}_\Omega, \\ v = 0, \quad \text{on } \partial_L \mathcal{C}_\Omega, \\ \lim_{y\to 0^+} y^{1-\alpha} \frac{\partial w}{\partial \nu} = |w(x,0)|^{2_\alpha^* - 2} w(x,0) + \lambda w(x,0), \quad \text{on } \Omega \times \{0\}, \end{gathered} \tag{1.4}$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative of $\partial \mathcal{C}_\Omega$. Hence, critical points of the functional

$$\begin{aligned} J(w) = \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w|^2 \,dx\,dy - \frac{1}{2_\alpha^*} \int_{\Omega\times\{0\}} |w(x,0)|^{2_\alpha^*} \,dx \\ - \frac{\lambda}{2} \int_{\Omega\times\{0\}} |w(x,0)|^2 \,dx \end{aligned} \tag{1.5}$$

defined on $H^1_{0,L}(\mathcal{C}_\Omega)$ correspond to solutions of (1.4), and the trace $u = tr\, w$ of w is a solution of (1.1). A critical point of the functional $J(u)$ at the mountain pass level was found in [2, 12]. On the other hand, it can be shown by using the Pohozaev type identity that the problem

$$\begin{gathered}(-\Delta)^{\alpha/2}u = |u|^{p-1}u \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega\end{gathered}$$

has no nontrivial solution if $p+1 \geq \frac{2N}{N-\alpha}$ and Ω is star-shaped, see for example [3] and [12].

It is well-known that if Ω has a rich topology, (1.1) with $\alpha = 2$, $\lambda = 0$ has a solution, see [1, 8, 14] etc. In this paper, we assume $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of the fractional Laplacian $(-\Delta)^{\alpha/2}$. We investigate the existence of multiple solutions of problem (1.1). Let A be a closed subset of a topology space X. The category of A is the least integer n such that there exist n closed subsets $A_1, \dots, A_n$ of X satisfying $A = \cup_{j=1}^n A_j$ and $A_1, \dots, A_n$ are contractible in X. Our main result is as follows.

Theorem 1.1. *If Ω is a smooth bounded domain of $\mathbb{R}^{\mathbb{N}}$, $N \geq 4$, $0 < \alpha < 2$ and $0 < \lambda < \lambda_1$, problem* (1.4) *has at least* $\operatorname{cat}_\Omega(\Omega)$ *nontrivial solutions. Equivalently,* (1.1) *possesses at least* $\operatorname{cat}_\Omega(\Omega)$ *positive solutions.*

We say that $w \in H^1_{0,L}(\mathcal{C}_\Omega)$ is a solution to (1.4) if for every function $\varphi \in H^1_{0,L}(\mathcal{C}_\Omega)$, we have

$$k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\langle \nabla w, \nabla \varphi\rangle \,dx\,dy = \int_\Omega (\lambda w + w^{\frac{N+\alpha}{N-\alpha}})\varphi\,dx. \tag{1.6}$$

We will find solutions of J at energy levels below a value related to the best Sobolev constant $S_{\alpha,N}$, where

$$S_{\alpha,N} = \inf_{w\in H^1_{0,L}(\mathcal{C}_\Omega), w\neq 0} \frac{k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dx\,dy}{(\int_\Omega |w(x,0)|^{2^*_\alpha}\,dx)^{2/(2^*_\alpha)}}, \tag{1.7}$$

which is not achieved in any bounded domain and is indeed achieved in the case $\Omega = \mathbb{R}^{N+1}_+$. We know from [2] that the trace $u_\epsilon(x) = w_\epsilon(x,0)$ of the family of minimizers w_ϵ of $S_{\alpha,N}$ takes the form

$$u(x) = u_\epsilon(x) = \frac{\epsilon^{\frac{N-\alpha}{2}}}{(|x|^2+\epsilon^2)^{\frac{N-\alpha}{2}}}, \tag{1.8}$$

with $\epsilon > 0$. Using this property, we are able to find critical values of J in a right range.

In section 2, we prove the $(PS)_c$ condition and the main result is shown in section 3.

2. Palais-Smale condition

In this section, we show that the functional $J(w)$ satisfies $(PS)_c$ condition for c in certain interval. By a $(PS)_c$ condition for the functional $J(w)$ we mean that a sequence $\{w_n\} \subset H^1_{0,L}(\mathcal{C}_\Omega)$ such that $J(w_n) \to c$, $J'(w_n) \to 0$ contains a convergent subsequence.

Define on the space $H^1_{0,L}(\mathcal{C}_\Omega)$ the functionals

$$\psi(w) = \int_\Omega (w^+(x,0))^{2^*_\alpha}\,dx,$$

$$\varphi_\lambda(w) = k_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla w|^2 dx\,dy - \lambda \int_\Omega |w(x,0)|^2 dx.$$

We may verify as in [14] that on the manifold

$$V = \{w \in H^1_{0,L}(\mathcal{C}_\Omega) : \psi(w) = 1\},$$

$\psi'(w) \neq 0$ for every $w \in V$. Hence, the tangent space of V at v is given by

$$T_v V := \{w \in H^1_{0,L}(\mathcal{C}_\Omega) : \langle \psi'(v), w\rangle = 0\},$$

and the norm of the derivative of $\varphi_\lambda(w)$ at v restricted to V is defined by

$$\|\varphi'_\lambda(v)\|_* = \sup_{w\in T_v V, \|w\|=1} |\langle \varphi'_\lambda(v), w\rangle|.$$

It is well known that

$$\|\varphi'_\lambda(w)\|_* = \min_{\mu\in\mathbb{R}} \|\varphi'_\lambda(w) - \mu\psi'(w)\|.$$

A critical point $v \in V$ of φ_λ is a point such that $\|\varphi'_\lambda(v)\|_* = 0$.

Since λ_1 is the first eigenvalue of the fractional Laplacian $(-\Delta)^{\alpha/2}$, it can be characterized as

$$\lambda_1 = \inf_{w\in H^1_{0,L}(C_\Omega), w\neq 0} \frac{k_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla w|^2\,dx\,dy}{\int_\Omega |w(x,0)|^2\,dx}.$$

If $0 < \lambda < \lambda_1$, we see that

$$\|w\|_1 := \Big(k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dx\,dy - \lambda \int_\Omega w^2(x,0)\,dx\Big)^{1/2}$$

is an equivalent norm on $H^1_{0,L}(\mathcal{C}_\Omega)$.

Lemma 2.1. *Any sequence $\{v_n\} \subset H^1_{0,L}(\mathcal{C}_\Omega)$ such that*

$$d := \sup_n J(v_n) < C^* := \frac{\alpha}{2N} S^{\frac{N}{\alpha}}_{\alpha,N}, \quad J'(v_n) \to 0 \quad \text{in } H^{-1}_{0,L}(\mathcal{C}_\Omega)$$

contains a convergent subsequence.

Proof. It is easy to show from the assumptions that

$$\begin{aligned} d + 1 + \|v_n\|_1 &\geq J(v_n) - \frac{1}{2^*_\alpha}\langle J'(v_n), v_n\rangle \\ &= (\frac{1}{2} - \frac{1}{2^*_\alpha})\Big(\int_{\mathcal{C}_\Omega} k_\alpha y^{1-\alpha}|\nabla v_n|^2\,dx\,dy - \lambda\int_\Omega |v_n|^2\,dx\Big) \\ &= (\frac{1}{2} - \frac{1}{2^*_\alpha})\|v_n\|_1^2; \end{aligned}$$

that is, $\|v_n\|_1$ is bounded. We may assume that

$$\begin{gathered} v_n(x,y) \rightharpoonup v(x,y) \quad \text{in } H^1_{0,L}(\mathcal{C}_\Omega), \\ v_n(x,0) \to v(x,0) \quad \text{in } L^2(\Omega), \\ v_n(x,0) \to v(x,0) \quad \text{a.e. in } \Omega. \end{gathered}$$

Therefore, for every $\varphi \in H^1_{0,L}(\mathcal{C}_\Omega)$,

$$\langle J'(v_n), \varphi\rangle \to \langle J'(v), \varphi\rangle = 0$$

as $n \to +\infty$. We also have that $J(v) \geq 0$. By Brézis-Lieb's lemma,

$$J(v) + \frac{1}{2}\|v_n - v\|_1^2 - \frac{1}{2^*_\alpha}\int_\Omega (v_n - v)_+^{2^*_\alpha}\,dx = J(v_n) + o(1) = C + o(1).$$

Since $\langle J'(v_n), v_n\rangle \to 0$, we obtain

$$\begin{aligned}
&\|v_n - v\|_1^2 - 2^*_\alpha \int_\Omega (v_n - v)_+^{2^*_\alpha}\,dx \\
&= \|v_n\|_1^2 - \|v\|_1^2 - 2^*_\alpha \int_\Omega \big((v_n)_+^{2^*_\alpha} - v_+^{2^*_\alpha}\big)\,dx + o(1) \\
&= -\|v\|_1^2 + 2^*_\alpha \int_\Omega v_+^{2^*_\alpha}\,dx \\
&= -\langle J'(v), v\rangle = 0.
\end{aligned}$$

Hence, there exist a constant b such that

$$\|v_n - v\|_1^2 \to b, \quad 2^*_\alpha \int_\Omega (v_n)_+^{2^*_\alpha}\,dx \to b, \quad \text{as } n \to +\infty.$$

It follows by $v_n \to v$ in $L^2(\Omega)$ that

$$\int_{\mathcal{C}_\Omega} k_\alpha y^{1-\alpha}|\nabla(v_n - v)|^2\,dx\,dy \to b.$$

The trace inequality

$$\int_{\mathcal{C}_\Omega} k_\alpha y^{1-\alpha}|\nabla(v_n - v)|^2\,dx\,dy \geq S_{\alpha,N}\|(v_n - v)(x,0)\|^2_{L^{2^*_\alpha}(\Omega)}$$

implies $b \geq S_{\alpha,N} b^{\frac{2}{2^*_\alpha}}$. Hence, either $b = 0$ or $b \geq S_{\alpha,N}^{\frac{N}{\alpha}}$.

If $b = 0$, then $v_n \to v$ in $H^1_{0,L}(\mathcal{C}_\Omega)$, and the proof is complete. If $b \geq S_{\alpha,N}^{\frac{N}{\alpha}}$, we deduce that

$$\begin{aligned}
C^* &= \frac{\alpha}{2N} S_{\alpha,N}^{\alpha/N} \\
&\leq (\frac{1}{2} - \frac{1}{2^*_\alpha})b \\
&= (\frac{1}{2} - \frac{1}{2^*_\alpha})\|v_n - v\|_1^2 + o(1) \\
&\leq J(v) + \frac{1}{2}\|v_n - v\|_1^2 - \frac{1}{2^*_\alpha}\|v_n - v\|_1^2 + o(1) \\
&= J(v) + \frac{1}{2}\|v_n - v\|_1^2 - \frac{1}{2^*_\alpha}\int_\Omega (v_n - v)_+^{2^*_\alpha} + o(1) \\
&= C \leq d < C^*,
\end{aligned}$$

which is a contradiction. □

Alternatively, we have the following result.

Lemma 2.2. *Every sequence $\{w_n\} \in V$ satisfying $\varphi_\lambda(w_n) \to c < S_{\alpha,N}$ and $\|\varphi'_\lambda(w_n)\|_* \to 0$, as $n \to +\infty$, contains a convergent subsequence.*

Proof. Since

$$\|\varphi'_\lambda(w_n)\|_* = \min_{\mu\in\mathbb{R}} \|\varphi'_\lambda(w_n) - \mu\psi'(w_n)\|,$$

there exists a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $\|\varphi'_\lambda(w_n) - \alpha_n\psi'(w_n)\| \to 0$. It follows that for every $h \in H^1_{0,L}(\mathcal{C}_\Omega)$,

$$k_\alpha \int_{C_\Omega} y^{1-\alpha}\nabla w_n \nabla h \,dx\,dy - \lambda \int_\Omega w_n h\,dx - \mu_n \int_\Omega (w_n^+)^{2^*_\alpha - 1} h\,dx \to 0, \tag{2.1}$$

where $\mu_n = \frac{\alpha_n 2^*_\alpha}{2}$. Choosing $h = w_n$ in (2.1) and using the fact $\psi(w_n^+) = 1$, we obtain

$$\varphi_\lambda(w_n) - \mu_n = k_\alpha \int_{C_\Omega} y^{1-\alpha}|\nabla w_n|^2 - \lambda \int_\Omega |w_n|^2 - \mu_n \int_\Omega (w_n^+)^{2^*_\alpha} \to 0.$$

Whence by $\varphi_\lambda(w_n) \to c$, $\mu_n \to c$ as $n \to +\infty$. Setting $v_n := \mu_n^{\frac{N-\alpha}{2\alpha}} w_n$, we obtain

$$\begin{aligned} J(v_n) &= \frac{1}{2}\mu_n^{\frac{N-\alpha}{\alpha}} \Big(\int_{C_\Omega} k_\alpha y^{1-\alpha}|\nabla w_n|^2\,dx\,dy - \lambda\int_\Omega |w_n|^2\,dx\Big) - \frac{1}{2^*_\alpha}\mu_n^{\frac{N}{\alpha}} \int_\Omega (w_n^+)^{2^*_\alpha} \\ &= \frac{1}{2}\mu_n^{\frac{N-\alpha}{\alpha}} \varphi_\lambda(w_n) - \frac{N-\alpha}{2N}\mu_n^{\frac{N}{\alpha}}. \end{aligned}$$

Thus,

$$J(v_n) \to \frac{\alpha}{2N} c^{\frac{N}{\alpha}} < \frac{\alpha}{2N} S^{\frac{N}{\alpha}}.$$

In the same way, for every $h \in H^1_{0,L}(\mathcal{C}_\Omega)$, by (2.1),

$$\begin{aligned} &\langle J'(v_n), h\rangle \\ &= \mu_n^{\frac{N-\alpha}{2\alpha}} \Big(k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}\nabla w_n\nabla h\,dx\,dy - \lambda\int_\Omega w_n h - \mu_n \int_\Omega (w_n^+)^{2^*_\alpha-1} h\,dx\Big) \to 0. \end{aligned}$$

Now, the assertion follows by Lemma 2.1. □

Let us define

$$Q_\lambda = \inf_{w\in V} \varphi_\lambda(w) = \inf_{w\in V} \big\{k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2 dx\,dy - \lambda\int_\Omega |w(x,0)|^2\,dx\big\}.$$

Denote by $\eta_0(t) \in C^\infty(\mathbb{R}_+)$ a cut-off function, which is non-increasing and satisfies

$$\eta_0(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2}, \\ 0 & \text{if } t \ge 1. \end{cases}$$

Assume $0 \in \Omega$, for fixed $\rho > 0$ small enough such that $\overline{B}_\rho \subseteq C_\Omega$, we define the function $\eta(x,y) = \eta_\rho(x,y) = \eta_0(\frac{|(x,y)|}{\rho})$. Then $\eta w_\epsilon \in H^1_{0,L}(C_\Omega)$. It is standard to establish the following estimates, see [2] for details.

Lemma 2.3. *The family $\{\eta w_\epsilon\} \subset H^1_{0,L}(\mathcal{C}_\Omega)$ and its trace on $y = 0$ satisfy*

$$\|\eta w_\epsilon\|^2 = \|w_\epsilon\|^2 + O(\epsilon^{N-\alpha}), \tag{2.2}$$

If $N > 2\alpha$,

$$\|\eta w_\epsilon\|^2_{L^2(\Omega)} = C\epsilon^\alpha + O(\epsilon^{N-\alpha}), \tag{2.3}$$

If $N = 2\alpha$,

$$\|\eta w_\epsilon\|^2_{L^2(\Omega)} = C\epsilon^\alpha \log(\frac{1}{\epsilon}) + O(\epsilon^\alpha) \tag{2.4}$$

for $\epsilon > 0$ small enough and some $C > 0$.

Lemma 2.4. *Assume $N \geq 2\alpha$, $0 < \lambda < \lambda_1$, then*

$$Q_\lambda = \inf_{w \in V} \varphi_\lambda(w) < S_{\alpha,N}. \tag{2.5}$$

Moreover, there exists $u \in V$ such that $\varphi_\lambda(u) = Q_\lambda$.

Proof. We first show that (2.5) holds if $N \geq 2\alpha$ and $0 < \lambda < \lambda_1$. Since

$$\int_{|x|>\frac{\rho}{2}} |u_\epsilon|^{2_\alpha^*}\,dx = \int_{\{|x|\geq\frac{\rho}{2}\}} \frac{\epsilon^N}{(|x|^2+\epsilon^2)^N}\,dx \leq \frac{N2^N}{\rho^N}\epsilon^N,$$

we have

$$\begin{aligned}
\int_\Omega |\eta u_\epsilon|^{2_\alpha^*}\,dx &\geq \int_{\{|x|\leq\frac{\rho}{2}\}} |u_\epsilon|^{2_\alpha^*} dx = \|u_\epsilon\|^{2_\alpha^*}_{L^{2_\alpha^*}(\Omega)} - \int_{\{|x|\geq\frac{\rho}{2}\}} |u_\epsilon|^{2_\alpha^*} dx \\
&\geq \|u_\epsilon\|^{2_\alpha^*}_{L^{2_\alpha^*}(\Omega)} + O(\epsilon^N).
\end{aligned}$$

By Lemma 2.3, for $N > 2\alpha$, we have

$$\begin{aligned}
&\frac{k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla(\eta w_\epsilon)|^2\,dx\,dy - \lambda\int_\Omega |\eta u_\epsilon|^2\,dx}{(\int_\Omega |\eta u_\epsilon|^{2_\alpha^*}\,dx)^{\frac{2}{2_\alpha^*}}} \\
&\leq \frac{k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w_\epsilon|^2\,dx\,dy - \lambda C\epsilon^\alpha + O(\epsilon^{N-\alpha})}{\|u_\epsilon\|^2_{L^{2_\alpha^*}(\Omega)} + O(\epsilon^N)} \\
&\leq S_{\alpha,N} - \frac{\lambda C\epsilon^\alpha}{\|u_\epsilon\|^2_{L^{2_\alpha^*}(\Omega)}} + O(\epsilon^{N-\alpha}) < S_{\alpha,N}.
\end{aligned}$$

Similarly, for $N = 2\alpha$, we find for ϵ small enough such that

$$Q_\lambda \leq S_{\alpha,N} - \frac{\lambda C\epsilon^\alpha \log(\frac{1}{\epsilon})}{\|u_\epsilon\|^2_{L^{2_\alpha^*}(\Omega)}} + O(\epsilon^\alpha) < S_{\alpha,N}.$$

Consequently, inequality (2.5) holds.

Next, we show that Q_λ is achieved if $0 < \lambda < \lambda_1$. Obviously, $Q_\lambda > 0$. Now, let $\{w_n\} \subset H^1_{0,L}(\mathcal{C}_\Omega)$ be a minimizing sequence of $Q_\lambda > 0$ such that $w_n \geq 0$ and $\|w_n(x,0)\|_{L^{2_\alpha^*}(\Omega)} = 1$. The boundedness of $\{w_n\}$ implies that

$$\begin{gathered}
w_n(x,y) \rightharpoonup w(x,y) \quad \text{in } H^1_{0,L}(\mathcal{C}_\Omega), \\
w_n(x,0) \to w(x,0) \quad \text{in } L^q(\Omega), \\
w_n(x,0) \to w(x,0) \quad \text{a.e. in} \quad \Omega,
\end{gathered}$$

where $1 \leq q \leq 2_\alpha^*$. Since

$$\|w_n\|^2 = \|w_n - w\|^2 + \|w\|^2 + o(1),$$

by the Brezis-Lieb Lemma,

$$\begin{aligned}
&\|w_n\|^2 - \lambda\|w_n(x,0)\|^2_{L^2(\Omega)} \\
&= \|w_n - w\|^2 + \|w\|^2 - \lambda\|w_n(x,0)\|^2_{L^2(\Omega)} + o(1) \\
&\geq S_{\alpha,N}\|w_n(x,0) - w(x,0)\|^2_{L^{2_\alpha^*}(\Omega)} + Q_\lambda\|w(x,0)\|^2_{L^{2_\alpha^*}(\Omega)} + o(1) \\
&\geq (S_{\alpha,N} - Q_\lambda)\|w_n(x,0) - w(x,0)\|^{2_\alpha^*}_{L^{2_\alpha^*}(\Omega)} + Q_\lambda\|w_n(x,0)\|^{2_\alpha^*}_{L^{2_\alpha^*}(\Omega)} + o(1) \\
&= (S_{\alpha,N} - Q_\lambda)\|w_n(x,0) - w(x,0)\|^{2_\alpha^*}_{L^{2_\alpha^*}(\Omega)} + Q_\lambda + o(1).
\end{aligned}$$

Hence, we obtain

$$o(1) + Q_\lambda \geq (S_{\alpha,N} - Q_\lambda)\|w_n(x,0) - w(x,0)\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + Q_\lambda + o(1).$$

The $S_{\alpha,N} > Q_\lambda$ implies $w_n(x,0) \to w(x,0)$ in $L^{2^*_\alpha}(\Omega)$ and $\|w(x,0)\|_{L^{2^*_\alpha}(\Omega)} = 1$. This yields

$$Q_\lambda \leq \|w\|^2 - \lambda\|w(x,0)\|^2_{L^2(\Omega)} \leq \lim_{n\to+\infty}(\|w_n\|^2 - \lambda\|w_n(x,0)\|^2_{L^2(\Omega)}) \leq Q_\lambda;$$

that is, w is a minimizer for Q_λ. □

3. Proof of main theorem

Taking into account the concentration-compactness principle in [11], we may derive the following result, its proof can be found in [2].

Lemma 3.1. *Suppose $w_n \rightharpoonup w$ in $H^1_{0,L}(\mathcal{C}_\Omega)$, and the sequence $\{y^{1-\alpha}|\nabla w_n|^2\}$ is tight, i.e. for any $\eta > 0$ there exists $\rho_0 > 0$ such that for all n,*

$$\int_{\{y>\rho_0\}}\int_\Omega y^{1-\alpha}|\nabla w_n|^2\,dx\,dy < \eta.$$

Let $u_n = t_r w_n$ and $u = t_r w$ and let μ, ν be two non negative measures such that

$$y^{1-\alpha}|\nabla w_n|^2 \to \mu \quad \text{and} \quad |u_n|^{2^*_\alpha} \to \nu \tag{3.1}$$

in the sense of measures as $n \to \infty$. Then, there exist an at most countable set I and points $x_i \in \Omega$ with $i \in I$ such that

(1) $\nu = |u|^{2^*_\alpha} + \Sigma_{k\in I}\nu_k\delta_{x_k}$, $\nu_k > 0$,
(2) $\mu = y^{1-\alpha}|\nabla w|^2 + \Sigma_{k\in I}\mu_k\delta_{x_k}$, $\mu_k > 0$,
(3) $\mu_k \geq S_{\alpha,N}\nu_k^{\frac{2}{2^*_\alpha}}$.

On the manifold V, we define the mapping $\beta : V \to \mathbb{R}^N$ by

$$\beta(w) := \int_\Omega x(w^+(x,0))^{2^*_\alpha}\,dx,$$

which has the following properties.

Lemma 3.2. *Let $\{w_n\} \subset V$ be a sequence such that*

$$\|w_n\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} = \int_{\mathcal{C}_\Omega} k_\alpha y^{1-\alpha}|\nabla w_n|^2\,dx\,dy \to S_{\alpha,N}$$

as $n \to \infty$, then $\operatorname{dist}(\beta(w_n),\Omega) \to 0$, as $n \to \infty$.

Proof. Suppose by contradiction that $\operatorname{dist}(\beta(w_n),\Omega) \not\to 0$ as $n \to \infty$. We may verify that $\{w_n\}$ is tight. By Lemma 3.1, there exist sequences $\{\mu_k\}$ and $\{\nu_k\}$ such that

$$S_{\alpha,N} = \lim_{n\to\infty}\|w_n\|^2 = k_\alpha\int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dx\,dy + \Sigma_{k\in I}\mu_k, \tag{3.2}$$

$$1 = \lim_{n\to\infty}\int_\Omega \|u_n\|^2 = \int_\Omega |u|^{2^*_\alpha}\,dx + \Sigma_{k\in I}\nu_k. \tag{3.3}$$

By the Sobolev inequality and Lemma 3.1, from (3.2) we deduce that

$$S_{\alpha,N} = \|w\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + \Sigma_{k\in I}\mu_k \geq S_{\alpha,N}\|u\|^2_{L^{2^*_\alpha}(\Omega)} + S_{\alpha,N}(\Sigma_{k\in I}\nu_k)^{\frac{2}{2^*_\alpha}}.$$

Hence,

$$\|u\|^2_{L^{2^*_\alpha}(\Omega)} + (\Sigma_{k\in I}\nu_k)^{\frac{2}{2^*_\alpha}} \le 1. \tag{3.4}$$

Equations (3.3) and (3.4) imply either $\Sigma_{k\in I}\nu_k = 0$ or $\|u\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} = 0$.

If $\Sigma_{k\in I}\nu_k = 0$, that is $\|u\|^{2^*_\alpha}_{L^{2^*_\alpha}}(\Omega) = 1$, the lower semi-continuity of norms yields

$$S_{\alpha,N} \ge \|w\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} = \frac{k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dx\,dy}{(\int_\Omega |u|^{2^*_\alpha}\,dx)^{\frac{2}{2^*_\alpha}}}.$$

While by the Sobolev trace inequality,

$$S_{\alpha,N} \le \frac{k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha}|\nabla w|^2\,dx\,dy}{(\int_\Omega |u|^{2^*_\alpha}\,dx)^{\frac{2}{2^*_\alpha}}},$$

it then implies that $S_{\alpha,N}$ is achieved, which is a contradiction to the fact that $S_{\alpha,N}$ is not achieved unless $\mathcal{C}_\Omega = \mathbb{R}_+{}^{N+1}$. Thus, $\|u\|^{2^*_\alpha}_{L^{2^*_\alpha}}(\Omega) \ne 1$. Consequently, $\Sigma_{k\in I}\nu_k = 1$ and $u = 0$. Furthermore, by the uniqueness of the extension of u, we have $w = 0$. Now, it is standard to show that ν is concentrated at a single x_0 of $\bar\Omega$. So we have

$$\beta(w_n) \to \int_\Omega x\,d\nu(x) = x_0 \in \bar\Omega,$$

this is a contradiction. □

Since Ω is a smooth bounded domain of $\mathbb{R}^{\mathbb{N}}$, we choose $r > 0$ small enough so that

$$\Omega_r^+ = \{x \in \mathbb{R}^{\mathbb{N}} : \operatorname{dist}(x,\Omega)) < r\} \quad \text{and} \quad \Omega_r^- = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > r\}$$

are homotopically equivalent to Ω. Moreover we assume that the ball $B_r(0) \subset \Omega$, and then $\mathcal{C}_{B_r(0)} := B_r(0) \times (0,+\infty) \subset \mathcal{C}_\Omega$. We define

$$V_0 := \{w \in H^1_{0,L}(\mathcal{C}_{B_r(0)}) : \int_{\mathcal{C}_{B_r(0)}} w_+^{2^*_\alpha}(x,0)\,dx = 1\} \subset V$$

as well as

$$Q_0 = \inf_{w\in V_0} \varphi_\lambda(w).$$

Denote by $\varphi_\lambda^{Q_0} := \{w \in V : \varphi_\lambda(w) < Q_0\}$ the level set below Q_0. We may verify as in Lemma 3.2 that $Q_0 < S_{\alpha,N}$.

Lemma 3.3. *There exists a λ^*, $0 < \lambda^* < \lambda_1$ such that for $0 < \lambda < \lambda^*$, if $w \in \varphi_\lambda^{Q_0}$, then $\beta(w) \in \Omega_r^+$.*

Proof. By Hölder's inequality, for every $w \in V$,

$$\int_\Omega |w(x,0)|^2\,dx \le \Big(\int_\Omega |w(x,0)|^{2^*_\alpha}\,dx\Big)^{\frac{2}{2^*_\alpha}} |\Omega|^{\alpha/N} = |\Omega|^{\alpha/N}.$$

Let $\lambda^* = \frac{\epsilon}{|\Omega|^{\alpha/N}}$. If $0 < \lambda < \lambda^*$ and $w \in \varphi_\lambda^{Q_0}$, we have

$$\|w\|^2 \le \lambda \int_\Omega |w(x,0)|^2\,dx + Q_0 \le \lambda^*|\Omega|^{\alpha/N} + S_{\alpha,N} = S_{\alpha,N} + \epsilon.$$

Therefore, we conclude by Lemma 3.2 that $\beta(w) \in \Omega_r^+$. □

Now, we establish the relation of category between the domain Ω and the level set $\varphi_\lambda^{Q_0}$.

Lemma 3.4. *If $N \geq 2\alpha$ and $0 < \lambda < \lambda^*$, then we have $\operatorname{cat}_{\varphi_\lambda^{Q_0}} \varphi_\lambda^{Q_0} \geq \operatorname{cat}_\Omega(\Omega)$.*

Proof. Let $w_0 \in H^1_{0,L}(\mathcal{C}_{B(0,r)})$ be a minimizer of Q_0. Hence, we may assume that $w_0 > 0$ is cylinder symmetric and $\|w_0\|_{L^{2^*_\alpha}(B_r(0))} = 1$,

$$Q_0 = \int_{\mathcal{C}_{B_r(0)}} k_\alpha y^{1-\alpha} |\nabla w_0|^2 \, dx\, dy - \lambda \int_{B_r(0)} |w_0(x,0)|^2 \, dx.$$

For $z \in \Omega_r^-$, we define $\gamma : \Omega_r^- \to \varphi_\lambda^{Q_0}$ by

$$\gamma(z) = \begin{cases} w_0(x-z,y), & (x,y) \in B_r(z) \times (0,+\infty), \\ 0, & (x,y) \notin B_r(z) \times (0,+\infty). \end{cases}$$

Since $w_0(x,0)$ is a radial function,

$$\beta \circ \gamma(z) = \int_{B_r(z)} x (w_0)_+^{2^*_\alpha}(x-z,0))\, dx = \int_{B_r(0)} x (w_0)_+^{2^*_\alpha}(x,0)\, dx + z = z.$$

Hence, $\beta \circ \gamma = id$.

Assume that $\varphi_\lambda^{Q_0} = A_1 \cup A_2 \cup \dots \cup A_n$, where $A_j, j = 1, 2 \dots n$, is closed and contractible in $\varphi_\lambda^{Q_0}$, i.e. there exists $h_j \in C([0,1] \times A_j, \varphi_\lambda^{Q_0})$ such that, for every u, $v \in A_j$,

$$h_j(0,u) = u, \quad h_j(1,u) = h_j(1,v).$$

Let $B_j := \gamma^{-1}(A_j)$, $1 \leq j \leq n$. The sets B_j are closed and $\Omega_r^- = B_1 \cup B_2 \dots \cup B_n$. By Lemma 3.3, we know $\beta(h_j(t,\gamma(x))) \in \Omega_r^+$. Using the deformation $g_j(t,x) = \beta(h_j(t,\gamma(x)))$, we see that B_j is contractible in Ω_r^+. Indeed, for every x, $y \in B_j$, there exist $\gamma(x)$, $\gamma(y) \in A_j$ such that

$$\begin{gathered} g_j(0,x) = \beta(h_j(0,\gamma(x))) = \beta(\gamma(x)) = x, \\ g_j(1,x) = \beta(h_j(1,\gamma(x))) = \beta(h_j(1,\gamma(y))) = g_j(1,y). \end{gathered}$$

It follows that $\operatorname{cat}_{\varphi_\lambda^{Q_0}} \varphi_\lambda^{Q_0} \geq \operatorname{cat}_{\Omega_r^+}(\Omega_r^-) = \operatorname{cat}_\Omega(\Omega)$. □

Lemma 3.5. *If $\varphi_\lambda|_V$ is bounded from below and satisfies the $(PS)_c$ condition for any*

$$c \in [\inf_{w \in V} \varphi_\lambda, Q_0],$$

then $\varphi_\lambda|_V$ has a minimum and level set $\varphi_\lambda^{Q_0}$ contains at least $\operatorname{cat}_{\varphi_\lambda^{Q_0}} \varphi_\lambda^{Q_0}$ critical points of $\varphi_\lambda|_V$.

The proof of the above lemma can be found in [14].

Proof of Theorem 1.1. By Lemma 3.5, for $0 < \lambda < \lambda^*$, the level set $\varphi_\lambda^{Q_0}$ contains at least $m := \operatorname{cat}_{\varphi_\lambda^{Q_0}} \varphi_\lambda^0$ critical points $w_1, w_2, \dots, w_m$ of $\varphi_\lambda|_V$.

For $j = 1, 2, \dots, m$, there exist $\mu_j \in \mathbb{R}$ such that, for $h \in H^1_{0,L}(\mathcal{C}_\Omega)$,

$$k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} \nabla w_j \nabla h \, dx\, dy - \lambda \int_\Omega w h \, dx - \mu_j \int_\Omega (w_j^+)^{2^*_\alpha - 1} h \, dx = 0.$$

Choosing $h = w_j^-$, we have

$$0 = k_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla w_j^-|^2 dx\, dy - \lambda \int_\Omega |w_j^-|^2 dx.$$

Since $0 < \lambda < \lambda_1$, it implies $w_j^- = 0$ and

$$k_\alpha \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w_j|^2 \, dx\, dy - \lambda \int_\Omega |w_j|^2 \, dx - \mu_j \int_\Omega (w_j^+)^{2_\alpha^*} \, dx = 0.$$

Therefore, $\mu_j = \varphi_\lambda(w_j)$ and $v_j := \mu_j^{\frac{N-\alpha}{2\alpha}} w_j$ is a positive solution of (1.4), $tr_\Omega(v_j)$ is a solution of (1.1). By Lemma 3.4, problems (1.4) and (1.1) have at least $\operatorname{cat}_\Omega(\Omega)$ positive solutions. □

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