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WEAK ASYMPTOTIC SOLUTION FOR A NON-STRICTLY HYPERBOLIC SYSTEM OF CONSERVATION LAWS-II

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Abstract. In this article we introduce a concept of entropy weak asymptotic solution for a system of conservation laws and construct the same for a prolonged system of conservation laws which is highly non-strictly hyperbolic. This is first done for Riemann type initial data by introducing δ, δ' and δ'' waves along a discontinuity curve and then for general initial data by piecing together the Riemann solutions.

1. Introduction

The general theory of systems of conservation laws assumes systems to be strictly hyperbolic, see; Lax [10], Glimm [3] and Bressan [1]. In general for non-strictly hyperbolic system, the solution does not lie in the class L^p, $1 \le p < \infty$. For existence and uniqueness result for a particular system which is not strictly hyperbolic we refer Floch [9]. In general one has to admit solution space as the space of distributions. Then due to the appearance of product of distributions, it is difficult to define the notion of product. One way to overcome this is to work with the generalized space of Colmbeau, for details see [2, 12].

Recently a new notion of solution is introduced by Panov and Shelkovich [13], called weak asymptotic solution. We would also like to cite another concept of solution by Marko Nedelkjov [11], called shadow wave solution. This is same as the *solution in the sense of association* introduced by Colombeau but author also allows non smooth functions.

We study the following system of conservation laws which is studied in [6, 11, 14, 15, 16]; for different values of n, namely;

$$(u_j)_t + \sum_{i=1}^{j} (\frac{u_i u_{j-i+1}}{2})_x = 0, \quad j = 1, 2, \dots, n. \tag{1.1}$$

For $n = 1$, system (1.1) is Burger's equation, which is well studied by Hopf [4]. For $n = 2$ case, (1.1) is a one dimensional model for the large scale structure formation of universe, see [17]. Using vanishing viscosity approach it is observed by Joseph [5] that the second component contain δ measure concentrated along the line of

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discontinuity. The case $n = 3$ is studied in [7] in Colombeau setting. If $u_1 = u$, $u_2 = v$, $u_3 = w$, (1.1) becomes

$$u_t + (\frac{u^2}{2})_x = 0, \quad v_t + (uv)_x = 0, \quad w_t + (\frac{v^2}{2} + uw)_x = 0. \tag{1.2}$$

A system similar to the one above, namely

$$u_t + (u^2)_x = 0, \quad v_t + (2uv)_x = 0, \quad w_t + 2(v^2 + uw)_x = 0. \tag{1.3}$$

is studied in [14]. There, a concept of weak asymptotic solution is introduced and using this a generalized integral formulation is given. Note that the system (1.3) can be obtained from (1.2) using the transformation $(u, v, w) \to (2u, v, \frac{w}{2})$. The case $n = 4$ is studied by Joseph and Sahoo [8], using vanishing viscosity approach. In that paper a solution is constructed for Riemann type initial data and based on this a weak integral formulation is given. In [16], weak asymptotic solution is constructed for the case $n = 4$ when u develops shock and initial data are of Riemann type. Using this, weak asymptotic solution is constructed for the case $n = 4$ when the initial data for the first component is monotonic increasing and initial data for the other components are of general type.

In this paper we define entropy weak asymptotic solution for a general conservation law and construct the same for the case $n = 4$, for Riemann type initial data. Using Riemann type solutions we construct entropy weak asymptotic solution for some special general initial data.

For $n = 4$, $u_1 = u, u_2 = v, u_3 = w, u_4 = z$ and followed by a linear transformation, the system (1.1) leads to the system

$$\begin{gathered} u_t + (u^2)_x = 0, \quad v_t + (2uv)_x = 0 \\ w_t + 2(v^2 + uw)_x = 0, \quad z_t + 2((3vw + uz)_x) = 0. \end{gathered} \tag{1.4}$$

We take initial conditions as

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad z(x, 0) = z_0(x). \tag{1.5}$$

The contents of this article is as follows. We define entropy weak asymptotic solution for any general system of conservation laws. For Riemann type data we construct the same for the system (1.4) by using asymptotic analysis of the product of regularized singular waves, namely δ, δ' and δ''. Then we construct a weak asymptotic solution when the initial data for u is a monotonic function and initial data for v, w and z are bounded measurable functions.

2. Weak asymptotic solution for Riemann type initial data

First of all we define weak asymptotic solution [14, 15] and entropy weak asymptotic solution as follows.

Definition 2.1. A sequence of smooth functions $u_j^\epsilon(x, t)$, $j = 1, 2, dotsn$; is said to be weak asymptotic solution to the system of conservation laws

$$\begin{gathered} (u_j)_t + (f_j(u_1, u_2 ... u_n))_x = 0 \\ u_j(x, 0) = u_{0j}(x) \end{gathered} \tag{2.1}$$

if the following identity hold

$$\begin{gathered}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\big[(u_j^\epsilon)_t+(f_j(u_1^\epsilon,u_2^\epsilon,\dots,u_n^\epsilon))_x\big]\psi(x)dx=0\\ \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\big[u_j(x,0)-u_{0j}(x)\big]\psi(x)dx=0\end{gathered}\tag{2.2}$$

uniformly in the variable t lying in any compact subset of $(0,\infty)$ and for all test function $\psi\in C_c^\infty(R)$.

Definition 2.2. A sequence of smooth functions $u_j^\epsilon(x,t)$, $j=1,2,\dots,n$ is said to be entropy admissible weak asymptotic solution to the following system of conservation laws

$$\begin{gathered}(u_j)_t+(f_j(u_1,u_2,\dots,u_n))_x=0\\ u_j(x,0)=u_{0j}(x)\end{gathered}\tag{2.3}$$

if it is a weak asymptotic solution with the following extra condition: For any $\eta,q\in C^2(\mathbb{R}^n;\mathbb{R})$, with η convex and $D\eta(u)DF(u)=Dq(u)$,

$$\limsup_{\epsilon\to 0}\int_{-\infty}^{\infty}\big[\eta((U^\epsilon)_t+(q(U^\epsilon))_x\big]\psi(x)dx\le 0,\tag{2.4}$$

for all test functions $0\le\psi\in C_c^\infty(R)$. Here $U^\epsilon=(u_1^\epsilon,u_2^\epsilon,\dots,u_n^\epsilon)$ and $F(u)=(f_1(u),\dots,f_n(u))$.

In accordance with the above definition, let us define

$$\begin{gathered}L_1(u)=u_t+(u^2)_x,\quad L_2(u,v)=v_t+(2uv)_x\\ L_3(u,v,w)=w_t+2(v^2+uw)_x,\quad L_4(u,v,w,z)=z_t+2((3vw+uz)_x).\end{gathered}$$

The expression $(u^\epsilon,v^\epsilon,w^\epsilon,z^\epsilon)$ is said to be weak asymptotic solution to (1.4) with initial data (1.5) if

$$\begin{gathered}\int L_1[u(x,t,\epsilon)]\psi(x)dx=o(1)\quad\int L_2[u(x,t,\epsilon),v(x,t,\epsilon)]\psi(x)dx=o(1)\\ \int L_3[u(x,t,\epsilon),v(x,t,\epsilon),w(x,t,\epsilon)]\psi(x)dx=o(1)\\ \int L_4[u(x,t,\epsilon),v(x,t,\epsilon),w(x,t,\epsilon),z(x,t,\epsilon)]\psi(x)dx=o(1),\end{gathered}\tag{2.5}$$

and initial conditions satisfy

$$\begin{gathered}\int\Big(u(x,0,\epsilon)-u_0(x)\Big)\psi(x)dx=o(1),\quad\int\Big(v(x,0,\epsilon)-v_0(x)\Big)\psi(x)dx=o(1)\\ \int\Big(w(x,0,\epsilon)-w_0(x)\Big)\psi(x)dx=o(1),\quad\int\Big(z(x,0,\epsilon)-z_0(x)\Big)\psi(x)dx=o(1),\end{gathered}$$

for all $\psi\in D(R)$.

With the similar lines we can have entropy weak asymptotic solution. To study weak asymptotic analysis first we need the following Lemma as in [14], regarding the superpositions of singular waves; δ,δ',δ'' and δ'''.

Lemma 2.3. *Let $\{w_i\}_{i\in I}$ is an indexed family of Friedrich mollifiers satisfying*

$$w_i(x)=w_i(-x)\quad\textit{and}\quad\int w_i=1.$$

Define $H_i(x,\epsilon) = w_{0i}(\frac{x}{\epsilon}) = \int_{-\infty}^{\frac{x}{\epsilon}} w_i(y)dy$, $\delta_i(x,\epsilon) = \frac{1}{\epsilon} w_i(\frac{x}{\epsilon})$ *and* $\delta_i^{''\dots(k\ times)}(x,\epsilon) = \frac{1}{\epsilon^{k+1}} w_i^{''\dots(k\ times)}(\frac{x}{\epsilon})$. *The above assumptions implies the following asymptotic expansions, in the sense of distribution.*

$$
\begin{gathered}
(H_i(x,\epsilon))^r = H(x) + O_{D'}(\epsilon), \quad (H_i(x,\epsilon)(H_j(x,\epsilon)) = H(x) + O_{D'}(\epsilon) \\
(H_i(x,\epsilon))^r \delta_j(x,\epsilon) = \delta(x) \int w_{0i}^r(y) w_j(y) dy + O_{D'}(\epsilon) \\
(\delta_i(x,\epsilon))^2 = \frac{1}{\epsilon}\delta(x) \int w_i^2(y) dy + O_{D'}(\epsilon) \\
H_i(x,\epsilon)\delta_j'(x,\epsilon) = -\frac{1}{\epsilon}\delta(x) \int w_i(y) w_j(y) dy + \delta'(x) \int w_{0i}(y) w_j(y) dy + O_{D'}(\epsilon) \\
H_i(x,\epsilon)\epsilon^2 \delta_j'''(x,\epsilon) = \frac{1}{\epsilon}\delta(x) \int w_i'(y)\delta_j'(y) dy + O_{D'}(\epsilon), \\
\delta_i(x,\epsilon).\delta_j(x,\epsilon) = \frac{1}{\epsilon}\delta(x) \int w_i(y) w_j(y) dy + O_{D'}(\epsilon) \\
\delta_i(x,\epsilon)\delta_j'(x,\epsilon) = \frac{1}{\epsilon}\delta'(x) \int y w_i(y) w_j'(y) dy + O_{D'}(\epsilon), \\
H_i(x,\epsilon)\delta_j''(x,\epsilon) = \frac{1}{\epsilon}\delta(x) \int w_{0i}(y) w_j(y) dy + \frac{1}{2}\delta''(x) \int y^2 w_{0i}(y) w_j(y) dy + O_{D'}(\epsilon) \\
\delta_i(x,\epsilon))\epsilon^2 \delta_j'''(x,\epsilon) = \frac{1}{\epsilon}\delta'(x) \int y w_i(y) w_j'''(y) dy + O_{D'}(\epsilon) \\
H_i(x,\epsilon)\epsilon^2 \delta_j''''(x,\epsilon) = \frac{1}{\epsilon}\delta'(x) \int y w_{0i}(y) w_j''''(y) dy + O_{D'}(\epsilon),
\end{gathered}
$$

where $O_{D'}(\epsilon)$ *is the error term satisfying* $\lim_{\epsilon\to 0}\langle O_{D'}(\epsilon), \psi(x)\rangle = 0$, *for any test function* ψ.

Proof. Let $\psi \in D(\mathbb{R})$ be any test function. Relations from $1-6$ can be found in [14]. We prove the asymptotic expansions from the seventh onward.

Now we prove the seventh asymptotic expansion. Using change of variable formula ($x = \epsilon y$), employing third order Taylor expansion, $\psi(\epsilon y) = \psi(0) + \epsilon y \psi'(0) + \epsilon^2 y^2 \psi''(0) + \epsilon^3 y^3 O(1)$ and the fact that $\int y w_i(y) w_j(y) dy = 0$, we have

$$
\begin{aligned}
\langle \delta_i(x,\epsilon)\delta_j(x,\epsilon), \psi(x)\rangle &= \int \frac{1}{\epsilon} w_i(\frac{x}{\epsilon}) \frac{1}{\epsilon} w_j(\frac{x}{\epsilon}) \psi(x) dx \\
&= \frac{1}{\epsilon} \int w_i(y) w_j(y) \psi(\epsilon y) dy \\
&= \frac{1}{\epsilon}\psi(0) \int w_i(y) w_j(y) dy + \psi'(0) \int y w_i(y) w_j(y) dy + O(\epsilon) \\
&= \frac{1}{\epsilon}\delta(x) \int w_i(y) w_j(y) dy + O(\epsilon).
\end{aligned}
$$

Now we prove the eighth asymptotic expansion. Using change of variable formula ($x = \epsilon y$), employing third order Taylor expansion, $\psi(\epsilon y) = \psi(0) + \epsilon y \psi'(0) + \epsilon^2 y^2 \psi''(0) + \epsilon^3 y^3 O(1)$ and the fact that $\int y w_i(y) w_j(y) dy = 0$, we have

$$
\begin{aligned}
\langle \delta_i(x,\epsilon)\delta_j'(x,\epsilon), \psi(x)\rangle &= \frac{1}{\epsilon^2} \int w_i(y) w_j'(y) \psi(\epsilon y) dy \\
&= \frac{1}{\epsilon^2}\psi(0) \int w_i(y) w_j'(y) dy + \frac{1}{\epsilon}\psi'(0) \int y w_i(y) w_j'(y) dy
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\psi''(0)\int y^2 w_i(y)w_j'(y)dy + O(\epsilon)\\
&= \frac{1}{\epsilon}\delta'(x)\int y w_i(y)w_j'(y)dy + O(\epsilon).
\end{aligned}
$$

In the above calculation we also used the identity

$$
\int w_i(y)w_j'(y)dy = \int y^2 w_i(y)w_j'(y)dy = 0.
$$

Following the analysis similar as above, we prove the remaining identities. Details are as follows:

$$
\begin{aligned}
&\langle H_i(x,\epsilon)\delta_j''(x,\epsilon), \psi(x)\rangle\\
&= \int w_{0i}(y)\frac{1}{\epsilon^2}w_j''(y)\psi(\epsilon y)dy\\
&= \int w_{0i}(y)\frac{1}{\epsilon^2}w_j''(y)(\psi(0)+\epsilon y\psi'(0)+\frac{\epsilon^2 y^2}{2}\psi''(0))dy + O(\epsilon)\\
&= \frac{1}{\epsilon}\delta'(x)\int y w_{0i}(y)w_j''(y)dy + \frac{1}{2}\delta''(x)\int y^2 w_{0i}(y)w_j''(y)dy + O(\epsilon).
\end{aligned}
$$

$$
\begin{aligned}
&\langle \delta_i(x,\epsilon))\epsilon^2 w_j'''(x,\epsilon), \psi(x)\rangle\\
&= \frac{1}{\epsilon^2}\int w_i(y)w_j'''(y)\psi(\epsilon y)dy\\
&= \frac{1}{\epsilon^2}\int w_i(y)w_j'''(y)(\psi(0)+\epsilon y\psi'(0)+\frac{\epsilon^2 y^2}{2}\psi''(0))dy + O(\epsilon)\\
&= \frac{1}{\epsilon^2}\delta(x)\int w_i(y)w_j'''(y)dy + \frac{1}{\epsilon}\delta'(x)\int y w_i(y)w_j'''(y)dy\\
&\quad + \frac{1}{2}\delta''(x)\int y^2 w_i(y)w_j'''(y)dy + O(\epsilon)\\
&= \frac{1}{\epsilon}\delta'(x)\int y w_i(y)w_j'''(y)dy + O(\epsilon).
\end{aligned}
$$

$$
\begin{aligned}
&\langle H_i(x,\epsilon)\epsilon^2\delta_j''''(x,\epsilon), \psi(x)\rangle\\
&= \int w_{0i}(y)\frac{1}{\epsilon^2}w_j''''(y)\psi(\epsilon y)(\psi(0)+\epsilon y\psi'(0)+\frac{\epsilon^2 y^2}{2}\psi''(0))dy + O(\epsilon)\\
&= \frac{1}{\epsilon}\delta'(x)\int y w_{0i}(y)w_j''''(y)dy + O(\epsilon).
\end{aligned}
$$

□

In [8], it is observed that the vanishing viscosity limit for the component z admits combinations of δ and δ' waves when u develops rarefaction. So we choose ansatz as the combination of the above singular waves along the discontinuity curve. But this is not enough as it is clear in the construction of w, see [14]. In [14], a correction term is added in the component w to construct weak asymptotic solution for the shock case. Here in this rarefaction case we will add correction terms in the component z, but no correction terms in the third component w. Note that this correction term is required to balance the unexpected term coming in the product of singular waves.

Theorem 2.4. *The ansatz*

$$\begin{aligned}
u(x,t,\epsilon) &= u_1 H_u(-x+2u_1t,\epsilon) + \frac{x}{2t}\{H_u(-x+2u_2t,\epsilon) \\
&\quad - H_u(-x+2u_1t,\epsilon)\} + u_2\{1-H_u(-x+2u_2t,\epsilon)\}. \\
v(x,t,\epsilon) &= v_1 H_v(-x+2u_1t,\epsilon) + v_2\{1-H_v(-x+2u_2t,\epsilon)\}. \\
w(x,t,\epsilon) &= w_1 H_w(-x+2u_1t,\epsilon) + w_2\{1-H_w(-x+2u_2t,\epsilon)\} \\
&\quad + e_1(t)\delta_e(-x+2u_1t,\epsilon) + e_2(t)\delta_e(-x+2u_2t,\epsilon). \\
z(x,t,\epsilon) &= z_1 H_z(-x+2u_1t,\epsilon) + z_2\{1-H_z(-x+2u_2t,\epsilon)\} \\
&\quad + g_1(t)\delta_e(-x+2u_1t,\epsilon) + g_2(t)\delta_e(-x+2u_2t,\epsilon) \\
&\quad + h_1(t)\delta_g'(-x+2u_1t,\epsilon) + h_2(t)\delta_g'(-x+2u_2t,\epsilon) \\
&\quad + R_1(-x+2u_1t,\epsilon) + R_2(-x+2u_2t,\epsilon),
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
R_1(-x+2u_1t,\epsilon) &= \epsilon^2 P_1(t)\delta_P'''(-x+2u_1t,\epsilon), \\
R_2(-x+2u_2t,\epsilon) &= \epsilon^2 P_2(t)\delta_P'''(-x+2u_2t,\epsilon).
\end{aligned}$$

is entropy weak asymptotic solution to the problem (1.4) *if the following relations hold.*

$$\begin{gathered}
\dot{e_1}(t) = 2v_1^2, \quad \dot{e_2}(t) = -2v_2^2, \\
\int w_{0u}(y)w_e(y)dy = \int w_{0v}(y)w_e(y)dy = \int w_{0u}(y)w_g(y)dy = 1, \\
\dot{g_1}(t) = 6v_1w_1, \quad \dot{g_2}(t) = -6v_2w_2, \\
\dot{h_1}(t) = 6v_1e_1(t), \quad \dot{h_2}(t) = 6v_2e_2(t), \\
\int w_u(y)w_g(y)dy = 1, \quad \int w_u'(y)w_P'(y)dy = 1 \\
P_1(t) = h_1(t), \quad P_2(t) = h_2(t).
\end{gathered} \tag{2.7}$$

Remark 2.5. In Theorem 2.4, R_1 and R_2 are called correction terms which is needed to adjust the odd terms appearing in the asymptotic expansion of the product of regularized singular waves.

Proof of Theorem 2.4. Using lemma 2.3, we obtain

$$\begin{aligned}
&u^2(x,t,\epsilon) \\
&\approx u_1^2 H(-x+2u_1t,\epsilon) + \frac{x^2}{4t^2}\{H(-x+2u_2t,\epsilon) - H(-x+2u_1t,\epsilon)\} \\
&+ u_2^2\{1-H(-x+2u_2t,\epsilon)\}.
\end{aligned} \tag{2.8}$$

Taking the distributional derivative of u with respect to t, we obtain

$$\begin{aligned}
u_t(x,t,\epsilon) &\approx \frac{x}{2t^2}\{H(-x+2u_1t,\epsilon) - H(-x+2u_2t,\epsilon)\} \\
&\quad + \{2u_1^2 - \frac{u_1x}{t}\}\delta(-x+2u_1t,\epsilon) - \{2u_2^2 - \frac{u_2x}{t}\}\delta(-x+2u_2t,\epsilon).
\end{aligned} \tag{2.9}$$

Putting (2.8) and (2.9) in the first equation of (1.4), we obtain

$$\begin{aligned}
&u_t(x,t,\epsilon) + (u^2)_x(x,t,\epsilon) \\
&\approx \{u_1^2 - \frac{u_1x}{t} + \frac{x^2}{4t^2}\}\delta(-x+2u_1t,\epsilon) - \{u_2^2 - \frac{u_2x}{t} + \frac{x^2}{4t^2}\}\delta(-x+2u_2t,\epsilon) = 0.
\end{aligned}$$

Now we find the asymptotic expansion of terms appearing in the second equation of (1.4). Applying lemma 2.3,

$$\begin{gathered} u(x,t,\epsilon)v(x,t,\epsilon) \approx u_1v_1H(-x+2u_1t,\epsilon)+u_2v_2\{1-H(-x+2u_2t,\epsilon)\}. \\ v_t(x,t,\epsilon) \approx 2u_1v_1\delta(-x+2u_1t,\epsilon)-2u_2v_2\delta(-x+2u_2t,\epsilon). \end{gathered} \tag{2.10}$$

Using expansions (2.10) in the second equation of (1.4), we obtain

$$\begin{aligned} &v_t(x,t,\epsilon)+(2u(x,t,\epsilon)v(x,t,\epsilon))_x \\ &\approx \{2u_1v_1-2u_1v_1\}\delta(-x+2u_1t,\epsilon)+\{2u_2v_2-2u_2v_2\}\delta(-x+2u_2t,\epsilon)=0 \end{aligned}$$

Now we find the asymptotic expansion of terms appearing in the third equation of (1.4). Applying lemma 2.3,

$$v^2(x,t,\epsilon) \approx v_1^2H(-x+2u_1t,\epsilon)+v_2^2\{1-H(-x+2u_2t,\epsilon)\}. \tag{2.11}$$

$$\begin{aligned} &u(x,t,\epsilon)w(x,t,\epsilon) \\ &\approx u_1w_1H(-x+2u_1t,\epsilon)+u_2w_2\{1-H(-x+2u_2t,\epsilon)\} \\ &\quad + u_1e_1(t)\int w_{0u}(y)w_e(y)dy\delta(-x+2u_1t,\epsilon) \\ &\quad + u_2e_2(t)\int w_{0u}(y)w_e(y)dy\delta(-x+2u_2t,\epsilon), \end{aligned} \tag{2.12}$$

Using (2.11) and (2.12), we obtain

$$\begin{aligned} &2(v^2(x,t,\epsilon)+u(x,t,\epsilon)w(x,t,\epsilon)) \\ &\approx (2v_1^2+2u_1w_1)H(-x+2u_1t,\epsilon) \\ &\quad + (2v_2^2+2u_2w_2)\{1-H(-x+2u_2t,\epsilon)\} \\ &\quad + 2u_1e_1(t)\int w_{0u}(y)w_e(y)dy\delta(-x+2u_1t,\epsilon) \\ &\quad + 2u_2e_2(t)\int w_{0u}(y)w_e(y)dy\delta(-x+2u_2t,\epsilon). \end{aligned} \tag{2.13}$$

Taking the distributional derivative of w with respect to t, we obtain

$$\begin{aligned} &w_t(x,t,\epsilon) \\ &\approx \{\dot{e_1}(t)+2u_1w_1\}\delta(-x+2u_1t,\epsilon) \\ &+ \{\dot{e_2}(t)-2u_2w_2\}\delta(-x+2u_2t,\epsilon)+2u_1e_1(t)\delta'(-x+2u_1t,\epsilon) \\ &+ 2u_2e_2(t)\delta'(-x+2u_2t,\epsilon). \end{aligned} \tag{2.14}$$

Taking the distributional derivative of the expression in (2.13), with respect to x, we obtain

$$\begin{aligned} &2(v^2(x,t,\epsilon)+u(x,t,\epsilon)w(x,t,\epsilon))_x \\ &\approx -(2v_1^2+2u_1w_1)\delta(-x+2u_1t,\epsilon) \\ &\quad + (2v_2^2+2u_2w_2)\delta(-x+2u_2t,\epsilon) \\ &- 2u_1e_1(t)\int w_{0u}(y)w_e(y)dy\delta'(-x+2u_1t,\epsilon) \\ &\quad - 2u_2e_2(t)\int w_{0u}(y)w_e(y)dy\delta'(-x+2u_2t,\epsilon). \end{aligned} \tag{2.15}$$

Using expansions (2.14) and (2.15) in the third equation of (1.4), we obtain

$$\begin{aligned} &w_t(x,t,\epsilon)+2(v^2(x,t,\epsilon)+u(x,t,\epsilon)w(x,t,\epsilon))_x \\ &\approx \{\dot{e_1}(t)-2v_1^2\}\delta(-x+2u_1t)+\{\dot{e_2}(t)+2v_2^2\}\delta(-x+2u_2t) \\ &\quad +2u_1e_1(t)\{1-\int w_{0u}(y)w_e(y)dy\}\delta'(-x+2u_1t) \\ &\quad +2u_2e_2(t)\{1-\int w_{0u}(y)w_e(y)dy\}\delta'(-x+2u_2t). \end{aligned} \tag{2.16}$$

So expression (2.16) is zero, if the coefficients of $\delta(-x+2u_1t)$ and $\delta'(-x+2u_1t)$ are zero. Which implies

$$\int w_{0u}(y)w_e(y)dy=1, \quad \dot{e_1}(t)=2v_1^2 \quad \dot{e_2}(t)=-2v_2^2. \tag{2.17}$$

Now we find the asymptotic expansion of terms appearing in the fourth equation of (1.4). Applying lemma 2.3,

$$\begin{aligned} v(x,t,\epsilon)w(x,t,\epsilon) \approx\ & v_1w_1H(-x+2u_1t,\epsilon)+v_2w_2\{1-H(-x+2u_2t,\epsilon)\} \\ &+v_1e_1(t)\int w_{0v}(y)w_e(y)dy\delta(-x+2u_1t,\epsilon) \\ &+v_2e_2(t)\int w_{0v}(y)w_e(y)dy\delta(-x+2u_2t,\epsilon). \end{aligned} \tag{2.18}$$

$$\begin{aligned} &u(x,t,\epsilon)z(x,t,\epsilon) \\ &\approx u_1z_1H(-x+2u_1t,\epsilon)+u_2z_2\{1-H(-x+2u_2t,\epsilon)\} \\ &\quad +u_1g_1(t)\int w_{0u}(y)w_g(y)dy\delta(-x+2u_1t,\epsilon) \\ &\quad +u_2g_2(t)\int w_{0u}(y)w_g(y)dy\delta(-x+2u_2t,\epsilon) \\ &\quad +u_1h_1(t)\int w_{0u}(y)w_g(y)dy\delta'(-x+2u_1t,\epsilon) \\ &\quad +u_2h_2(t)\int w_{0u}(y)w_g(y)dy\delta'(-x+2u_2t,\epsilon) \\ &\quad +\{u_1P_1(t)\int w_u'(y)w_P'(y)dy-u_1h_1(t)\int w_u(y)w_g(y)dy\}\frac{1}{\epsilon}\delta(-x+2u_1t,\epsilon) \\ &\quad +\{u_2P_2(t)\int w_u'(y)w_P'(y)dy-u_2h_2(t)\int w_u(y)w_g(y)dy\}\frac{1}{\epsilon}\delta(-x+2u_2t,\epsilon). \end{aligned} \tag{2.19}$$

Using (2.18) and (2.19), we have

$$\begin{aligned} &3v(x,t,\epsilon)w(x,t,\epsilon)+u(x,t,\epsilon)z(x,t,\epsilon) \\ &\approx \{3v_1w_1+u_1z_1\}H(-x+2u_1t,\epsilon)+\{3v_2w_2+u_2z_2\}\{1-H(-x+2u_2t,\epsilon)\} \\ &\quad +\{3v_1e_1(t)\int w_{0v}(y)w_e(y)dy+u_1g_1(t)\int w_{0u}(y)w_g(y)dy\}\delta(-x+2u_1t,\epsilon) \\ &\quad +\{3v_2e_2(t)\int w_{0v}(y)w_e(y)dy+u_2g_2(t)\int w_{0u}(y)w_g(y)dy\}\delta(-x+2u_2t,\epsilon) \\ &\quad +u_1h_1(t)\int w_{0u}(y)w_g(y)dy\delta'(-x+2u_1t,\epsilon) \end{aligned}$$

$$
\begin{aligned}
&+ u_2h_2(t)\int w_{0u}(y)w_g(y)dy\delta'(-x+2u_2t,\epsilon)\\
&+ \{u_1P_1(t)\int w_u'(y)w_P'(y)dy - u_1h_1(t)\int w_u(y)w_g(y)dy\}\\
&\times \frac{1}{\epsilon}\delta(-x+2u_1t,\epsilon)\\
&+ \{u_2P_2(t)\int w_u'(y)w_P'(y)dy - u_2h_2(t)\int w_u(y)w_g(y)dy\}\\
&\times \frac{1}{\epsilon}\delta(-x+2u_2t,\epsilon). \qquad (2.20)
\end{aligned}
$$

Taking the distributional derivative of the expression in (2.20), with respect to x, we obtain

$$
\begin{aligned}
&2((3v(x,t,\epsilon)w(x,t,\epsilon)+u(x,t,\epsilon)z(x,t,\epsilon))_x)\\
&\approx -2\{3v_1w_1+u_1z_1\}\delta(-x+2u_1t,\epsilon)+2\{3v_2w_2+u_2z_2\}\delta(-x+2u_2t,\epsilon)\\
&\quad - 2\{3v_1e_1(t)\int w_{0v}(y)w_e(y)dy + u_1g_1(t)\int w_{0u}(y)w_g(y)dy\}\delta'(-x+2u_1t,\epsilon)\\
&\quad - 2\{3v_2e_2(t)\int w_{0v}(y)w_e(y)dy + u_2g_2(t)\int w_{0u}(y)w_g(y)dy\}\delta'(-x+2u_2t,\epsilon)\\
&\quad - 2u_1h_1(t)\int w_{0u}(y)w_g(y)dy\delta''(-x+2u_1t,\epsilon)\\
&\quad - 2u_2h_2(t)\int w_{0u}(y)w_g(y)dy\delta''(-x+2u_2t,\epsilon)\\
&\quad - 2\{u_1P_1(t)\int w_u'(y)w_P'(y)dy - u_1h_1(t)\int w_u(y)w_g(y)dy\}\frac{1}{\epsilon}\delta'(-x+2u_1t,\epsilon)\\
&\quad - 2\{u_2P_2(t)\int w_u'(y)w_P'(y)dy - u_2h_2(t)\int w_u(y)w_g(y)dy\}\frac{1}{\epsilon}\delta'(-x+2u_2t,\epsilon). \qquad (2.21)
\end{aligned}
$$

Taking the distributional derivative of z with respect to t, we obtain

$$
\begin{aligned}
z_t(x,t,\epsilon) &\approx \{\dot{g_1}(t)+2u_1z_1\}\delta(-x+2u_1t,\epsilon)\{\dot{g_2}(t)-2u_2z_2\}\delta(-x+2u_2t,\epsilon)\\
&\quad + \{\dot{h_1}(t)+2u_1g_1(t)\}\delta'(-x+2u_1t,\epsilon)\\
&\quad + \{\dot{h_2}(t)+2u_2g_2(t)\}\delta'(-x+2u_2t,\epsilon)\\
&\quad + 2u_1h_1(t)\delta''(-x+2u_1t,\epsilon)+2u_2h_2(t)\delta''(-x+2u_2t,\epsilon).
\end{aligned} \qquad (2.22)
$$

Using expansions (2.21) and (2.22) in the fourth equation of (1.4), we obtain

$$
\begin{aligned}
&z_t(x,t,\epsilon)+2((3v(x,t,\epsilon)w(x,t,\epsilon)+u(x,t,\epsilon)z(x,t,\epsilon))_x)\\
&\approx \{\dot{g_1}(t)+2u_1z_1-2(3v_1w_1+u_1z_1)\}\delta(-x+2u_1t,\epsilon)\\
&\quad + \{\dot{g_2}(t)-2u_2z_2+2(3v_2w_2+u_2z_2)\}\delta(-x+2u_2t,\epsilon)\\
&\quad + \{\dot{h_1}(t)2u_1g_1(t)-2(3v_1e_1(t)\int w_{0v}(y)w_e(y)dy\\
&\quad + u_1g_1(t)\int w_{0u}(y)w_g(y)dy)\}\delta'(-x+2u_1t,\epsilon)\\
&\quad + \{\dot{h_2}(t)2u_2g_2(t)-2(3v_2e_2(t)\int w_{0v}(y)w_e(y)dy
\end{aligned}
$$

$$
\begin{aligned}
&+ u_2 g_2(t) \int w_{0u}(y) w_g(y) dy)\} \delta'(-x + 2u_2 t, \epsilon) \\
&+ 2u_1 h_1(t) \{1 - \int w_{0u}(y) w_g(y) dy\} \delta''(-x + 2u_1 t, \epsilon) \\
&+ 2u_2 h_2(t) \{1 - \int w_{0u}(y) w_g(y) dy\} \delta''(-x + 2u_2 t, \epsilon) \\
&- 2\{u_1 P_1(t) \int w'_u(y) w'_P(y) dy - u_1 h_1(t) \int w_u(y) w_g(y) dy\} \\
&\times \frac{1}{\epsilon} \delta'(-x + 2u_1 t, \epsilon) \\
&- 2\{u_2 P_2(t) \int w'_u(y) w'_P(y) dy - u_2 h_2(t) \int w_u(y) w_g(y) dy\} \\
&\times \frac{1}{\epsilon} \delta'(-x + 2u_2 t, \epsilon).
\end{aligned} \tag{2.23}
$$

Similarly equating the coefficients of $\delta(-x + 2u_i t)$, $\delta'(-x + 2u_i t)$, $\delta''(-x + 2u_i t)$,$\frac{1}{\epsilon}\delta$ and $\frac{1}{\epsilon}\delta'$, for $i = 1, 2$ to zero in (2.23), we obtain

$$
\begin{gathered}
\int w_{0v}(y) w_e(y) dy = \int w_{0u}(y) w_g(y) dy = 1, \\
\dot{g_1}(t) = 2(3v_1 w_1 + u_1 z_1) - 2u_1 z_1, \dot{g_2}(t) = 2u_2 z_2 - 2(3v_2 w_2 + u_2 z_2), \\
\dot{h_1}(t) = 6v_1 e_1(t), \quad \dot{h_2}(t) = 6v_2 e_2(t) \\
\int w_u(y) w_g(y) dy = 1, \quad \int w'_u(y) w'_P(y) dy = 1 \\
P_1(t) = h_1(t), \quad P_2(t) = h_2(t).
\end{gathered} \tag{2.24}
$$

The conditions (2.17) and (2.24) together constitute the condition (2.7).

To prove it is entropy admissible. Let η be a convex entropy with entropy flux q. For this general system it is of the form, see [13].

$$
\begin{gathered}
\eta(u, v, w, z) = \bar{\eta}(u) + c_1 v + c_2 w + c_3 z \\
q(u, v, w, z) = \bar{q}(u) + c_1 uv + c_2(uw + \frac{v^2}{2}) + c_3(uz + vw),
\end{gathered} \tag{2.25}
$$

where $\bar{\eta}(u)$ is a convex function in the variable u and satisfies $2u\bar{\eta}'(u) = \bar{q}'(u)$. So entropy admissible condition is

$$
\limsup_{\epsilon \to 0} \int_{-\infty}^{\infty} \big[\eta(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)_t + (q(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon))_x\big] \psi(x) dx \leq 0.
$$

for any positive test function ψ. This means

$$
\limsup_{\epsilon \to 0} \int_{-\infty}^{\infty} \big[\bar{\eta}(u^\epsilon)_t + (\bar{q}(u^\epsilon))_x\big] \psi(x) dx \leq 0.
$$

for any test function $\psi \geq 0$. We know from the general theory of conservation laws,

$$
\begin{aligned}
&\limsup_{\epsilon \to 0} \int_0^\infty \int_{-\infty}^{\infty} \big[\eta((u^\epsilon)_t + (q(u^\epsilon))_x\big] \phi(x, t) dx dt \\
&= \int_0^\infty \int_{-\infty}^{\infty} \big[\eta((u)_t + (q(u))_x\big] \phi(x, t) dx dt \leq 0,
\end{aligned}
$$

where u is the limit of u^ϵ in the sense of distributions and $\phi \in D(\mathbb{R} \times (0,\infty))$. The last inequality holds because u is an entropy solution for the conservation law(Burgers equation).

For any $\epsilon_0 > 0$, there exists $\delta > 0$ such that for $0 < \epsilon < \delta$, we have

$$\int_0^\infty \int_{-\infty}^\infty \big[\eta((u^\epsilon)_t + (q(u^\epsilon))_x\big]\phi(x,t)dxdt \le \epsilon_0$$

Choose $\phi(x,t) = \psi(x)\frac{1}{\eta}k(\frac{t-t_0}{\eta})$, $\int k(t)dt = 1$ and $k \ge 0$ is a smooth function. Then

$$\int_0^\infty \int_{-\infty}^\infty \big[\eta((u^\epsilon)_t + (q(u^\epsilon))_x\big]\psi(x)\frac{1}{\eta}k(\frac{t-t_0}{\eta})dxdt \le \epsilon_0, \text{for all } \eta > 0.$$

Passing to the limit as $\eta \to 0$, we have

$$\int_{-\infty}^\infty \big[\eta((u^\epsilon)_t + (q(u^\epsilon))_x\big]\psi(x)dxdt \le \epsilon_0 \quad \text{for } t = t_0.$$

Hence

$$\limsup_{\epsilon\to 0} \int_{-\infty}^\infty \big[\eta((u^\epsilon)_t + (q(u^\epsilon))_x\big]\psi(x)dx \le \epsilon_0.$$

for all $t \ge 0$. Since ϵ_0 is arbitrary, we have $(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)$ is an entropy weak asymptotic solution.

Same analysis as above can be used to prove $(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)$ is an entropy weak asymptotic solution when u develops shock; can be seen from the structure of the solution given in [16, Theorem 2.3]. Hence the proof. ☐

Piecing together the Riemann problems we construct a weak asymptotic solution for general type initial data under the assumption that u is a monotonic decreasing function.

Theorem 2.6. *If u_0, v_0, w_0 and z_0 are bounded measurable functions on $\mathbb{R}$, and u_0 is monotonic, then there exists entropy weak asymptotic solution $(u^\epsilon, v^\epsilon, w^\epsilon, z^\epsilon)$ to the system* (1.4) *in $[-K,K] \times [0,\infty)$, for any $K > 0$.*

Proof. If u_0 is a monotonic increasing function, then the result of [16, Theorem 2.5] is an entropy solution as can be seen easily from the structure given in the [16, Theorem 2.5]. Lets assume u_0 is a monotonic decreasing function. Let ϕ be a test function on $\mathbb{R}$ having support in $[-K,K]$. Given $\epsilon > 0$, there exist piecewise constant functions $(u_{0\epsilon}, v_{0\epsilon}, w_{0\epsilon}, z_{0\epsilon})$ such that

$$\begin{gathered}\int_{[-K,K]} |u_0(x) - u_{0\epsilon}(x)|dx < \epsilon, \quad \int_{[-K,K]} |v_0(x) - v_{0\epsilon}(x)|dx < \epsilon \\ \int_{[-K,K]} |w_0(x) - w_{0\epsilon}(x)|dx < \epsilon, \quad \int_{[-K,K]} |z_0(x) - z_{0\epsilon}(x)|dx < \epsilon.\end{gathered} \tag{2.26}$$

In addition to this we can take $u_{0\epsilon}$ monotonic decreasing and all functions have same points of discontinuities. $(u_{0\epsilon}, v_{0\epsilon}, w_{0\epsilon}, z_{0\epsilon})$ in $[-K, K]$ can be represented as

$$\begin{aligned}
u_{0\epsilon} &= \sum_{i=1}^{n} u_{0i}(H(x-a_{i-1}) - H(x-a_i)) \\
v_{0\epsilon} &= \sum_{i=1}^{n} v_{0i}(H(x-a_{i-1}) - H(x-a_i)) \\
w_{0\epsilon} &= \sum_{i=1}^{n} w_{0i}(H(x-a_{i-1}) - H(x-a_i)) \\
z_{0\epsilon} &= \sum_{i=1}^{n} z_{0i}(H(x-a_{i-1}) - H(x-a_i)).
\end{aligned} \tag{2.27}$$

Since $u_{0\epsilon}$ is a monotonic decreasing function, discontinuity curve arising in the solution of (u, v, w, z) do not intersect for any time. So the following functions are weak asymptotic solutions

$$\begin{aligned}
u(x,t,\epsilon) &= \sum_{i=1}^{n-1} \Big[u_{0i} H_u(-x + u_{0i}t + x_i, \eta) \\
&\quad + \frac{x - x_i}{t}(H_u(-x + u_{0i+1}t + x_i, \eta) - H_u(-x + (u_{0i}t + x_i), \eta)) \\
&\quad + u_{0i+1}(1 - H_u(-x + u_{0i+1}t + x_i, \eta)) \Big], \\
v(x,t,\epsilon) &= \sum_{i=1}^{n-1} \Big[v_{0i} H_u(-x + u_{0i}t + x_i, \eta) + v_{0i+1}(1 - H_u(-x + u_{0i+1}t + x_i, \eta)) \Big], \\
w(x,t,\epsilon) &= \sum_{i=1}^{n-1} \Big[w_{0i} H_u(-x + u_{0i}t + x_i, \eta) + w_{0i+1}(1 - H_u(-x + u_{0i+1}t + x_i, \eta) \Big] \\
&\quad + \sum_{i=1}^{n-1} \Big[e_{1i}(t)\delta_e(-x + u_{0i}t, \eta) + e_{2i}(t)\delta_e(-x + u_{0i+1}t, \eta) \Big] \\
z(x,t,\epsilon) &= \sum_{i=1}^{n-1} \Big[z_{0i} H_u(-x + u_{0i}t + x_i, \eta) + z_{0i+1}(1 - H_u(-x + u_{0i+1}t + x_i, \eta)) \Big] \\
&\quad + \sum_{i=1}^{n-1} \Big[g_{1i}(t)\delta_e(-x + u_{0i}t, \eta) + g_{2i}(t)\delta_e(-x + u_{0i+1}t, \eta) \Big] \\
&\quad + \sum_{i=1}^{n-1} \Big[h_{1i}(t)\delta_h'(-x + u_{0i}t, \eta) + h_{2i}(t)\delta_h'(-x + u_{0i+1}t, \eta) \Big] \\
&\quad + \sum_{i=1}^{n-1} [R_{1i}(-x + 2u_1 t, \epsilon) + R_{2i}(-x + 2u_2 t, \epsilon)]
\end{aligned}$$

where e_{1i}, e_{2i}, g_{1i}, g_{2i}, h_{1i}, h_{2i}, R_{1i} and R_{2i} satisfy (2.7) with u_1, u_2, v_1, v_2, w_1, w_2, z_1, z_2, e_1, e_2, g_1, g_2, h_1, 1, h_2, R_1 and R_2 replaced by u_{i-1}, u_i, v_{i-1}, v_i, w_{i-1}, w_i, z_{i-1}, z_i, e_{1i}, e_{2i}, g_{1i}, g_{2i}, h_{1i}, e_{2i}, h_{1i}, h_{2i}, R_{1i} and R_{2i} respectively. In fact, given $\epsilon > 0$, we have the following estimates.

$$\Big|\int L_1[u(x,t,\epsilon)]\psi(x)dx\Big| = O(\epsilon), \quad \Big|\int L_2[u(x,t,\epsilon), v(x,t,\epsilon)]\psi(x)\Big| = O(\epsilon)$$

$$\Big|\int L_3[u(x,t,\epsilon), v(x,t,\eta(\epsilon)), w(x,t,\epsilon)]\psi(x)dx\Big| = O(\epsilon)$$

$$\Big|\int L_4[u(x,t,\epsilon), v(x,t,\eta(\epsilon)), w(x,t,\eta(\epsilon)), z(x,t,\epsilon)]\psi(x)dx\Big| = O(\epsilon)$$

$$\Big|\int \Big(u(x,0,\epsilon) - u_0(x)\Big)\psi(x)dx\Big| = O(\epsilon), \quad \Big|\int \Big(v(x,0,\epsilon) - v_0(x)\Big)\psi(x)dx\Big| = O(\epsilon)$$

$$\Big|\int \Big(w(x,0,\epsilon) - w_0(x)\Big)\psi(x)dx\Big| = O(\epsilon), \quad \Big|\int \Big(z(x,0,\epsilon) - z_0(x)\Big)\psi(x)dx\Big| = O(\epsilon),$$

Therefore (u, v, w, z) is a weak asymptotic solution to system (1.4)–(1.5). Similar proof as in Theorem 2.4 gives (u, v, w, z) is an entropy weak asymptotic solution. $\square$

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