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BASIC EXISTENCE AND A PRIORI BOUND RESULTS FOR SOLUTIONS TO SYSTEMS OF BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. This article examines the qualitative properties of solutions to systems of boundary value problems involving fractional differential equations. Our primary interest is in forming new results that involve sufficient conditions for the existence of solutions. To do this, we formulate some new ideas concerning *a priori* bounds on solutions, which are then applied to produce the novel existence results. The main techniques of the paper involve the introduction of novel fractional differential inequalities and the application of the fixed-point theorem of Schäfer. We conclude the work with several new results that link the number of solutions to our problem with a fractional initial value problem, akin to an abstract shooting method. A YouTube video from the author that is designed to complement this research is available at `www.youtube.com/watch?v=cDUrLsQLGvA`

1. Introduction

"Although fractional differential equations are centuries old, it is surprising to discover that much of the basic qualitative and quantitative foundational theory is yet to be fully developed" [18]. Motivated by the above, in this work, our discussion is centred around the following system of boundary value problems (BVPs) for fractional differential equations of arbitrary order $0 < q < 1$

$$D^q\left(\mathbf{x} - \mathbf{x}(0)\right) = \mathbf{f}(t, \mathbf{x}); \tag{1.1}$$

$$M\mathbf{x}(0) + N\mathbf{x}(a) = \mathbf{b}. \tag{1.2}$$

Above: D^q represents the Riemann-Liouville fractional differentiation operator of arbitrary order $0 < q < 1$ (a precise definition is found in (2.1) a little later); $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$; $a > 0$; the M and N are constant matrices in $\mathbb{R}^{n\times n}$; and $\mathbf{b}$ is a constant vector in $\mathbb{R}^n$.

In particular, this work addresses the following questions:

- What are sufficient conditions under which (1.1), (1.2) will have bounds on its solutions?

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- What are sufficient conditions under which (1.1), (1.2) will have at least one solution?

Investigations into these kinds of questions shed light on the basic theory that supports advanced studies and applications of fractional differential equations to nonlinear phenomena – and thus are of significant interest.

Recent articles that have examined existence and uniqueness of solutions to fractional BVPs include [3, 5, 14, 16, 17, 27] where a variety of important methods have been used, including differential inequalities and fixed-point techniques. The approach herein differs from theirs in the sense that we formulate distinct differential inequalities and use different sufficient conditions in our theorems.

This work is organised as follows: In Section 2 the preliminary notation is presented. Section 3 contains new *a priori* bound results for solutions to (1.1), (1.2). The ideas rely on new differential inequalities applied to a particular Liapunov function. Our results are novel for the vector ($n > 1$) and scalar cases ($n = 1$). In Section 4, the *a priori* bound results from Section 3 are applied to provide new existence results for solutions to (1.1), (1.2). The method employs the fixed-point theorem of Schäfer [13, pp. 70–71]. Our results are new for both the vector and scalar cases and we include several example that illustrates how to apply the new ideas. Finally, Section 5 contains several new results that connect the number of solutions to our problem with a fractional initial value problem, akin to an abstract shooting method. For more recent and related research on qualitative and quantitative properties of solutions to fractional differential equations, the reader is referred to [6, 10, 11, 12, 18, 21, 22, 23, 24, 25] and the books [8, 9, 15].

2. Preliminaries

To understand the notation used in this work we now present some preliminary ideas and definitions. Define the Riemann-Liouville fractional derivative and integral of order $0 < q < 1$ of a vector-valued function $\mathbf{y}$ at a point $t > 0$, respectively, by:

$$\begin{aligned} D^q\mathbf{y}(t) &:= \frac{d}{dt}\frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\mathbf{y}(s)\,ds; \\ I^q\mathbf{y}(t) &:= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathbf{y}(s)\,ds. \end{aligned} \tag{2.1}$$

A solution to (1.1), (1.2) on the interval $[0, a]$ is defined to be a function $\mathbf{x} : [0, a] \to \mathbb{R}^n$ such that $\mathbf{x}(t)$ satisfies: (1.1) for all $t \in [0, a]$; and (1.2).

For $\mathbf{u} \in \mathbb{R}^n$ we define the inner product as

$$\langle \mathbf{u}, \mathbf{u}\rangle := \|\mathbf{u}\|^2$$

where $\|\cdot\|$ is the usual Euclidean norm, that is, $\|\mathbf{u}\| := (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}$.

If A is a matrix, then we understand $\|A\|$ to represent any norm that is compatible with the above Euclidean norm.

We now provide a theorem on the equivalence between the fractional BVP (1.1), (1.2) and a particular integral equation. The integral equation is of a more tractable nature than the original problem (1.1), (1.2) and will be used in latter sections.

Theorem 2.1. *Let* $\mathbf{f} : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous. If the matrix* $(M+N)$ *is invertible then the fractional BVP* (1.1), (1.2) *is equivalent to the integral equation*

$$\begin{aligned} \mathbf{x}(t) = (M+N)^{-1}\mathbf{b} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathbf{f}(s, \mathbf{x}(s))\, ds \\ - (M+N)^{-1} N \frac{1}{\Gamma(q)} \int_0^a (a-s)^{q-1} \mathbf{f}(s, \mathbf{x}(s))\, ds. \end{aligned} \tag{2.2}$$

Proof. For completeness, we provide a proof. An application of the fractional integral operator I^q to both sides of (1.1) and then an application of the Fundamental Theorem of Fractional Calculus yields

$$\mathbf{x}(t) = \mathbf{x}(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mathbf{f}(s, \mathbf{x}(s))\, ds, \quad t \in [0,a] \tag{2.3}$$

and so substituting $t = a$ into (2.3) we obtain

$$\mathbf{x}(a) = \mathbf{x}(0) + \frac{1}{\Gamma(q)} \int_0^a (a-s)^{q-1} \mathbf{f}(s, \mathbf{x}(s))\, ds.$$

From the boundary conditions (1.2) we then obtain

$$M\mathbf{x}(0) + N \left[\mathbf{x}(0) + \frac{1}{\Gamma(q)} \int_0^a (a-s)^{q-1} \mathbf{f}(s, \mathbf{x}(s))\, ds \right] = \mathbf{b}$$

which can be rearranged to form

$$\mathbf{x}(0) = (M+N)^{-1}\mathbf{b} - (M+N)^{-1} N \frac{1}{\Gamma(q)} \int_0^a (a-s)^{q-1} \mathbf{f}(s, \mathbf{x}(s))\, ds.$$

Back-substitution into (2.3) then yields (2.2) as required. □

3. *A priori* bounds on solutions

We now examine *a priori* bounds on solutions for (1.1), (1.2). These results give us geometric insight into potential solutions (1.1), (1.2) by providing us with an estimate on their size and location without having explicit knowledge of solutions. The ideas will be applied to prove the existence of solutions in the following section. Our new *a priori* bound result for (1.1), (1.2) is now presented.

Theorem 3.1. *Let* $\mathbf{f} : [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and let the matrices* M *and* $(M+N)$ *be invertible. If there exist non-negative constants* β *and* L *such that*

$$\|\mathbf{f}(t, \mathbf{u})\| \le -2\beta \langle \mathbf{u}, \mathbf{f}(t, \mathbf{u}) \rangle + L, \quad \text{for all } (t, \mathbf{u}) \in [0,a] \times \mathbb{R}^n; \tag{3.1}$$

$$2\|M^{-1}N\|^2 (1 + \|(M+N)^{-1}N\|) \le \|(M+N)^{-1}N\| \tag{3.2}$$

then all solutions $\mathbf{x}$ *to* (1.1), (1.2) *satisfy the* a priori *bound*

$$\begin{aligned} \|\mathbf{x}(t) - (M+N)^{-1}\mathbf{b}\| \le 2\beta \|M^{-1}\mathbf{b}\|^2 (1 + \|(M+N)^{-1}N\|) \\ + \ L(1 + \|(M+N)^{-1}N\|) a^q / \Gamma(q+1), \end{aligned} \tag{3.3}$$

for all $t \in [0,a]$.

Proof. Let $\mathbf{x}$ be a solution to (1.1), (1.2) on $[0,a]$ and define the Liapunov function

$$r(t) := \|\mathbf{x}(t)\|^2, \quad t \in [0,a].$$

Now, by [2, Lemma 1], for all $t \in [0,a]$, we have

$$D^q (r(t) - r(0)) \le 2 \langle \mathbf{x}(t), D^q (\mathbf{x}(t) - \mathbf{x}(0)) \rangle$$

and so

$$D^q(r(t)-r(0)) \le 2\langle \mathbf{x}(t), \mathbf{f}(t,\mathbf{x}(t))\rangle. \tag{3.4}$$

From Theorem 2.1 the equivalent integral representation for (1.1), (1.2) is given in (2.2). Thus, for all $t \in [0,a]$ we have

$$\begin{aligned}
&\|\mathbf{x}(t)-(M+N)^{-1}\mathbf{b}\| \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|\mathbf{f}(s,\mathbf{x}(s))\|\,ds \\
&\quad + \|(M+N)^{-1}N\|\frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1}\|\mathbf{f}(s,\mathbf{x}(s))\|\,ds \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[-2\beta\langle \mathbf{x}(s),\mathbf{f}(s,\mathbf{x}(s))\rangle + L\right]\,ds \\
&\quad + \|(M+N)^{-1}N\|\frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1}\left[-2\beta\langle \mathbf{x}(s),\mathbf{f}(s,\mathbf{x}(s))\rangle + L\right]\,ds \\
&= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[-2\beta\langle \mathbf{x}(s), D^q(\mathbf{x}(s)-\mathbf{x}(0))\rangle + L\right]\,ds \\
&\quad + \|(M+N)^{-1}N\|\frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1}\left[-2\beta\langle \mathbf{x}(s), D^q(\mathbf{x}(s)-x(0))\rangle + L\right]\,ds \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[-\beta D^q(r(s)-r(0))\right]\,ds + L(1+\|(M+N)^{-1}N\|)a^q/\Gamma(q+1) \\
&\quad + \|(M+N)^{-1}N\|\frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1}\left[-\beta D^q(r(s)-r(0))\right]\,ds \\
&= -\beta I^q[D^q(r(t)-r(0))] + \|(M+N)^{-1}N\|(-\beta I^q[D^q(r(a)-r(0))]) \\
&\quad + L(1+\|(M+N)^{-1}N\|)a^q/\Gamma(q+1).
\end{aligned}$$

Where we have applied (3.1) and (3.4).

Applying the Fundamental Theorem of Fractional Calculus, we see

$$\begin{aligned}
&\|\mathbf{x}(t)-(M+N)^{-1}\mathbf{b}\| \\
&\le \beta[r(0)-r(t)] + \beta\|(M+N)^{-1}N\|[r(0)-r(a)] \\
&\quad + L(1+\|(M+N)^{-1}N\|)a^q/\Gamma(q+1) \\
&\le \beta\|\mathbf{x}(0)\|^2 + \beta\|(M+N)^{-1}N\|[\|\mathbf{x}(0)\|^2 - \|\mathbf{x}(a)\|^2] \\
&\quad + L(1+\|(M+N)^{-1}N\|)a^q/\Gamma(q+1).
\end{aligned}$$

Using the boundary conditions (1.2) in the previous line we have

$$\begin{aligned}
&\|\mathbf{x}(t)-(M+N)^{-1}\mathbf{b}\| \\
&\le \beta\|M^{-1}(\mathbf{b}-N\mathbf{x}(a))\|^2 + \beta\|(M+N)^{-1}N\|[\|M^{-1}(\mathbf{b}-N\mathbf{x}(a))\|^2 - \|\mathbf{x}(a)\|^2] \\
&\quad + L(1+\|(M+N)^{-1}N\|)a^q/\Gamma(q+1).
\end{aligned}$$

Now, using the inequality

$$\|M^{-1}(\mathbf{b}-N\mathbf{x}(a))\|^2 \le 2\|M^{-1}\mathbf{b}\|^2 + 2\|M^{-1}N\mathbf{x}(a)\|^2$$

we obtain

$$\begin{aligned}
&\|\mathbf{x}(t)-(M+N)^{-1}\mathbf{b}\| \\
&\le 2\beta\|M^{-1}\mathbf{b}\|^2(1+\|(M+N)^{-1}N\|)
\end{aligned}$$

$$
\begin{aligned}
&+ \beta\|\mathbf{x}(a)\|^2(2\|M^{-1}N\|^2 + \|(M+N)^{-1}N\|(2\|M^{-1}N\|^2 - 1))) \\
&+ L(1 + \|(M+N)^{-1}N\|)a^q/\Gamma(q+1) \\
\le\ & 2\beta\|M^{-1}\mathbf{b}\|^2(1 + \|(M+N)^{-1}N\|) + L(1 + \|(M+N)^{-1}N\|)a^q/\Gamma(q+1).
\end{aligned}
$$

Above, we have employed (3.2).

Thus, the *a priori* bound (3.3) holds for all solutions to (1.1), (1.2). □

In the scalar case ($n = 1$) in (1.1), (1.2) we have the following new result as a corollary of Theorem 3.1.

Corollary 3.2. *Let $f : [0,a] \times \mathbb{R} \to \mathbb{R}$ be continuous and with (the numbers) $M \neq 0$ and $M + N \neq 0$. If there exist non-negative constants β and L such that*

$$
|f(t,u)| \le -2\beta u f(t,u) + L, \quad \text{for all } (t,u) \in [0,a] \times \mathbb{R},
$$
$$
2|N/M|^2(1 + |N/(M+N)|) \le |N/(M+N)|
$$

then all solutions x to (1.1), (1.2) *satisfy the* a priori *bound*

$$
\begin{aligned}
|x(t) - b/(M+N)| &\le 2\beta|b/M|^2(1 + |N/(M+N)|) \\
&\quad + L(1 + |N/(M+N)|)a^q/\Gamma(q+1), \quad \text{for all } t \in [0,a].
\end{aligned}
$$

Remark 3.3. If (3.1) holds with $\beta = 0$ then we enter the classically-important territory of fractional differential equations with uniformly bounded right-hand sides, with the bound on solutions simplified accordingly in (3.3).

4. Existence of solutions

We now apply the results of Section 3 to generate new existence results for solutions to (1.1), (1.2). Our main existence results employs the ideas of Theorem 3.1.

Theorem 4.1. *If the conditions of Theorem 3.1 hold, then the fractional boundary value problem* (1.1), (1.2) *has at least one solution.*

Proof. We apply Schäfer's fixed-point theorem [13, pp.70–71]. Consider the normed space

$$
(C([0,a];\mathbb{R}^n), \|\cdot\|_0)
$$

which consists of the space of continuous, vector-valued functions on $[0,a]$ and the maximum norm

$$
\|\mathbf{x}\|_0 := \max_{t\in[0,a]} \|\mathbf{x}(t)\|.
$$

Also consider the family of equations

$$
\mathbf{x} = \lambda \mathbf{F}\mathbf{x}, \quad \lambda \in [0,1) \tag{4.1}
$$

where $\mathbf{F} : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ is defined by

$$
\begin{aligned}
[\mathbf{F}\mathbf{x}](t) : &= (M+N)^{-1}\mathbf{b} + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\mathbf{f}(s,\mathbf{x}(s))\,ds \\
&\quad - (M+N)^{-1}N\frac{1}{\Gamma(q)}\int_0^a (a-s)^{q-1}\mathbf{f}(s,\mathbf{x}(s))\,ds.
\end{aligned}
$$

Now, showing there is an $\mathbf{x} \in C([0,a];\mathbb{R}^n)$ such that $\mathbf{x} = \mathbf{F}\mathbf{x}$ is equivalent to showing (1.1), (1.2) has at least one solution.

To apply Schäfer's theorem, we note that $\mathbf{F}$ is continuous and compact, see, for example, [18]. In addition, we show that the set of solutions to the family (4.1) is bounded. This is equivalent to showing that, for each $\lambda \in [0,1)$, solutions to the following family of boundary value problems

$$D^q(\mathbf{x} - \mathbf{x}(0)) = \lambda \mathbf{f}(t,\mathbf{x}), \tag{4.2}$$

$$M\mathbf{x}(0) + N\mathbf{x}(a) = \lambda \mathbf{b} \tag{4.3}$$

are bounded, with the bound independent of λ. Let $\mathbf{x}_\lambda$ be a solution to (4.2), (4.3) for fixed $\lambda \in [0,1)$. We show that $\lambda\mathbf{f}$ satisfies the conditions of Theorem 3.1. If (3.1) holds then we multiply both sides by $\lambda \in [0,1)$ to obtain

$$\|\lambda\mathbf{f}(t,\mathbf{u})\| \le -2\beta\langle \mathbf{u}, \lambda\mathbf{f}(t,\mathbf{u})\rangle + \lambda L \le -2\beta\langle \mathbf{u}, \lambda\mathbf{f}(t,\mathbf{u})\rangle + L$$

and so $\lambda\mathbf{f}$ satisfies the conditions of Theorem 3.1. Thus,

$$\begin{aligned}\|\mathbf{x}_\lambda(t) - (M+N)^{-1}\mathbf{b}\| &\le 2\beta\|M^{-1}N\|^2(1+\|(M+N)^{-1}N\|)\\ &\quad + L(1+\|(M+N)^{-1}N\|)a^q/\Gamma(q+1), \quad \text{for all } t\in[0,a].\end{aligned}$$

with the bound independent of λ.

Schäfer's theorem now can be applied to yield the existence of at least one $\mathbf{x} \in C([0,a];\mathbb{R}^n)$ such that $\mathbf{x} = \mathbf{F}\mathbf{x}$ – equivalently showing (1.1), (1.2) has at least one solution. □

Remark 4.2. If the conditions of Theorem 3.1 hold with $\beta = 0$ then we obtain a result of classical importance: if $\mathbf{f}$ is continuous and bounded (by L) on $[0,a]\times\mathbb{R}^n$ then the system of fractional BVPs (1.1), (1.2) has at least one solution.

We now present some examples of vector-valued $\mathbf{f}$ that satisfy the conditions of Theorem 4.1.

Example 4.3. Let $\mathbf{f}$ be defined on $[0,1]\times\mathbb{R}^2$ by

$$\mathbf{f}(t,x_1,x_2) := \begin{pmatrix} -x_1 - x_2 \\ -x_2 + x_1 \end{pmatrix}.$$

This $\mathbf{f}$ satisfies the conditions of Theorem 4.1 with $\beta = 1/\sqrt{2}$ and $L = 2$.

Proof. We show that (3.1) holds with $\beta = 1$ and $L = 2$. We have

$$\|\mathbf{f}(t,x_1,x_2)\| = \sqrt{2x_1^2 + 2x_2^2} \le \sqrt{2}(|x_1| + |x_2|).$$

Also,

$$-2\beta\langle (x_1,x_2), \mathbf{f}(t,x_1,x_2)\rangle + L = -2\beta(-x_1^2 - x_2^2) + L \ge \sqrt{2}(|x_1| + |x_2|)$$

for the choices $\beta = 1/\sqrt{2}$ and $L = 2$. □

Example 4.4. Let $\mathbf{f}$ be defined on $[0,1]\times\mathbb{R}^2$ by

$$\mathbf{f}(t,x_1,x_2) := \begin{pmatrix} \cos(x_1x_2t) \\ \sin(x_1x_2t) \end{pmatrix}.$$

This $\mathbf{f}$ satisfies the conditions of Theorem 4.1 with $\beta = 0$ and $L = 1$.

Proof. We show that (3.1) holds with $\beta = 0$ and $L = 1$. We have

$$\|\mathbf{f}(t,x_1,x_2)\| = \sqrt{\cos^2(x_1x_2t) + \sin^2(x_1x_2t)} \le 1.$$

□

Corollary 4.5. *Consider the scalar version of* (1.1), (1.2) *so that* $n = 1$*. If the conditions of Corollary 3.2 hold then the fractional boundary value problem* (1.1), (1.2) *has at least one solution.*

We conclude this section with an example that highlights how our *a priori* bound and existence ideas come together for the scalar case ($n = 1$) which is new in its own right.

Example 4.6. Consider the scalar BVP

$$D^q\left(x - x(0)\right) = -x^7, \quad x(0) + 0.25x(1/2) = 1.$$

All solutions are bounded *a priori* on $[0, 1]$ and the problem has at least one solution.

Proof. Here we have a scalar-valued problem with: $f(u) := -u^7$; $M = 1$; $N = 1/4$; and $b = 1$. We show that the conditions of Corollary 3.2 hold. It is clear that $\mathbf{f}$ is continuous. Consider

$$-2\beta uf(t, u) + L = 2\beta u^8 + L \geq |-u^7|$$

for the choices $\beta = 1/2$ and $L = 1$. Finally,

$$2|N/M|^2(1 + |N/(M + N)|) = 3/20 \leq 1/5 = |N/(M + N)|.$$

Hence all of the conditions of Corollary 3.2 hold and conclude, by Corollary 4.5, that our example has at least one solution. $\square$

5. Counting solutions of boundary value problems

We conclude the work with several new results that link the number of solutions to our problem (1.1), (1.2) with a fractional initial value problem, akin to an abstract shooting method.

Theorem 5.1. *Let* $\mathbf{f} : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ *be continuous and let* M *be invertible. If there is a constant* $P > 0$ *such that*

$$\|\mathbf{f}(t, \mathbf{u}) - \mathbf{f}(t, \mathbf{v})\| \leq P\|\mathbf{u} - \mathbf{v}\|, \quad \text{for all } (t, \mathbf{u}),\ (t, \mathbf{v}) \in [0, a] \times \mathbb{R}^n \tag{5.1}$$

then the fractional BVP (1.1), (1.2) *has as many solutions as there are distinct roots of the equation*

$$\mathbf{G}(\mathbf{s}) := M\mathbf{s} + N\mathbf{x}(a; \mathbf{s}) = \mathbf{b} \tag{5.2}$$

where $\mathbf{x}(t; \mathbf{s})$ *is the unique solution of the initial value problem*

$$D^q\left(\mathbf{x} - \mathbf{x}(0)\right) = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{s}. \tag{5.3}$$

Proof. Part (i): Since $\mathbf{f}$ satisfies the Lipschitz condition (5.1), we know that for each $\mathbf{s}$ the fractional IVP (5.3) has a unique solution for all $t \in [0, a]$, which we denote by $\mathbf{x}(t; \mathbf{s})$ (see, for example, [6, 18]).

Now, if $\mathbf{s}$ satisfies equation (5.2) then we claim that $\mathbf{x}(t; \mathbf{s})$ will also satisfy the fractional BVP (1.1), (1.2). This follows because: the fractional differential equations in (5.3) and (1.1) are identical; and (1.2) holds since

$$\mathbf{b} = M\mathbf{s} + N\mathbf{x}(a; \mathbf{s}) = M\mathbf{x}(0; \mathbf{s}) + N\mathbf{x}(a; \mathbf{s}).$$

If $\mathbf{b}_1$ and $\mathbf{b}_2$ are distinct roots of (5.2) then for all $t \in [0, a]$ we have

$$\mathbf{x}(t; \mathbf{b}_1) \neq \mathbf{x}(t; \mathbf{b}_2)$$

because of the uniqueness of solutions to (5.3), so each distinct root of (5.2) will yield a distinct solution of the fractional BVP (1.1), (1.2).

Part (ii): Now let $\mathbf{x} = \mathbf{x}(t)$ be a solution to the fractional BVP (1.1), (1.2). From (1.2) it follows that $\mathbf{x}$ satisfies the fractional IVP (5.3) for the value of

$$\mathbf{s} = M^{-1}(\mathbf{b} - N\mathbf{x}(a)).$$

The above value of $\mathbf{s}$ also satisfies (5.2) so that every solution to the fractional BVP (1.1), (1.2) yields a root of (5.2). □

As can be seen from the previous proof, the continuity and Lipschitz assumptions on $\mathbf{f}$ ensure existence and uniqueness of solutions to the fractional IVP (5.3). We now generalize this idea.

Theorem 5.2. *Let $\mathbf{f} : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$. If, for each $\mathbf{s}$, solutions to the fractional IVP* (5.3) *exist and are unique on $[0, a]$, then the fractional BVP* (1.1), (1.2) *has as many solutions as there are distinct roots of the equation* (5.2).

For the proof of the above theorem, the ideas mirror those of the proof of Theorem 5.1 and so are omitted.

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