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UNIFORM DECAY OF SOLUTIONS FOR COUPLED VISCOELASTIC WAVE EQUATIONS

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Abstract. In this article, we consider a system of two coupled viscoelastic equations with Dirichlet boundary conditions. By using the perturbed energy method, we obtain a general decay result which depends on the behavior of the relaxation functions and source terms.

1. Introduction

In this article, we study the coupled system of quasi-linear viscoelastic equations

$$
\begin{gathered}
|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds + f_1(u,v) = 0, \quad (x,t) \in \Omega \times (0,\infty), \\
|v_t|^\rho v_{tt} - \Delta v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)ds + f_2(u,v) = 0, \quad (x,t) \in \Omega \times (0,\infty), \\
u = v = 0, \quad (x,t) \in \partial\Omega \times (0,\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \bar{\Omega}, \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \bar{\Omega},
\end{gathered} \tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \ge 1)$ with smooth boundary $\partial\Omega$, ρ satisfies

$$
\begin{gathered}
0 < \rho \le \frac{2}{n-2}, \quad n \ge 3, \\
\rho > 0, \quad n = 1, 2.
\end{gathered} \tag{1.2}
$$

The functions u_0, u_1, v_0, v_1 are given initial data. The functions g_1, g_2, f_1, f_2 will be specified later.

The study of the asymptotic behavior of viscoelastic problems has attracted lots of interest of researchers. The pioneer work of Dafermos [4] studied a one-dimensional viscoelastic problem, established some existence and asymptotic stability results for smooth monotone decreasing relaxation functions. Muñoz Rivera [18] considered equations for linear isotropic viscoelastic solids of integral type, and established exponential decay and polynomial decay in a bounded domain and in

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the whole space respectively. Messaoudi [12] considered a nonlinear viscoelastic wave equation with source and damping terms

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + u_t|u_t|^{m-1} = u|u|^{p-1}. \tag{1.3}$$

He established blow-up result for solutions with negative initial energy and $m < p$, and gave a global existence result for arbitrary initial if $m \geq p$. This work was later improved by Messaoudi [13].

Liu [10] considered the equation with initial-boundary value conditions

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)|u_t|^m u_t + b|u|^r u = 0, \tag{1.4}$$

he established exponential or polynomial decay result which depends on the rate of the decay of the relaxation function g. Song et al [23] studied the problem (1.4) with replacing $a(x)|u_t|^m u_t$ by $a(x)u_t$, they obtained general decay result.

Cavalcanti et al [3] discussed the wave equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)div[a(x)\nabla u(\tau)]d\tau + b(x)f(u_t) = 0 \tag{1.5}$$

on a compact Riemannian manifold $(M, \mathbf{g})$ subject to a combination of locally distributed viscoelastic and frictional dissipations. It is shown that the solutions decay according to the law dictated by the decay rates corresponding to the slowest damping.

Muñoz Rivera and Naso [19] studied a viscoelastic systems with nondissipative kernels, and showed that if the kernel function decays exponentially to zero, then the solution decays exponentially to zero. On the other hand, if the kernel function decays polynomially as t^{-p}, then the corresponding solution also decays polynomially to zero with the same rate of decay.

Wu [24] considered the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}u, \tag{1.6}$$

he also improved some results to obtain the decay rate of the energy under the suitable conditions.

Cavalcanti et al [2] discussed a quasilinear initial-boundary value problem of equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(\tau)d\tau - \gamma\Delta u_t = bu|u|^{p-2}, \tag{1.7}$$

with Dirichlet boundary condition, where $\rho > 0$, $\gamma \geq 0$, $p \geq 2$, $b = 0$. An exponential decay result for $\gamma > 0$ and $b = 0$ has been obtained. For $\gamma = 0$ and $b > 0$, Messaoudi and Tatar [16], [17] showed that there exists an appropriate set, called stable set, such that if the initial data are in stable set, the solution continuous to live there forever, and the solution approaches zero with an exponential or polynomial rate depending on the decay rate of relaxation function. For other related single wave equation, we refer the reader to [7, 14, 21].

Han and Wang [6] studied the initial-boundary value problem for a coupled system of nonlinear viscoelastic equations

$$u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-1}u_t = f_1(u,v), \quad (x,t) \in \Omega \times (0,T),$$

$$
\begin{gathered}
v_{tt}-\Delta v+\int_0^t g_2(t-\tau)\Delta v(\tau)d\tau+|v_t|^{m-1}v_t=f_2(u,v),\quad (x,t)\in\Omega\times(0,T),\\
u=v=0,\quad (x,t)\in\partial\Omega\times(0,T),\\
u(x,0)=u_0(x),u_t(x,0)=u_1(x),\quad x\in\Omega,\\
v(x,0)=v_0(x),v_t(x,0)=v_1(x),\quad x\in\Omega.
\end{gathered}\tag{1.8}
$$

Existence of local and global solutions, uniqueness, and blow up in finite time were obtained when f_1, f_2, g_1, g_2 and the initial values satisfy some conditions.

Messaoudi and Said-Houari [15] dealt with the problem (1.8) and proved a global nonexistence of solutions for a large class of initial data for which the initial energy takes positive values. Also, Said-Houari et al [22] discussed (1.8) and proved a general decay result.

Liu [9] studied the coupled equations

$$
\begin{gathered}
u_{tt}-\Delta u+\int_0^t g(t-s)\Delta u(x,s)ds+f_1(u,v)=0,\\
v_{tt}-\Delta v+\int_0^t h(t-s)\Delta v(x,s)ds+f_2(u,v)=0,
\end{gathered}\tag{1.9}
$$

he proved that the decay rate of the solution energy is similar to that of relaxation functions which is not necessarily of exponential or polynomial type. Others similar problems were considered in [1, 20].

Motivated by the above researches, we consider the system (1.1). Liu [8] already considered the system (1.1), and obtained the exponential or polynomial decay of the solutions energy depending on the decay rate of the relaxation functions. In [8], the relaxation functions $g_i(t)$ $(i=1,2)$ satisfy $g_i'(t)\le -\xi_i g_i^{p_i}(t)$ for all $t\ge 0$, $p_i\in[1,3/2)$ and some constants ξ_1, ξ_2. In this paper, the conditions have been replaced by $g_i'(t)\le -\xi_i(t)g_i(t)$ where $\xi_i(t)$ are positive non-increasing functions. This allow us to obtain a general decay rate than just exponential or polynomial type. We use the perturbed energy method to obtain a general decay of solutions energy. The rest of this article is organized as follows. Some preparation and main result are given in Section 2. In Section 3, we give the proof of our main result.

2. Preliminaries and statement of main results

We denote the norm in $L^\rho(\Omega)$ by $\|\cdot\|_\rho, 1\le\rho<\infty$. The Dirichlet norm in $H_0^1(\Omega)$ is $\|\nabla\cdot\|_2$. C and C_i denote general constants, which may be different in different estimates.

Throughout this paper, we use the following notation,

$$
(\phi\circ\psi)(t)=\int_0^t\phi(t-\tau)\|\psi(t)-\psi(\tau)\|_2^2d\tau.
$$

To state our main result, we need the following assumptions.

(A1) $g_i:R^+\to R^+, i=1,2$, are differentiable functions such that

$$
g_i(0)>0,\quad 1-\int_0^{+\infty}g_i(s)ds=l_i>0,
$$

and there exist non-increasing functions $\xi_1,\xi_2:R^+\to R^+$ satisfying

$$
g_i'(t)\le -\xi_i(t)g_i(t),\quad t\ge 0.
$$

(A2) There exists nonnegative function $F(u,v)$ such that
$$f_1(u,v)=\frac{\partial F(u,v)}{\partial u},\quad f_2(u,v)=\frac{\partial F(u,v)}{\partial v},$$
and there exist constants $C,d>0$ such that
$$\begin{gathered} uf_1(u,v)+vf_2(u,v)\geq CF(u,v),\\ |f_1(u,v)|\leq d(|u+v|^{p-1}+|u|^{\frac{p}{2}-1}|v|^{\frac{p}{2}}),\\ |f_2(u,v)|\leq d(|u+v|^{p-1}+|u|^{\frac{p}{2}}|v|^{\frac{p}{2}-1}), \end{gathered}$$
where $p>2$ if $n=1,2$ and $2<p\leq\frac{2(n-1)}{n-2}$ if $n\geq 3$.

By using the Galerkin method, as in [11], we can obtain the existence of a local weak solution to (1.1). We omit the proof here.

Theorem 2.1. *Assume that* (A1), (A2) *hold. For the initial data* $(u_0,v_0,u_1,v_1)\in(H_0^1(\Omega))^4$*, there exists at least one weak local solution* (u,v) *such that for some* $T>0$*,*
$$u,v\in L^\infty(0,T;H_0^1(\Omega)),\quad u_t,v_t\in L^\infty(0,T;H_0^1(\Omega)),\quad u_{tt},v_{tt}\in L^2(0,T;H_0^1(\Omega)).$$

We introduce the energy functional of system (1.1),
$$\begin{aligned} E(t)=&\frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2}+\frac{1}{\rho+2}\|v_t\|_{\rho+2}^{\rho+2}+\frac12\|\nabla u_t\|_2^2+\frac12\|\nabla v_t\|_2^2\\ &+\frac12(g_1\circ\nabla u)+\frac12(g_2\circ\nabla v)+\frac12\Big(1-\int_0^t g_1(s)ds\Big)\|\nabla u\|_2^2\\ &+\frac12\Big(1-\int_0^t g_2(s)ds\Big)\|\nabla v\|_2^2+\int_\Omega F(u,v)dx. \end{aligned}\tag{2.1}$$
It is easy to prove that
$$E'(t)=\frac12(g_1'\circ\nabla u)(t)+\frac12(g_2'\circ\nabla v)(t)-\frac12 g_1(t)\|\nabla u(t)\|_2^2-\frac12 g_2(t)\|\nabla v\|_2^2\leq 0.\tag{2.2}$$
Then we have
$$\|\nabla u_t\|_2^2+\|\nabla v_t\|_2^2+l_1\|\nabla u\|_2^2+l_2\|\nabla v\|_2^2\leq 2E(0).\tag{2.3}$$
Our main result reads as follows.

Theorem 2.2. *Assume that* (A1), (A2) *hold. Let* $(u_0,v_0,u_1,v_1)\in(H_0^1(\Omega))^4$ *be given, and* (u,v) *be the solution to* (1.1)*. Then for any* $t_1>0$ *there exist positive constants* C *and* α *such that for all* $t\geq t_1$*,*
$$E(t)\leq Ce^{-\alpha\int_{t_1}^t\xi(\tau)d\tau},$$
where $\xi(t)=\min\{\xi_1(t),\xi_2(t)\}$*.*

3. Decay result

To prove the general decay result, we define the perturbed modified energy functional
$$L(t)=ME(t)+\varepsilon I(t)+J(t),$$
where M and ε are positive constants to be specified later and
$$I(t)=\frac{1}{\rho+1}\int_\Omega|u_t|^\rho u_t u dx+\frac{1}{\rho+1}\int_\Omega|v_t|^\rho v_t v dx+\int_\Omega\nabla u_t\nabla u dx+\int_\Omega\nabla v_t\nabla v dx,$$

$$J(t) = J_1(t) + J_2(t),$$

where

$$J_1(t) = \int_\Omega \Big(\Delta u_t - \frac{|u_t|^\rho u_t}{\rho+1}\Big) \int_0^t g_1(t-\tau)\left(u(t)-u(\tau)\right) d\tau\, dx,$$
$$J_2(t) = \int_\Omega \Big(\Delta v_t - \frac{|v_t|^\rho v_t}{\rho+1}\Big) \int_0^t g_2(t-\tau)\big(v(t)-v(\tau)\big) d\tau\, dx.$$

Firstly, we have the following lemmas.

Lemma 3.1 ([5]). *Under assumption* (A1), *if* (u,v) *is the solution of* (1.1), *then the following hold for* $i=1,2$*:*

$$\int_\Omega \Big(\int_0^t g_i(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right) d\tau\Big)^2 dx \le C_i(g_i \circ \nabla u), \tag{3.1}$$
$$\int_\Omega \Big(\int_0^t -g_i'(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right) d\tau\Big)^2 dx \le -C_i(g_i' \circ \nabla u). \tag{3.2}$$

Lemma 3.2. *Let* (A1), (A2) *hold and* (u,v) *be the solution of* (1.1). *Then*

$$\begin{aligned} I'(t) \le& -\frac{l_1}{2}\|\nabla u\|_2^2 + \frac{C_1}{4\delta}(g_1\circ\nabla u) + \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \\ &+ \|\nabla u_t\|_2^2 - \frac{l_2}{2}\|\nabla v\|_2^2 + \frac{C_2}{4\delta}(g_2\circ\nabla v) + \frac{1}{\rho+1}\|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v_t\|_2^2 \\ &- C\int_\Omega F(u,v)dx, \end{aligned} \tag{3.3}$$

in which $\delta = \min\{\frac{l_1}{2}, \frac{l_2}{2}\}$.

Proof. Differentiating $I(t)$ and using (1.1), we obtain

$$\begin{aligned} I'(t) =& \frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} - \|\nabla u\|_2^2 + \int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\nabla u(\tau)\, d\tau\, dx \\ &+ \frac{1}{\rho+1}\|v_t\|_{\rho+2}^{\rho+2} - \|\nabla v\|_2^2 + \int_\Omega \nabla v(t)\int_0^t g_2(t-\tau)\nabla v(\tau)\, d\tau\, dx \\ &- \int_\Omega (uf_1+vf_2)dx + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2. \end{aligned} \tag{3.4}$$

Using (3.1) and Young's inequality, we can estimate the third term of (3.4) as follows

$$\begin{aligned} &\int_\Omega \nabla u(t)\int_0^t g_1(t-\tau)\nabla u(\tau)\, d\tau\, dx \\ &= \int_\Omega \nabla u\int_0^t g_1(t-\tau)\left(\nabla u(\tau)-\nabla u(t)+\nabla u(t)\right) d\tau\, dx \\ &= \|\nabla u\|_2^2\int_0^t g_1(\tau)d\tau + \int_\Omega \nabla u\int_0^t g(t-\tau)\left(\nabla u(\tau)-\nabla u(t)\right) d\tau\, dx \\ &\le \|\nabla u\|_2^2\int_0^t g_1(\tau)d\tau + \delta\|\nabla u\|_2^2 + \frac{1}{4\delta}\int_\Omega \Big(\int_0^t g_1(t-\tau)|\nabla u(\tau)-\nabla u(t)| d\tau\Big)^2 dx \\ &\le \|\nabla u\|_2^2\int_0^t g_1(\tau)d\tau + \delta\|\nabla u\|_2^2 + \frac{C_1}{4\delta}(g_1\circ\nabla u). \end{aligned} \tag{3.5}$$

Similarly, we obtain

$$\begin{aligned} &\int_\Omega \nabla v(t)\int_0^t g_2(t-\tau)\nabla v(\tau)\,d\tau\,dx \\ &\le \|\nabla v\|_2^2\int_0^t g_2(\tau)d\tau+\delta\|\nabla v\|_2^2+\frac{C_2}{4\delta}(g_2\circ\nabla v). \end{aligned} \tag{3.6}$$

From (3.4)–(3.6), we obtain

$$\begin{aligned} I'(t) &\le -\Big(1-\int_0^t g_1(\tau)d\tau-\delta\Big)\|\nabla u\|_2^2+\frac{C_1}{4\delta}(g_1\circ\nabla u)+\frac{1}{\rho+1}\|u_t\|_{\rho+2}^{\rho+2} \\ &\quad+\|\nabla u_t\|_2^2-\Big(1-\int_0^t g_2(\tau)d\tau-\delta\Big)\|\nabla v\|_2^2+\frac{C_2}{4\delta}(g_2\circ\nabla v) \\ &\quad+\frac{1}{\rho+1}\|v_t\|_{\rho+2}^{\rho+2}+\|\nabla v_t\|_2^2-\int_\Omega(uf_1+vf_2)dx. \end{aligned} \tag{3.7}$$

We can choose $\delta=\min\{\frac{l_1}{2},\frac{l_2}{2}\}$. From (3.7) and (A1), (3.3) follows. □

Lemma 3.3. *Under assumptions* (A1), (A2)*, we have*

$$\begin{aligned} J'(t) &\le \big(\delta+2\delta(1-l_2)^2+2C\delta\big)\|\nabla v\|_2^2+\big(\delta+2\delta(1-l_1)^2+2C\delta\big)\|\nabla u\|_2^2 \\ &\quad+\Big(\frac{3C_2}{4\delta}+2\delta C_2\Big)(g_2\circ\nabla v)+\Big(\frac{3C_1}{4\delta}+2\delta C_1\Big)(g_1\circ\nabla u) \\ &\quad+\Big(\frac{C_2}{4\delta}+\frac{C_2}{4\delta(\rho+1)}\Big)(g_2'\circ\nabla v)+\Big(\frac{C_1}{4\delta}+\frac{C_1}{4\delta(\rho+1)}\Big)(g_1'\circ\nabla u) \\ &\quad-\Big(\int_0^t g_2(s)ds-\delta-\frac{\delta}{\rho+1}\left(2E(0)\right)^\rho\Big)\|\nabla v_t\|_2^2 \\ &\quad-\Big(\int_0^t g_1(s)ds-\delta-\frac{\delta}{\rho+1}\left(2E(0)\right)^\rho\Big)\|\nabla u_t\|_2^2 \\ &\quad-\frac{1}{\rho+1}\Big(\int_0^t g_2(s)ds\Big)\|v_t\|_{\rho+2}^{\rho+2}-\frac{1}{\rho+1}\Big(\int_0^t g_1(s)ds\Big)\|u_t\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.8}$$

Proof. Differentiating $J_1(t)$ and using (1.1), we obtain

$$\begin{aligned} J_1'(t) &= \int_\Omega \nabla u(t)\Big(\int_0^t g_1(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)d\tau\Big)dx \\ &\quad-\int_\Omega\Big(\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g_1(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)d\tau\Big)dx \\ &\quad+\int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)\left(u(t)-u(\tau)\right)\,d\tau\,dx-\Big(\int_0^t g_1(s)ds\Big)\|\nabla u_t\|_2^2 \\ &\quad-\int_\Omega \nabla u_t\int_0^t g_1'(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)\,d\tau\,dx \\ &\quad-\frac{1}{\rho+1}\int_\Omega|u_t|^\rho u_t\int_0^t g_1'(t-\tau)\left(u(t)-u(\tau)\right)\,d\tau\,dx \\ &\quad-\frac{1}{\rho+1}\Big(\int_0^t g_1(s)ds\Big)\|u_t\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.9}$$

By using Young's inequality and (3.1), we obtain that for some $\delta > 0$,

$$\int_\Omega \nabla u(t)\Big(\int_0^t g_1(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)d\tau\Big)dx \le \delta\|\nabla u\|_2^2 + \frac{C_1}{4\delta}(g_1\circ\nabla u). \tag{3.10}$$

For the second term of (3.9), employing Young's inequality, (A1) and (3.1), we have for some $\delta > 0$,

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t g_1(t-\tau)\nabla u(\tau)d\tau\Big)\Big(\int_0^t g_1(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)d\tau\Big)dx\\
&\le \delta\int_\Omega\Big(\int_0^t g_1(t-\tau)\left(\nabla u(\tau)-\nabla u(t)\right)d\tau + \int_0^t g_1(t-\tau)|\nabla u(t)|d\tau\Big)^2dx\\
&\quad + \frac{1}{4\delta}\int_\Omega\Big(\int_0^t g_1(t-\tau)\left(\nabla u(t)-\nabla u(\tau)\right)d\tau\Big)^2 dx\\
&\le (2\delta+\frac{1}{4\delta})\int_\Omega\Big(\int_0^t g_1(t-\tau)|\nabla u(t)-\nabla u(\tau)|d\tau\Big)^2dx\\
&\quad + 2\delta\int_\Omega\Big(\int_0^t g_1(t-\tau)d\tau\Big)^2|\nabla u(t)|^2dx\\
&\le (2\delta+\frac{1}{4\delta})C_1(g_1\circ\nabla u) + 2\delta(1-l_1)^2\|\nabla u\|_2^2.
\end{aligned} \tag{3.11}$$

Thanks to Young's inequality, Sobolev embedding theorem and (3.1), for some $\delta > 0$ we have

$$\begin{aligned}
&\int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)\left(u(t)-u(\tau)\right)\,d\tau\,dx\\
&\le \delta\int_\Omega f_1^2(u,v)dx + \frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(t-\tau)\left(u(t)-u(\tau)\right)d\tau\Big)^2dx\\
&\le \delta\int_\Omega f_1^2(u,v)dx + \frac{C_1}{4\delta}(g_1\circ u)\\
&\le \delta\int_\Omega f_1^2(u,v)dx + \frac{C_1}{4\delta}(g_1\circ\nabla u).
\end{aligned}$$

Using (A2) and the Sobolev embedding theorem and (2.3), we have

$$\begin{aligned}
\int_\Omega f_1^2(u,v)dx &\le C\Big(\int_\Omega |u+v|^{2(p-1)}dx + \int_\Omega |u|^{p-2}|v|^p dx\Big)\\
&\le C\Big(\|u\|_{2(p-1)}^{2(p-1)} + \|v\|_{2(p-1)}^{2(p-1)} + \|u\|_{n(p-2)}^{2(p-2)} + \|v\|_{\frac{np}{n-1}}^{2p}\Big)\\
&\le C\Big(\left(\|\nabla u\|_2^2\right)^{p-2}\|\nabla u\|_2^2 + \left(\|\nabla v\|_2^2\right)^{p-2}\|\nabla v\|_2^2\Big)\\
&\quad + C\Big(\left(\|\nabla u\|_2^2\right)^{p-3}\|\nabla u\|_2^2 + \left(\|\nabla v\|_2^2\right)^{p-1}\|\nabla v\|_2^2\Big)\\
&\le C\Big(\Big(\frac{2E(0)}{l_1}\Big)^{p-2}\|\nabla u\|_2^2 + \Big(\frac{2E(0)}{l_2}\Big)^{p-2}\|\nabla v\|_2^2\Big)\\
&\quad + C\Big(\Big(\frac{2E(0)}{l_1}\Big)^{p-3}\|\nabla u\|_2^2 + \Big(\frac{2E(0)}{l_2}\Big)^{p-1}\|\nabla v\|_2^2\Big)\\
&\le C\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right).
\end{aligned}$$

Then we obtain

$$\begin{aligned} &\int_\Omega f_1(u,v)\int_0^t g_1(t-\tau)\left(u(t)-u(\tau)\right)\,d\tau\,dx \\ &\le C\delta\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+\frac{C_1}{4\delta}(g_1\circ\nabla u). \end{aligned} \tag{3.12}$$

The fifth term of (3.9) yields

$$\int_\Omega \nabla u_t\int_0^t g_1'\left(\nabla u(t)-\nabla u(\tau)\right)\,d\tau\,dx \le \delta\|\nabla u_t\|_2^2+\frac{C_1}{4\delta}(g_1'\circ\nabla u). \tag{3.13}$$

We estimate the sixth term of (3.9) by using Young's inequality, Sobolev embedding theorem and (2.3) as follows

$$\begin{aligned} &\frac{1}{\rho+1}\int_\Omega |u_t|^\rho u_t\int_0^t g_1'(t-\tau)\left(u(t)-u(\tau)\right)\,d\tau\,dx \\ &\le \frac{\delta}{\rho+1}\|u_t\|_{2(\rho+1)}^{2(\rho+1)}+\frac{C_1}{4\delta(\rho+1)}(g_1'\circ\nabla u) \\ &\le \frac{\delta}{\rho+1}\left(2E(0)\right)^\rho\|\nabla u_t\|_2^2+\frac{C_1}{4\delta(\rho+1)}(g_1'\circ\nabla u). \end{aligned} \tag{3.14}$$

Inserting (3.10)–(3.14) into (3.9), we obtain

$$\begin{aligned} J_1'(t) &\le \left(\delta+2\delta(1-l_1)^2+C\delta\right)\|\nabla u\|_2^2+C\delta\|\nabla v\|_2^2 \\ &\quad+\Big(\frac{3C_1}{4\delta}+2\delta C_1\Big)(g_1\circ\nabla u)+\Big(\frac{C_1}{4\delta}+\frac{C_1}{4\delta(\rho+1)}\Big)(g_1'\circ\nabla u) \\ &\quad-\Big(\int_0^t g_1(s)ds-\delta-\frac{\delta}{\rho+1}\left(2E(0)\right)^\rho\Big)\|\nabla u_t\|_2^2 \\ &\quad-\frac{1}{\rho+1}\Big(\int_0^t g_1(s)ds\Big)\|u_t\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.15}$$

In the same way, we conclude that

$$\begin{aligned} J_2'(t) &\le \left(\delta+2\delta(1-l_2)^2+C\delta\right)\|\nabla v\|_2^2+C\delta\|\nabla u\|_2^2 \\ &\quad+\Big(\frac{3C_2}{4\delta}+2\delta C_2\Big)(g_2\circ\nabla v)+\Big(\frac{C_2}{4\delta}+\frac{C_2}{4\delta(\rho+1)}\Big)(g_2'\circ\nabla v) \\ &\quad-\Big(\int_0^t g_2(s)ds-\delta-\frac{\delta}{\rho+1}\left(2E(0)\right)^\rho\Big)\|\nabla v_t\|_2^2 \\ &\quad-\frac{1}{\rho+1}\Big(\int_0^t g_2(s)ds\Big)\|v_t\|_{\rho+2}^{\rho+2}. \end{aligned} \tag{3.16}$$

Combining the estimates (3.15) and (3.16), we can obtain (3.8). □

Proof of Theorem 2.1. It is not difficult to find positive constants a_1, a_2 such that $a_1 E(t) \le L(t) \le a_2 E(t)$. Differentiating $L(t)$, we have

$$\begin{aligned}
L'(t) \le & -\frac{1}{\rho+1}\Big(\int_0^t g_1(s)ds - \varepsilon\Big)\|u_t\|_{\rho+2}^{\rho+2} \\
& - \Big(\int_0^t g_1(s)ds - \delta - \frac{\delta}{\rho+1}\,(2E(0))^\rho - \varepsilon\Big)\|\nabla u_t\|_2^2 \\
& - \frac{1}{\rho+1}\Big(\int_0^t g_2(s)ds - \varepsilon\Big)\|v_t\|_{\rho+2}^{\rho+2} \\
& - \Big(\int_0^t g_2(s)ds - \delta - \frac{\delta}{\rho+1}\,(2E(0))^\rho - \varepsilon\Big)\|\nabla v_t\|_2^2 \\
& + \Big(\frac{3C_1}{4\delta} + 2\delta C_1 + \frac{C_1\varepsilon}{4\delta}\Big)(g_1 \circ \nabla u) \\
& + \Big(\frac{3C_2}{4\delta} + 2\delta C_2 + \frac{C_2\varepsilon}{4\delta}\Big)(g_2 \circ \nabla v) \\
& - \Big(\Big(\frac{M}{2}g_1(t) + \frac{l_1}{2}\varepsilon\Big) - \delta - 2\delta(1-l_1)^2 - 2C\delta\Big)\|\nabla u\|_2^2 \\
& - \Big(\Big(\frac{M}{2}g_2(t) + \frac{l_2}{2}\varepsilon\Big) - \delta - 2\delta(1-l_2)^2 - 2C\delta\Big)\|\nabla v\|_2^2 \\
& - C\varepsilon\int_\Omega F(u,v)dx.
\end{aligned} \tag{3.17}$$

For any $t_0 > 0$ we can pick $\varepsilon, \delta > 0$ small enough, M so large such that for $t > t_0$ there exist constants $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, and

$$\begin{aligned}
L'(t) \le & -\eta_1(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}) - \eta_2(\|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2) \\
& + \eta_3\left((g_1 \circ \nabla u) + (g_2 \circ \nabla v)\right) - \eta_4(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) - \varepsilon C\int_\Omega F(u,v)dx.
\end{aligned} \tag{3.18}$$

Then, we can choose $t_1 > t_0$ such that $\eta, C > 0$ and (3.18) takes the form

$$L'(t) \le -\eta E(t) + C\left((g_1 \circ \nabla u) + (g_2 \circ \nabla v)\right), \quad t \ge t_1. \tag{3.19}$$

Multiplying (3.19) by $\xi(t)$, by using (A1) we have

$$\begin{aligned}
& \xi(t)L'(t) \\
& \le C\int_\Omega\int_0^t \xi_1(t-\tau)g_1(t-\tau)|\nabla u(t) - \nabla u(\tau)|^2\, d\tau\, dx \\
& \quad + C\int_\Omega\int_0^t \xi_2(t-\tau)g_2(t-\tau)|\nabla v(t) - \nabla v(\tau)|^2\, d\tau\, dx - \eta\xi(t)E(t) \\
& \le -C\int_\Omega\int_0^t g_1'(t-\tau)|\nabla u(t) - \nabla u(\tau)|^2\, d\tau\, dx \\
& \quad - C\int_\Omega\int_0^t g_2'(t-\tau)|\nabla v(t) - \nabla v(\tau)|^2\, d\tau\, dx - \eta\xi(t)E(t) \\
& \le -CE'(t) - \eta\xi(t)E(t).
\end{aligned} \tag{3.20}$$

where $\xi(t) = \min\{\xi_1(t), \xi_2(t)\}$. Thanks to (A1), we obtain

$$\frac{d}{dt}\left(\xi(t)L(t) + CE(t)\right) \le -\eta\xi(t)E(t), \quad t \ge t_1. \tag{3.21}$$

By defining the functional

$$F(t) := \xi(t)L(t) + CE(t) \sim E(t), \tag{3.22}$$

we have

$$F'(t) \leq -\alpha\xi(t)F(t). \tag{3.23}$$

Then integrating over (t_1, t), we have

$$F(t) \leq F(t_1)e^{-\alpha\int_{t_1}^{t}\xi(\tau)d\tau}.$$

By using (3.22) again, the decay result follows. □

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