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NULL CONTROLLABILITY OF A CASCADE SYSTEM OF SCHRÖDINGER EQUATIONS

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Abstract. This article presents a control problem for a cascade system of two linear N-dimensional Schrödinger equations. We address the problem of null controllability by means of a control supported in a region not satisfying the classical geometrical control condition. The proof is based on the application of a Carleman estimate with degenerate weights to each one of the equations and a careful analysis of the system in order to prove null controllability with only one control force.

1. Introduction

The controllability of coupled systems of PDE's has been intensely studied in recent years. In particular, very interesting problems arise when there are less controls than equations.

Null controllability results for systems of parabolic equations are reviewed in the survey [3]. About the systems of hyperbolic equations, we can mention [1, 2, 8], where the controllability of two coupled wave equations is proved with only one control, under the hypothesis of the geometric control condition. In [1, 2], the authors show that as a consequence, the same result is valid for a system of two Schrödinger equations. A boundary controllability result is proved in [13] for a cascade system of Schrödinger equations with periodic boundary conditions, also as a consequence of the controllability result for a cascade system of two wave equations. In this article we are interested in the null controllability of a linear system formed by two Schrödinger equations, controlling only one of them.

The controllability of (scalar) Schrödinger equations has been intensively studied in recent years. In [11] a general result about this problem was obtained: the author proved that if the wave equation is controllable at some time $T_0 > 0$ from controls supported in a subset of the domain, then the Schrödinger equation is controllable with controls supported in the given region, for any $T > 0$.

For the wave equation, boundary controllability at time $T > 0$ is equivalent (see [4]) to the fact that the zone of control meets every ray of the geometric optic in the domain in a smaller time than T. This is called the *geometric control condition*. In the case of the Schrödinger equation, this is no longer true: for some particular

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domains the equation is controllable by mean of controls acting in some open subset of the boundary that do not satisfies the geometric control condition ([6, 14, 15, 16]). In [18] a comprehensive review of related results is presented. It remains an open problem to find general controllability results for the Schrödinger equation with weaker geometric conditions.

A very important tool to prove controllability of evolution PDE's is given by Carleman estimates. To our knowledge the first paper to derive a global Carleman estimate for Schrödinger operators is [17], where was proved the exact controllability of a plate equation. Carleman estimates are also used to study a related problem: the stability of the inverse problem of retrieving a given coefficient in an equation, from observations of a trace of the solution. In [5] a Carleman estimate for Schrödinger operators is proved, with observations on a subset of the boundary satisfying the geometric control condition. This implies the stability of the stated inverse problem. Very interesting results are proved in [10] for non conservative Schrödinger equations in both cases: with observations on a set satisfying the geometrical condition and not satisfying it. In [12], some Carleman inequalities with an observation set not satisfying the geometric condition are proved, and then the stability of an inverse problem for a space-dependent coefficient is obtained. The results of [12] also imply the controllability of a scalar Schrödinger equation by means of an H^{-1} internal control acting in an open set not satisfying the geometric control condition. See Remark 3.3.

The main objective of this article is to use the Carleman estimates from [12] in order to prove the controllability of a coupled system of two Schrödinger equations from an open subset of the domain which does not satisfy the geometric control condition. We are able to prove the result with a control acting in only one of the equations. As far as we know it is the first distributed null controllability result for coupled Schrödinger equations that is not a consequence of a similar result for the wave equation.

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set with C^2 boundary, $N \geq 1$. Let ω and $\mathcal{O}$ be two nonempty open subsets of Ω. For $T > 0$ we set $Q = \Omega \times (0,T)$, and $\Sigma = \partial\Omega \times (0,T)$. We consider the following cascade system of Schrödinger equations:

$$
\begin{gathered}
ip_t + \Delta p = h_\omega \quad \text{in } \; Q,\\
iu_t + \Delta u = p\rho_{\mathcal{O}} \quad \text{in } Q,\\
p = 0, \quad u = 0 \quad \text{on } \Sigma,\\
p(x,0) = p^0(x), \quad u(x,0) = u^0(x) \quad \text{in } \Omega,
\end{gathered} \tag{1.1}
$$

where p^0, u^0 are given, h_ω is a control with support in $\omega \times (0,T)$ and $\rho_{\mathcal{O}}$ is a regular approximation of the characteristic function $1_{\mathcal{O}}$ of the set $\mathcal{O}$.

In this work we analyze the null controllability of the cascade system (1.1) with one interior control h_ω, i.e. we give conditions on T, ω, and $\mathcal{O}$ such that for every (p^0, u^0) in $L^2(\Omega)^2$ there exists a control h_ω with support in $\omega \times (0,T)$ such that the corresponding solution of (1.1) satisfies

$$
p(x,T) = 0, \quad u(x,T) = 0 \quad \text{in } \Omega. \tag{1.2}
$$

Throughout this article, we denote $X = D(-\Delta) = H^2 \cap H_0^1(\Omega)$ endowed with the usual norm

$$
\|v\|_X = \Big(\int_\Omega |\Delta v|^2 dx \Big)^{1/2} \quad \text{for all } v \in X.
$$

Here and throughout the paper $n(x)$ denotes the unitary exterior normal vector at $x \in \partial\Omega$, e_1 means the unitary vector $(1, 0, \dots, 0) \in \mathbb{R}^N$, and x_1 is the first component of $x \in \mathbb{R}^N$.

To state the hypothesis on the domain and the observability region, let $[a, d]$ be the x_1-projection of $\overline{\Omega}$. We shall assume that there exists an open set $\widehat{\omega} \subset \omega \cap \mathcal{O}$ with $\operatorname{dist}(\partial\widehat{\omega} \cap \Omega, \partial\omega \cap \Omega) \geq \alpha > 0$ and real numbers b, c with $a \leq b < c \leq d$ such that

$$([b, c] \times \mathbb{R}^{N-1}) \cap \Omega \subset \widehat{\omega}, \tag{1.3}$$

$$n(x) \cdot e_1 = 0 \quad \text{for all } x \in \partial\widehat{\omega} \cap \partial\Omega. \tag{1.4}$$

Also we assume that there exists a function $\psi \in C^4([a, d])$ satisfying

$$\begin{gathered} \psi' \neq 0 \text{ in } [a, b] \cup [c, d], \quad \psi'(x_1) n(x) \cdot e_1 \leq 0 \text{ for all } x \in \partial\Omega, \\ |\psi'|^2 + \psi'' \geq 0 \text{ in } [a, b] \cup [c, d], \quad \psi \geq \frac{2}{3} \|\psi\|_{L^\infty(a,d)} \text{ in } (a, d). \end{gathered} \tag{1.5}$$

Our main result reads as follows:

Theorem 1.1. *Suppose that there exists an open set $\widehat{\omega} \subset \omega \cap \mathcal{O}$ satisfying* (1.3)-(1.4) *and a function $\psi \in C^4([a, d])$ satisfying the hypotheses* (1.5). *Then for each $(p^0, u^0) \in L^2(\Omega)^2$ there exists a control $h_\omega \in L^2(0, T; X')$ supported in $\omega \times (0, T)$ such that the solution (p, u) of system* (1.1) *satisfies $(p(T), u(T)) = (0, 0)$.*

Remark 1.2. Hypotheses (1.3)-(1.5) are satisfied if Ω is an "stadium" (see Figure 1), and $\widehat{\omega}$ is a "strip" in $\mathbb{R}^N$, that is $\widehat{\omega} = ((b - \varepsilon, c + \varepsilon) \times \mathbb{R}^{N-1}) \cap \Omega$ for some $\varepsilon > 0$ ($\widehat{\omega}$ is the shaded region in Figure 1), and ψ is given by

$$\psi(x) = \begin{cases} x - a_1, & x \in [a, b], \\ d_1 - x, & x \in [c, d], \\ \rho(x), & x \in [b, c], \end{cases}$$

where $a_1 < a$ and $d < d_1$, and ρ is a suitable function. Note that (1.5) are fulfilled if a_1 and d_1 are chosen such that $a - a_1$ and $d_1 - d$ are large enough.

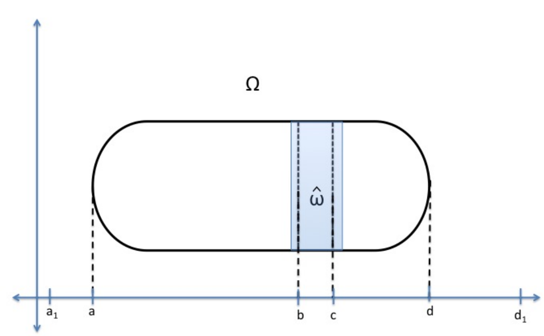

FIGURE 1. Observation region in a stadium $\Omega \subset \mathbb{R}^2$.

Remark 1.3. For any $h \in L^2(0,T;X')$ and any $(p^0,u^0) \in L^2(\Omega)^2 \subset (X')^2$, the cascade system (1.1) has exactly one solution (p,u) (in the sense specified in Section 2), with $(p,u) \in C([0,T];X')^2$, provided that $\rho_{\mathcal{O}} \in C^2(\overline{\Omega})$.

The proof of Theorem 1.1 is based on the existence of a constant $C > 0$ such that the observability inequality

$$\|z(0)\|_{L^2}^2 + \|q(0)\|_{L^2}^2 \le C \int_0^T \int_\Omega |\Delta(\rho_\omega q)|^2 \, dx \, dt \tag{1.6}$$

holds for any solution of the adjoint system

$$\begin{gathered} iz_t + \Delta z = 0 \quad \text{in } Q, \\ iq_t + \Delta q = z\rho_{\mathcal{O}} \quad \text{in } Q, \\ z = 0, \quad q = 0 \quad \text{on } \Sigma, \\ z(x,T) = z^0(x), \quad q(x,T) = q^0(x) \quad \text{in } \Omega, \end{gathered} \tag{1.7}$$

associated to $(z^0,q^0) \in X^2$. To have the appropriate regularity and support of the control, in (1.6) we consider a function $\rho_\omega \in C^2(\overline{\Omega})$, such that $\rho_\omega(x) = 0$ for all $x \in \Omega \backslash \omega$ and $\rho_\omega(x) = 1$ for all x in a large part of ω; in Section 3 we give the precise details.

The rest of this article is organized as follows: in Section 2 we state the functional framework where we will state the controllability problems. Section 3 is devoted to prove the observability inequality (1.6). Finally, Theorem 1.1 will be proved in Section 4.

2. Well posedness

In this section we recall some existence and regularity results for the Schrödinger equation. These results can be found in [7]. From now on, C stands for a generic positive constant depending only on Ω, T, ω and $\mathcal{O}$, which can take different values from line to line.

Let $k \in L^2(0,T;X)$ and $v^0 \in X$. Then the solution v of the linear problem

$$\begin{gathered} iv_t + \Delta v = k \quad \text{in } Q, \\ v = 0 \quad \text{on } \Sigma, \\ v(x,0) = v^0(x) \quad \text{in } \Omega, \end{gathered} \tag{2.1}$$

satisfies $v \in C([0,T];X)$. Moreover, there exists $C > 0$ such that

$$\|v\|_{L^\infty(0,T;X)} \le C(\|v^0\|_X + \|k\|_{L^2(0,T;X)}).$$

When $k \in C([0,T],L^2(\Omega))$ the corresponding solution satisfies $v \in C([0,T],X) \cap C^1([0,T],L^2(\Omega))$.

We need to solve (2.1) with $k \in L^2(0,T;X')$ and $v^0 \in L^2(\Omega)$. Under this assumption on k and considering $p^0 \in L^2(\Omega)$, the solution by transposition of

$$\begin{gathered} ip_t + \Delta p = k \quad \text{in } Q, \\ p = 0 \quad \text{on } \Sigma, \\ p(x,0) = p^0(x) \quad \text{in } \Omega, \end{gathered} \tag{2.2}$$

is, by definition, the unique function $p \in L^2(0,T;X')$ satisfying

$$\int_0^T \langle p(t), g(t)\rangle dt = \int_0^T \langle k(t), \varphi_g(\cdot,t)\rangle dt + i(p^0, \varphi_g(\cdot,0))_{L^2(\Omega)} \tag{2.3}$$

for all $g \in L^2(0,T;X)$, where $\langle \cdot, \cdot \rangle$ represents the duality between X and X', and for each $g \in L^2(0,T;X)$ we have denoted by φ_g the solution to the corresponding adjoint system

$$\begin{gathered} i\varphi_t + \Delta\varphi = g \quad \text{in } Q, \\ \varphi = 0 \quad \text{on } \Sigma, \\ \varphi(x,T) = 0 \quad \text{in } \Omega. \end{gathered} \tag{2.4}$$

Note that the solution φ_g of (2.4) satisfies

$$\begin{gathered} \varphi_g \in L^2(0,T;X) \cap C([0,T];L^2(\Omega)), \\ \|\varphi_g\|_{L^2(0,T;X)} + \|\varphi_g(0)\|_{L^2(\Omega)} \le C\|g\|_{L^2(0,T;X)}. \end{gathered}$$

Since X is a reflexive space, we have $L^2(0,T;X)' = L^2(0,T;X')$. Hence (2.3) makes sense, and we conclude that there exists a unique $p \in L^2(0,T;X')$ solution of (2.2), which satisfies

$$\|p\|_{L^2(0,T;X')} \le C(\|k\|_{L^2(0,T;X')} + \|p^0\|_{L^2(\Omega)}).$$

By energy estimates and density arguments we obtain that in fact $p \in C([0,T];X')$.

3. Observability inequality

In this section the observability inequality (1.6) for the adjoint system (1.7) will be proved. We use a Carleman estimate proved in [12]. To state the result, we introduce the following notation. For the function ψ given by (1.5), we set $C_\psi = 2\|\psi\|_{L^\infty(\Omega)}$ and we define the auxiliary functions

$$\theta(x,t) := \frac{e^{\lambda\psi(x_1)}}{t(T-t)}, \quad \varphi(x,t) := \frac{e^{\lambda C_\psi} - e^{\lambda\psi(x_1)}}{t(T-t)}, \quad \text{for all } (x,t) \in \Omega \times (0,T),$$

for $\lambda > 0$.

The following Carleman inequality for the Schrödinger equation is a particular case of [12, Corollary 3.3].

Proposition 3.1. *Let us define $\widetilde{\omega} = ((b,c) \times \mathbb{R}^{N-1}) \cap \Omega$ (see (1.3)), take $\psi \in C^4(\mathbb{R})$ be a function satisfying (1.5). Then there exists a constant $C > 0$ such that for $f \in C^{2,1}(\overline{\Omega} \times [0,T])$, with $f = 0$ on Σ, it holds*

$$\begin{aligned} &\int_Q [\theta|\frac{\partial f}{\partial x_1}|^2 + \theta^3|f|^2] e^{-2s\varphi}\,dx\,dt \\ &\le C\Big(\int_Q |f_t + i\Delta f|^2 e^{-2s\varphi}\,dx\,dt + \int_0^T \int_{\widetilde{\omega}} [\theta|\frac{\partial f}{\partial x_1}|^2 + \theta^3|f|^2] e^{-2s\varphi}\,dx\,dt\Big). \end{aligned} \tag{3.1}$$

Remark 3.2. The result in [12] is more general than Proposition 3.1, because it does not ask the weight function to depend only in one variable. Here we use such a function to estimate one of the observations of the two equations of the system, and to obtain the controllability with only one control (see Proposition 3.4).

Remark 3.3. It is not difficult to see that Proposition 3.1 implies the observability inequality

$$\|q(0)\|_{L^2(\Omega)} \le C \int_0^T \int_{\widetilde{\omega}} (|q|^2 + |\nabla q|^2)\,dx\,dt \tag{3.2}$$

for all $q^0 \in H_0^1(\Omega)$, where q solves the equation

$$\begin{gathered} iq_t + \Delta q = 0 \quad \text{in } \Omega, \\ q = 0 \quad \text{on } \Sigma, \\ q(0) = q^0 \quad \text{in } \Omega. \end{gathered} \tag{3.3}$$

In fact, inequality (3.2) follows from (3.1) (with $f = q$ solution to (3.3)) and the fact that

$$\int_Q \theta^3 |q|^2 e^{-2s\varphi}\,dx\,dt \geq C_T \int_{T/4}^{3T/4} \int_\Omega |q|^2\,dx\,dt = TC_T/2\|q(0)\|_{L^2(\Omega)}.$$

From (3.2) we have a controllability result: for every $u^0 \in L^2(\Omega)$ there exists a control $h \in L^2(0,T;H^{-1}(\Omega))$ supported on $\widetilde{\omega}$ such that the solution of the equation

$$\begin{gathered} iu_t + \Delta u = h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Sigma, \\ u(0) = u^0 \quad \text{in } \Omega, \end{gathered}$$

satisfies $u(T) = 0$.

As we said, the proof of Theorem 1.1 depends on an observability inequality for the adjoint system (1.7). The result is the following.

Proposition 3.4. *Assume the hypothesis of Theorem 1.1. There exists a constant $C > 0$ such that*

$$\|z(0)\|_{L^2}^2 + \|q(0)\|_{L^2}^2 \leq C \int_0^T \int_\Omega |\Delta(\rho_\omega q)|^2\,dx\,dt \tag{3.4}$$

for all $z^0, q^0 \in H^2 \cap H_0^1(\Omega)$, where (z,q) is the solution of (1.7), and ρ_ω is a cut-off function supported in ω.

Remark 3.5. Every solution of (1.7) with $z^0, q^0 \in H^2 \cap H_0^1(\Omega)$ satisfies

$$z, q \in C([0,T];X) \cap C^1([0,T];L^2(\Omega)).$$

Proof of Proposition 3.4. We will deal with regular solutions, getting the final result by standard density arguments. Let $\widehat{\omega}$ be the open satisfying hypothesis (1.3) and (1.4). We claim that there exists a constant C such that, for every $\varepsilon > 0$, the following estimates hold for solutions of the adjoint system (1.7).

Claim 1.

$$\begin{aligned} &\int_0^T \int_{\widetilde{\omega}} \theta^3 |z|^2 e^{-2s\varphi}\,dx\,dt \\ &\leq C\varepsilon^{-1} \int_0^T \int_{\widehat{\omega}} (|q|^2 + |\nabla q|^2)\,dx\,dt + \varepsilon \int_Q [\theta^3 |z|^2 + \theta |\frac{\partial z}{\partial x_1}|^2] e^{-2s\varphi}\,dx\,dt. \end{aligned} \tag{3.5}$$

Claim 2.

$$\begin{aligned} &\int_0^T \int_{\widetilde{\omega}} \theta |\frac{\partial z}{\partial x_1}|^2 e^{-2s\varphi}\,dx\,dt \\ &\leq C\varepsilon^{-1} \int_0^T \int_{\widehat{\omega}} [|\nabla q|^2 + |\nabla(\frac{\partial q}{\partial x_1})|^2]\,dx\,dt + \varepsilon \int_Q \theta |\frac{\partial z}{\partial x_1}|^2 e^{-2s\varphi}\,dx\,dt. \end{aligned} \tag{3.6}$$

Next we apply (3.1) to $\widetilde{z}(t) = iz(T-t)$ and $\widetilde{q}(t) = q(T-t)$ to obtain

$$\int_Q [\theta^3|z|^2 + \theta|\frac{\partial z}{\partial x_1}|^2]e^{-2s\varphi}\,dx\,dt \le C\int_0^T\int_{\widetilde{\omega}}[\theta^3|z|^2 + \theta|\frac{\partial z}{\partial x_1}|^2]e^{-2s\varphi}\,dx\,dt, \tag{3.7}$$

and

$$\begin{aligned}&\int_Q \theta^3|q|^2e^{-2s\varphi}\,dx\,dt\\ &\le C\Big(\int_0^T\int_{\widetilde{\omega}}[\theta^3|q|^2 + \theta|\frac{\partial q}{\partial x_1}|^2]e^{-2s\varphi}\,dx\,dt + \int_0^T\int_\Omega \rho_{\mathcal{O}}^2|z|^2e^{-2s\varphi}\,dx\,dt\Big).\end{aligned} \tag{3.8}$$

Taking ε small enough in (3.5), (3.6) and combining with (3.7) we obtain

$$\int_Q [\theta^3|z|^2 + \theta|\frac{\partial z}{\partial x_1}|^2]e^{-2s\varphi}\,dx\,dt \le C\int_0^T\int_{\widehat{\omega}}[|q|^2 + |\nabla q|^2 + |\nabla(\frac{\partial q}{\partial x_1})|^2]\,dx\,dt, \tag{3.9}$$

whereas a combination with (3.8) leads to

$$\int_Q \theta^3|q|^2e^{-2s\varphi}\,dx\,dt \le C\int_0^T\int_{\widehat{\omega}}[|q|^2 + |\nabla q|^2]\,dx\,dt + C\int_Q |z|^2\,dx\,dt, \tag{3.10}$$

for some constant $C > 0$.

We know that $-i\Delta$ generates a group of isometries $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on $L^2(\Omega)$ so $z(t) = \mathcal{T}(t-T)z^0$, and (3.9) yields

$$\begin{aligned}\|z(0)\|_{L^2}^2 &= \frac{2}{T}\int_{T/4}^{3T/4}\int_\Omega |z|^2\,dx\,dt\\ &\le C\int_Q \theta^3|z|^2e^{-2s\varphi}\,dx\,dt\\ &\le C\int_0^T\int_{\widehat{\omega}}[|q|^2 + |\nabla q|^2 + |\nabla(\frac{\partial q}{\partial x_1})|^2]\,dx\,dt.\end{aligned} \tag{3.11}$$

Multiplying the second equation in (1.7) by $i\overline{q}$ and integrating with respect to space we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |q(x,T-t)|^2dx - \mathrm{Re}\Big(i\int_\Omega |\nabla q(x,T-t)|^2dx\Big) = \mathrm{Re}\int_\Omega i\rho_{\mathcal{O}}(x)(z\overline{q})(x,T-t)dx;$$

therefore

$$\frac{d}{dt}\int_\Omega |q(x,T-t)|^2dx \le \int_\Omega |q(x,T-t)|^2dx + \int_\Omega |z(x,T-t)|^2dx$$

and the Gronwall inequality implies

$$\begin{aligned}\|q(0)\|_{L^2}^2 &\le e^{T-t}\int_\Omega |q(x,T-t)|^2dx + \int_t^T e^{T-s}\int_\Omega |z(x,T-s)|^2dxds\\ &\le e^T\int_\Omega |q(x,T-t)|^2dx + e^T\int_Q |z|^2\,dx\,dt.\end{aligned} \tag{3.12}$$

Integrating (3.12) with respect to time on $[T/4, 3T/4]$, and using (3.10) we obtain

$$\begin{aligned} \|q(0)\|_{L^2}^2 &\leq \frac{2e^T}{T}\int_{T/4}^{3T/4}\int_\Omega |q|^2\,dx\,dt + Te^T\|z(0)\|_{L^2}^2 \\ &\leq C\int_Q \theta^3|q|^2e^{-2s\varphi}\,dx\,dt + Te^T\|z(0)\|_{L^2}^2 \\ &\leq C\int_0^T\int_{\widehat{\omega}}[|q|^2+|\nabla q|^2]\,dx\,dt + C\|z(0)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.13}$$

From (3.11) and (3.13) we obtain

$$\begin{aligned} \|(q(0),z(0))\|_{(L^2(\Omega))^2}^2 &\leq C\int_0^T\int_{\widehat{\omega}}[|\nabla q|^2+|q|^2+|\nabla(\frac{\partial q}{\partial x_1})|^2]\,dx\,dt \\ &\leq C\int_0^T\int_\Omega |\Delta(\rho_\omega q)|^2\,dx\,dt \end{aligned}$$

where ρ_ω is a cut-off function with support in $\overline{\omega}$ and such that $\rho_\omega = 1$ in $\widehat{\omega}$. This completes the proof of Proposition 3.4, and it only remains to prove the two claims.

We consider a function $\sigma = \sigma(x_1) \in C_c^\infty(\mathbb{R})$ such that $\sigma \equiv 1$ in $[b, c]$ and

$$\operatorname{supp}(\sigma) \subset P_1(\widehat{\omega}), \tag{3.14}$$

where P_1 is the canonical projection onto the x_1-axis. Also we set

$$\eta(t) = t^{-1}(T-t)^{-1}.$$

Proof of Claim 1. Recall that z satisfies the homogeneous Schrödinger equation and $\theta^m e^{-2s\varphi}$ vanishes at $t = 0$, $t = T$ for every $m \geq 0$. Multiplying the second equation in (1.7) by $\sigma\theta^3\bar{z}e^{-2s\varphi}$ and integrating in Q we obtain

$$\int_0^T\int_{\mathcal{O}}\sigma\theta^3|z|^2e^{-2s\varphi}\,dx\,dt = \int_0^T\int_\Omega \sigma\theta^3e^{-2s\varphi}\overline{z}(iq_t+\Delta q)\,dx\,dt. \tag{3.15}$$

Integrating by parts the right hand side of (3.15), having in mind (3.14) and using that $\sigma\bar{z} = \sigma q = 0$ on $\partial\widehat{\omega}$, we obtain

$$\begin{aligned} &\int_0^T\int_\Omega \sigma\theta^3e^{-2s\varphi}\overline{z}(iq_t+\Delta q)\,dx\,dt \\ &= \int_0^T\int_{\widehat{\omega}} q\nabla(\sigma\theta^3e^{-2s\varphi})\cdot\nabla\overline{z} - \overline{z}\nabla(\sigma\theta^3e^{-2s\varphi})\cdot\nabla q\,dx\,dt \\ &\quad - i\int_0^T\int_{\widehat{\omega}}\sigma(\theta^3e^{-2s\varphi})_t\overline{z}q\,dx\,dt. \end{aligned} \tag{3.16}$$

Straightforward computations show that

$$\begin{gathered} (\theta^3e^{-2s\varphi})_t = \theta^3e^{-2s\varphi}(3-2s\varphi)\frac{2t-T}{t(T-t)}, \\ \nabla(\sigma\theta^3e^{-2s\varphi}) = \theta^3e^{-2s\varphi}(3\lambda\sigma\psi' + \sigma' + 2s\lambda\sigma\theta\psi')e_1. \end{gathered}$$

Note that $\theta \leq C\eta$ on $\overline{\Omega}$. Therefore,

$$\begin{gathered} |(\theta^3e^{-2s\varphi})_t| \leq C\theta^{3/2}\eta^{7/2}e^{-2s\varphi}, \\ |\nabla(\sigma\theta^3e^{-2s\varphi})| \leq C\theta^{1/2}\eta^{7/2}e^{-2s\varphi}. \end{gathered}$$

Hence, taking into account that $|e^{-2s\varphi}\eta^m| \le C$ for any $m > 0$, Claim 1 follows from (3.15), (3.16) and the Cauchy-Schwarz inequality $|ab| \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$, $\varepsilon > 0$.

Proof of Claim 2. The Green identity implies

$$\begin{aligned}
&\int_{\widehat{\omega}} \Delta(\sigma\theta e^{-2s\varphi} \frac{\partial \overline{q}}{\partial x_1}) \frac{\partial z}{\partial x_1} dx \\
&= \int_{\widehat{\omega}} \sigma\theta e^{-2s\varphi} \frac{\partial \overline{q}}{\partial x_1} \Delta(\frac{\partial z}{\partial x_1}) dx + \int_{\partial\widehat{\omega}} \frac{\partial z}{\partial x_1} \frac{\partial}{\partial \nu}(\sigma\theta e^{-2s\varphi} \frac{\partial \overline{q}}{\partial x_1}) dS \\
&\quad - \int_{\partial\widehat{\omega}} \sigma\theta e^{-2s\varphi} \frac{\partial \overline{q}}{\partial x_1} \frac{\partial}{\partial \nu}(\frac{\partial z}{\partial x_1}) dS,
\end{aligned}$$

and then, since $\frac{\partial q}{\partial x_1} = \frac{\partial z}{\partial x_1} = 0$ on $\partial\widehat{\omega} \cap \partial\Omega$ we obtain that

$$\begin{aligned}
&\int_0^T \int_{\widetilde{\omega}} \theta |\frac{\partial z}{\partial x_1}|^2 e^{-2s\varphi} \, dx \, dt \\
&\le \int_0^T \int_{\widehat{\omega}} \sigma\theta e^{-2s\varphi} \frac{\partial z}{\partial x_1} \frac{\partial}{\partial x_1} (\overline{iq_t + \Delta q}) \, dx \, dt \\
&= +i \int_0^T \int_{\widehat{\omega}} [\sigma(\theta e^{-2s\varphi})_t + i\Delta(\sigma\theta e^{-2s\varphi})] \frac{\partial z}{\partial x_1} \frac{\partial \overline{q}}{\partial x_1} \, dx \, dt \\
&\quad - 2 \int_0^T \int_{\widehat{\omega}} [\nabla(\sigma\theta e^{-2s\varphi}) \cdot \nabla(\frac{\partial \overline{q}}{\partial x_1})] \frac{\partial z}{\partial x_1} \, dx \, dt.
\end{aligned}$$

From these estimates and taking into account that

$$\begin{gathered}
|(\theta e^{-2s\varphi})_t| + |\Delta(\sigma\theta e^{-2s\varphi})| \le C\theta^{1/2}\eta^{5/2} e^{-2s\varphi}, \\
|\nabla(\sigma\theta e^{-2s\varphi})| \le C\theta^{1/2}\eta^{3/2} e^{-2s\varphi},
\end{gathered}$$

we obtain the proof of Claim 2. □

4. Proof of the main result

In this section we will deduce Theorem 1.1 from Proposition 3.4.

Proof of Theorem 1.1. Note that the observability inequality for $(z^0, q^0) \in X^2$ implies that we can define a norm

$$\|(z^0, q^0)\|_V^2 = \int_0^T \int_\Omega |\Delta(\rho_\omega q)|^2 \, dx \, dt$$

where q is the solution to

$$\begin{gathered}
iz_t + \Delta z = 0 \quad \text{in } Q, \\
iq_t + \Delta q = z\rho_{\mathcal{O}} \quad \text{in } Q, \\
z = 0, q = 0 \quad \text{on } \Sigma, \\
z(x,T) = z^0(x), \quad q(x,T) = q^0(x) \quad \text{in } \Omega.
\end{gathered}$$

Indeed, $\|(z^0, q^0)\|_V$ is clearly a semi-norm and the observability inequality implies that $\|(z^0, q^0)\|_V = 0$ takes to $(z(0), q(0)) = 0$. Using the conservation of energy for z we obtain easily that $z^0 = 0$ and then $q^0 = 0$. We set the space V as the completion of X^2 with this norm. For $(z^0, q^0) \in V$ and given fixed initial data $(p^0, u^0) \in L^2(\Omega)^2$ we define the functional

$$J(z^0, q^0) = \frac{1}{2} \int_0^T \int_\Omega |\Delta(\rho_\omega q)|^2 \, dx \, dt + \int_\Omega q(0) p^0 + \int_\Omega z(0) u^0.$$

Standard arguments show that J is continuous, convex and, thanks to the observability inequality, coercive. This implies that J reaches a minimum at a point $(\widehat{z}^0, \widehat{q}^0) \in V$. At this point the following optimality condition holds

$$\int_0^T \int_\Omega \Delta(\rho_\omega \widehat{q}) \Delta(\rho_\omega q) + \int_\Omega q(0) p^0 + \int_\Omega z(0) u^0 = 0 \tag{4.1}$$

for any $(z^0, q^0) \in V$. Now we propose as control $h_\omega = \Delta^2(\rho_\omega \widehat{q}) \rho_\omega$. Note that the corresponding solution to (1.1) satisfies $(p(T), u(T)) = 0$. In fact, (4.1) is valid for any $(z^0, q^0) \in X^2$ and (z, q) the corresponding solution to (1.7). So taking the duality product of (1.1) by (z, q) we obtain

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \Delta(\rho_\omega \widehat{q}) \Delta(\rho_\omega q) + \int_\Omega q(0) p^0 + \int_\Omega z(0) u^0 \\ &= \langle p(T), q^0 \rangle_{X', X} + \langle u(T), z^0 \rangle_{X', X} \end{aligned}$$

and the null controllability result is proved. □

5. Concluding remarks

It would be interesting to find similar results in the case of three or more coupled Schrödinger equations. In the case of parabolic equations there are various results. In particular in [9] the case of *cascade* system of parabolic equations was treated by a repeated argument of Carleman inequalities and local energy estimates. For the case of Schrödinger equations this is not the case, at least when the control set do not satisfy the geometric condition. Let us consider the case of three coupled Schrödinger equations in cascade. The adjoint system will be of the form

$$\begin{gathered} i z_t + \Delta z = 0 \quad \text{in } Q, \\ i q_t + \Delta q = z \rho_{\mathcal{O}} \quad \text{in } Q, \\ i v_t + \Delta v = q \rho_{\mathcal{O}} \quad \text{in } Q, \\ z = 0, \quad q = 0, \quad v = 0 \quad \text{on } \Sigma, \\ z(x, T) = z^0(x), \quad q(x, T) = q^0(x), \quad v(x, T) = v^0(x) \quad \text{in } \Omega. \end{gathered}$$

Using the arguments in this article it is not difficult to see that the following observability inequality holds

$$\|(z(0), q(0), v(0))\|^2_{(L^2(\Omega))^3} \le C \int_0^T \int_\Omega |\Delta(\rho_\omega q)|^2 + |\nabla(\rho_\omega v)|^2 \, dx \, dt. \tag{5.1}$$

This inequality implies that the system

$$\begin{gathered} i p_t + \Delta p = h^1_\omega \quad \text{in } Q, \\ i u_t + \Delta u = p \rho_{\mathcal{O}} + h^2_\omega \quad \text{in } Q, \\ i w_t + \Delta w = u \rho_{\mathcal{O}} \quad \text{in } Q, \\ p = 0, u = 0, w = 0 \quad \text{on } \Sigma, \\ p(x, 0) = p^0(x), \quad u(x, 0) = u^0(x), \quad w(x, 0) = w^0(x) \quad \text{in } \Omega, \end{gathered}$$

with controls $h^1_\omega \in L^2(0,T;X')$ and $h^2_\omega \in L^2(0,T;H^{-1}(\Omega))$ is null controllable. However to control only in the first equation it would be necessary to eliminate the term in q in the observability inequality (5.1). It seems to be an open problem that cannot be treated with the techniques used in this article. As far as we know this problem is still open even in the case in which ω satisfies the geometric condition.

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